5

Open systems of interacting quantum fields

As introduced in Chapter 1, for many problems in statistical mechanics one is interested in the detailed behavior of only a part of the overall system (call it *the* system) interacting with its surrounding (call it the environment). In field theory one can accordingly decompose the field describing the overall system $\phi = \phi_{\mathrm{S}} + \phi_{\mathrm{E}}$ into a sum of the system field ϕ_{S} and the environment field ϕ_{E}. This decomposition is always possible formally but only when there is a clear physical discrepancy between the two sectors will it be physically meaningful and technically implementable. The division could be made between slow and fast variables, low and high frequencies or light and heavy mass sectors. Drawing examples from cosmology, in the stochastic inflation scenario one regards the system field as containing only the lower modes and the environmental field as containing the higher modes with the division provided by the event horizon in de Sitter spacetime. A similar problem in quark–gluon plasma is to ascertain the effect of the hard thermal loops on the soft gluon modes. Another is the effect of the atoms in the noncondensate on the Bose–Einstein condensate (BEC). These cases will be discussed in later chapters.

Usually the reason for performing such a decomposition is because one is interested more in the details of the system (the "relevant" variables or the "distinguished" sector), and less in that of the environment (the "irrelevant" variables). Since the environment often contains many more degrees of freedom than the system the details of which are not of particular interest to us, introducing some way of coarse graining them and extracting their overall influence on the system is desirable. This procedure renders the original system an open system, and its behavior would then be describable by the open system conceptual and technical framework we introduced in Chapter 3. In particular, the quantity of special interest is the influence action obtained from the integration over the environment field in a CTP path integral.

We recall that when the time limits in this path integral are taken to infinity, the influence action turns into the so-called closed time path (CTP) coarse-grained effective action (CGEA). The idea behind this quantity which originated from studies in dynamical critical phenomena ("coarse-grained" free energy density) was transplanted to nonequilibrium quantum field theory by Hu and Zhang [Hu91, Zha90] first in the "in–out" (Schwinger–Dewitt) formulation and then by Sinha and Hu [SinHu91] in the "in–in" (Schwinger–Keldysh) formulation. A clear presentation of the CTP CGEA can be found in Lombardo and Mazzitelli

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[LomMaz96]. (See also [CaHuMa01] for a review.) We shall restrict usage of the term effective action (EA) to the particular case in which ϕ_S is the c-number part of the field operator. The so-called background field decomposition in quantum field theory, $\Phi = \phi_c + \phi_q$, on an interacting field Φ is a special case of this open system method, where the discrepancy parameter is the Planck constant $\hbar$, separating and systemizing the quantum contributions from the classical. The familiar loop expansion (in orders of $\hbar$) of the effective action is an example of the CGEA, with the special feature that the equations of motion it yields do not contain any dissipation (unless some causal condition like the factorizable initial state similar to the Boltzmann molecular chaos assumption is introduced). We will introduce the CTP CGEA in the language of influence functionals in this chapter and introduce more formal techniques for its development in the next chapter. The IF formalism and the CTP CGEA will be our main workhorse for the rest of the book.

Our goal in this chapter is to derive the influence action and the CGEA and the stochastic equations for two simple but fundamental quantum field scenarios. We treat first the case of two interacting scalar fields, one of which is chosen as the system and the other as its environment. This case is technically easier than the second case, that of a single quantum field split into two by separating the long and short wavelength sectors (to be defined precisely below), even though the CGEA was introduced historically for the latter situation, which exemplifies a broader class of statistical mechanical problems [Hu91]. For pedagogical reasons, we will stay within the technically simplest approach in quantum field theory familiar to the reader, using a straightforward perturbative expansion in powers of the coupling constants. More powerful methods will be introduced later in the book.

5.1 Influence functional: Two interacting quantum fields

In this section we study the problem of two quantum self-interacting scalar fields (one the system field, the other the environment field) interacting with each other in Minkowski spacetime. To do so we only need to generalize to quantum field theory the results for the quantum mechanical Brownian model based on the influence functional method introduced in Chapter 3. We first derive the influence functional, from which we identify the dissipation and noise kernels. We then derive a Langevin equation for the dissipative dynamics of the system field. The nonlinear mode–mode coupling between the system field and the environment field induces a nonlinear nonlocal dissipation and a coupled multiplicative colored noise source for the system field. Finally we write down the functional quantum master equation for the system field. Our presentation in this section follows [Hu94b, Zha90].

Consider two independent self-interacting scalar fields in Minkowski spacetime: $\phi(x)$ depicting the system, and $\psi(x)$ depicting the environment. The classical

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actions for these two fields are given respectively by:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial_\nu\phi(x)\partial^\nu\phi(x) - V(\phi) \right) = S_0[\phi] + S_I[\phi] \tag{5.1}$$

$$S[\psi] = \int d^4x \left(-\frac{1}{2}\partial_\mu\psi(x)\partial^\mu\psi(x) - V(\psi) \right) = S_0[\psi] + S_I[\psi] \tag{5.2}$$

where $V[\phi], V[\psi]$ are the self-interaction potentials. For a ϕ^4 interaction,

$$V[\phi] = \frac{1}{2}m_\phi^2\phi^2(x) + \frac{1}{4!}\lambda_\phi\phi^4(x), \tag{5.3}$$

and similarly for $V[\psi]$. Here, m_ϕ and m_ψ are the bare masses and λ_ϕ and λ_ψ are the bare self-coupling constants for the $\phi(x)$ and $\psi(x)$ fields respectively. In equation (5.2) we have written $S[\psi]$ in terms of a free part S_0 and an interacting part S_I which contains λ_ψ. Assume these two scalar fields interact via a polynomial coupling of the form

$$S_{\text{int}} = \int d^4x\, V_{\phi\psi}[\phi(x)]\psi^k(x) \tag{5.4}$$

where $V_{\phi\psi}[\phi(x)] \equiv -\lambda_{\phi\psi} f[\phi(x)]$ is the vertex function with coupling constant $\lambda_{\phi\psi}$, which we assume to be small and of the same order as λ_ϕ, λ_ψ.

The total classical action of the combined system is

$$S[\phi,\psi] = S[\phi] + S[\psi] + S_{\text{int}}[\phi,\psi] \tag{5.5}$$

The total density matrix of the combined system plus environment field is defined by

$$\rho[\phi^1,\psi^1,\phi^2,\psi^2,t] = \langle\phi^1,\psi^1|\ \hat\rho(t)\ |\phi^2,\psi^2\rangle \tag{5.6}$$

where the superscripts $1,2$ are the closed time path branches to integrate over as will be described in more detail in Chapter 6, and $|\phi\rangle$ and $|\psi\rangle$ are the eigenstates of the field operators $\hat\phi(x)$ and $\hat\psi(x)$, namely,

$$\hat\phi(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle, \qquad \hat\psi(\mathbf{x})|\psi\rangle = \psi(\mathbf{x})|\psi\rangle \tag{5.7}$$

Since we are primarily interested in the behavior of the system, and of the environment only to the extent in how it influences the system, the object of interest is the reduced density matrix defined by

$$\rho_r[\phi^1,\phi^2,t] = \int d\psi\, \rho[\phi^1,\psi^1,\phi^2,\psi^1,t] \tag{5.8}$$

For technical convenience, let us assume that the total density matrix at an initial time is factorized, i.e. that the system and environment are statistically independent,

$$\hat\rho(t_i) = \hat\rho_\phi(t_i) \times \hat\rho_\psi(t_i) \tag{5.9}$$

where $\hat\rho_\phi(t_i)$ and $\hat\rho_\psi(t_i)$ are the initial density matrix operator of the ϕ and ψ field respectively, the former being equal to the reduced density matrix $\hat\rho_r$ at t_i

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by this assumption. The reduced density matrix of the system field $\phi(x)$ evolves in time following

$$\rho_r[\phi_f^1, \phi_f^2, t] = \int d\phi_i^1 \int d\phi_i^2 \; \mathcal{J}_r\left[\phi_f^1, \phi_f^2, t \mid \phi_i^1, \phi_i^2, t_i\right] \; \rho_r[\phi_i^1, \phi_i^2, t_i] \tag{5.10}$$

As in Chapter 3, the propagator $\mathcal{J}_r[\phi_f^1, \phi_f^2, t \mid \phi_i^1, \phi_i^2, t_i]$ is given by a CTP Feynman integral of the exponent of the influence action

$$\mathcal{J}_r[\phi_f^1, \phi_f^2, t \mid \phi_i^1, \phi_i^2, t_i] = \int\limits_{\phi_i^1(\mathbf{x})}^{\phi_f^1(\mathbf{x})} D\phi^1 \int\limits_{\phi_i^2(\mathbf{x})}^{\phi_f^2(\mathbf{x})} D\phi^2 \; \exp\frac{i}{\hbar} S_{\text{eff}}[\phi^1, \phi^2] \tag{5.11}$$

where

$$S_{\text{eff}}[\phi^1, \phi^2] \equiv S[\phi^1] - S[\phi^2] + S_{\text{IF}}[\phi^1, \phi^2] \tag{5.12}$$

is the full influence functional (IF) effective action and S_{IF} is the influence action. The Feynman–Vernon influence functional $\mathcal{F}[\phi^1, \phi^2]$ is defined as

$$\begin{aligned}
\mathcal{F}[\phi^1, \phi^2] &= e^{\frac{i}{\hbar} S_{\text{IF}}[\phi^1, \phi^2]} \\
&= \int d\psi_f^1(\mathbf{x}) \int d\psi_i^1(\mathbf{x}) \int d\psi_i^2(\mathbf{x}) \; \rho_\psi[\psi_i^1, \psi_i^2, t_i] \int\limits_{\psi_i^1(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^1 \int\limits_{\psi_i^2(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^2 \\
&\quad \times \exp\frac{i}{\hbar}\left\{S[\psi^1] + S_{\text{int}}[\phi^1, \psi^1] - S[\psi^2] - S_{\text{int}}[\phi^2, \psi^2]\right\}
\end{aligned} \tag{5.13}$$

which summarizes the averaged effect of the bath on the system. For a zero-temperature bath (i.e. the environment field ψ is in a vacuum state, $\hat{\rho}_b(t_i) = |0\rangle\langle 0|$), the influence functional $\mathcal{F}$ is formally equivalent to the CTP vacuum generating functional, and the influence action S_{IF} in equation (5.12) is the usual CTP vacuum effective action, to be discussed in the next chapter.

5.1.1 Perturbation theory

The above formal framework is nice but often difficult to tackle. To evaluate the influence action we need to develop a perturbation theory. If $\lambda_{\phi\psi}$ and λ_ψ are assumed to be small parameters, the influence functional can be calculated perturbatively by making a power expansion of $\exp\frac{i}{\hbar}\left[S_{\text{int}} + S_I\right]$. In this section, we set $\lambda_\psi = 0$ for simplicity. Up to second order in $\lambda_{\phi\psi}$, and first order in $\hbar$ (one-loop), the influence action is given by

$$\begin{aligned}
S_{\text{IF}}[\phi^1, \phi^2] &= \langle S_{\text{int}}[\phi^1, \psi^1]\rangle_0 - \langle S_{\text{int}}[\phi^2, \psi^2]\rangle_0 \\
&\quad + \frac{i}{2\hbar}\left\{\langle S_{\text{int}}[\phi^1, \psi^1]^2\rangle_0 - \langle S_{\text{int}}[\phi^1, \psi^1]\rangle_0{}^2\right\} \\
&\quad - \frac{i}{\hbar}\left\{\langle S_{\text{int}}[\phi^1, \psi^1] S_{\text{int}}[\phi^2, \psi^2]\rangle_0 - \langle S_{\text{int}}[\phi^1, \psi^1]\rangle_0 \langle S_{\text{int}}[\phi^2, \psi^2]\rangle_0\right\} \\
&\quad + \frac{i}{2\hbar}\left\{\langle S_{\text{int}}[\phi^2, \psi^2]^2\rangle_0 - \langle S_{\text{int}}[\phi^2, \psi^2]\rangle_0{}^2\right\}
\end{aligned} \tag{5.14}$$

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where the quantum average of a physical variable $\mathcal{Q}[\psi^1, \psi^2]$ over the unperturbed action $S_0[\psi]$ is defined by

$$\begin{aligned}
\langle \mathcal{Q}[\psi^1, \psi^2] \rangle_0 &= \int d\psi_f^1(\mathbf{x}) \int d\psi_i^1(\mathbf{x}) \int d\psi_i^2(\mathbf{x}) \ \rho_\psi[\psi_i^1, \psi_i^2, t_i] \\
&\times \int_{\psi_i^1(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^1 \int_{\psi_i^2(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^2 \ \exp \frac{i}{\hbar} \{ S_0[\psi^1] - S_0[\psi^2] \} \times \mathcal{Q}[\psi^1, \psi^2] \\
&\equiv \mathcal{Q}\left[\frac{\hbar \delta}{i \delta J^1(x)}, -\frac{\hbar \delta}{i \delta J^2(x)} \right] \mathcal{F}^{(0)}[J^1, J^2] \bigg|_{J^1 = J^2 = 0}
\end{aligned} \tag{5.15}$$

Here, $\mathcal{F}^{(0)}[J^1, J^2]$ is the influence functional of the free environment field, assuming a linear coupling with external sources J^1 and J^2:

$$\begin{aligned}
&\mathcal{F}^{(0)}[J^1, J^2] \\
&= \int d\psi_f^1(\mathbf{x}) \int d\psi_i^1(\mathbf{x}) \int d\psi_i^2(\mathbf{x}) \ \rho_\psi[\psi_i^1, \psi_i^2, t_i] \int_{\psi_i^1(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^1 \int_{\psi_i^2(\mathbf{x})}^{\psi_f^1(\mathbf{x})} D\psi^2 \\
&\times \exp \frac{i}{\hbar} \left\{ S_0[\psi^1] + \int d^4x J^1(x) \psi^1(x) - S_0[\psi^2] - \int d^4x J^2(x) \psi^2(x) \right\}
\end{aligned} \tag{5.16}$$

Let us define the following free propagators of the ψ field

$$\langle T \psi^1(x) \psi^1(y) \rangle_0 = \Delta_F(x, y) \tag{5.17}$$
$$\langle \psi^1(x) \psi^2(y) \rangle_0 = \Delta^-(x, y) \tag{5.18}$$
$$\langle \tilde{T} \psi^2(x) \psi^2(y) \rangle_0 = \Delta_D(x, y) \tag{5.19}$$

As we have seen in Chapter 3, the CTP path integral time-orders fields in the first branch, anti-time-orders fields in the second branch, and puts fields on the second branch to the left of fields on the first branch. Therefore these are just the familiar Feynman, Dyson and negative-frequency Wightman propagators of a free scalar field given respectively by

$$\Delta_{F,D}(x, x') = \mp i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x - x')}}{k^2 + m_\psi^2 \mp i\varepsilon} \tag{5.20}$$

$$\Delta^-(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x - x')} \theta\left(-k^0\right) 2\pi\hbar\delta\left(k^2 + m_\psi^2\right) \tag{5.21}$$

The perturbation calculation by means of Feynman diagrams for the $\lambda\phi^4$ theory in the CTP formalism has been worked out before for quantum fluctuations [CalHu87, CalHu89] and for coarse-grained fields [Hu91, SinHu91]. For biquadratic coupling,

$$S_{\text{int}}[\phi, \psi] = -\int d^4x \, \lambda_{\phi\psi} \phi^2(x) \psi^2(x) \tag{5.22}$$

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the influence action up to the second order in λ is given by (cf. [HuPaZh93a])

$$\begin{aligned} S_{\rm IF}[\phi,\phi'] = &- \int d^4x\, \lambda_{\phi\psi}\, \Delta_F(x,x) \left[(\phi^1(x))^2 - (\phi^2(x))^2\right] \\ &+ i\hbar^{-1} \int d^4x \int d^4y\; \lambda^2_{\phi\psi}\, (\phi^1(x))^2\; [\Delta_F(x,y)]^2\; (\phi^1(y))^2 \\ &- 2i\hbar^{-1} \int d^4x \int d^4y\; \lambda^2_{\phi\psi}\, (\phi^1(x))^2\; [\Delta^-(x,y)]^2\; (\phi^2(y))^2 \\ &+ i\hbar^{-1} \int d^4x \int d^4y\; \lambda^2_{\phi\psi}\, (\phi^2(x))^2\; [\Delta_D(x,y)]^2\; (\phi^2(y))^2 \end{aligned} \tag{5.23}$$

We now evaluate each term in the perturbation expansion. It is well known that all one-loop diagrams in equation (5.23) contain ultraviolet divergences in spacetime dimension $d = 4 - \epsilon$. By dimensional regularization, one can show that the first one-loop bubble diagram for the ψ^1 field is

$$\begin{aligned} \Delta_F\,(x,x) &= \hbar \int \frac{d^dp}{(2\pi)^d} \frac{(-i)}{p^2 + m^2_\psi - i\varepsilon} \\ &= -\frac{\hbar m^2_\psi}{8\pi^2} \left[\frac{1}{\epsilon} + \text{ constant } - \frac{1}{2} \ln\left(\frac{m^2_\psi}{4\pi\mu^2}\right)\right] \end{aligned} \tag{5.24}$$

where μ^2 is the renormalization energy scale. The first term on the right-hand side is a singular part and must be canceled by mass renormalization. The counter action for this singular mass term is

$$\delta S_{r1}[\phi^1] = \int d^4x \frac{\hbar}{8\pi^2\epsilon} m^2_\psi \lambda_{\phi\psi} (\phi^1(x))^2 \tag{5.25}$$

The second term on the right-hand side is the one-loop finite mass renormalization term, which can be absorbed into the definition of the physical mass of the ϕ field.

For the one-loop bubble diagram for the ψ^2 field, since

$$\langle(\psi^2(x))^2\rangle_0 = \langle(\psi^1(x))^2\rangle_0 \tag{5.26}$$

the mass renormalization counter-action for the $\phi^2(x)$ field is the same as equation (5.25), so is the finite mass renormalization.

Next, for the "fish" diagram of the ψ^1 field, it also can be shown by dimensional regularization that

$$\begin{aligned} i\hbar^{-2}\Delta^2_F(x-y) = &\int \frac{d^4p}{(2\pi)^4}\; e^{ip(x-y)} \frac{1}{8\pi^2} \\ &\times \left[\frac{1}{\epsilon} + \text{ constant } - \frac{1}{2}\int_0^1 d\alpha\; \ln\left(\frac{m^2_\psi + \alpha\,(1-\alpha)\,(p^2 - i\varepsilon)}{4\pi\mu^2}\right)\right] \\ = &\frac{1}{8\pi^2\epsilon}\delta^4(x-y) + \frac{1}{16\pi^2}\left(2 + \psi(1) - \ln\frac{m^2_\psi}{4\pi\mu^2}\right)\delta^4(x-y) \\ &+ \frac{1}{2}U(x-y) + i\frac{1}{2}\nu(x-y) \end{aligned} \tag{5.27}$$

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with the following two real nonlocal kernels

$$U(x-y) = -\frac{2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)} \int_0^1 d\alpha\, \ln\left|1 - i\epsilon + \alpha(1-\alpha)\frac{p^2}{m_\psi^2}\right| \tag{5.28}$$

$$\nu(x-y) = \frac{2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \pi\, \sqrt{1 - \frac{4m_\psi^2}{(-p^2)}}\, \theta(-p^2 - 4m_\psi^2) \tag{5.29}$$

The first term on the right-hand side of equation (5.27) is another singular term. Its counter-action is

$$\delta S_2[\psi^1] = \int d^4x\, \lambda_{\phi\psi}^2 \frac{\hbar}{16\pi^2\epsilon}(\phi^1(x))^4 \tag{5.30}$$

The second term on the right-hand side of equation (5.27) represents a finite coupling constant renormalization, which can be absorbed into a redefinition of the physical coupling constant of the ϕ^1 field. The contribution from the fish diagram for the ψ^2 field is obtained from the above by changing the sign of the ν kernel; it can be renormalized with a counter-action similar to equation (5.30).

For the mixed "fish" diagram, we find

$$i\hbar^{-2}(\Delta^-(x,y))^2 = -\mu(x-y) + \frac{i}{2}\nu(x-y) \tag{5.31}$$

where Cutkowsky rules have been used. The kernel μ in equation (5.31)

$$\mu(x-y) = \frac{i}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \pi\sqrt{1 - \frac{4m_\psi^2}{(-p^2)}}\, \theta(-p^2 - 4m_\psi^2)\, \mathrm{sgn}(p_0) \tag{5.32}$$

is real.

Substituting equations (5.24), (5.27) and (5.31) into the influence action and adding the counter-action equations (5.25) and (5.30), finally we obtain the effective action for this biquadratically coupled system–environment scalar field model as follows

$$\begin{aligned} S_{\text{eff}}[\phi^1,\phi^2] = S_{\text{ren}}[\phi^1] &+ \hbar\int d^4x\int d^4y\, \frac{1}{2}\lambda_{\phi\psi}^2\, (\phi^1(x))^2\, V_{\phi\psi}(x-y)\, (\phi^1(y))^2 \\ &- S_{\text{ren}}[\phi^2] - \hbar\int d^4x\int d^4y\, \frac{1}{2}\lambda_{\phi\psi}^2\, (\phi^2(x))^2\, V_{\phi\psi}(x-y)\, (\phi^2(y))^2 \\ &- \hbar\int d^4x\int^{x^0} d^4y\, \lambda_{\phi\psi}^2\left[(\phi^1(x))^2 - (\phi^2(x))^2\right] \\ &\times \mu(x-y)\left[(\phi^1(y))^2 + (\phi^2(y))^2\right] \\ &+ \frac{i\hbar}{2}\int d^4x\int d^4y\, \lambda_{\phi\psi}^2\left[(\phi^1(x))^2 - (\phi^2(x))^2\right] \\ &\times \nu(x-y)\left[(\phi^1(y))^2 - (\phi^2(y))^2\right] \end{aligned} \tag{5.33}$$

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where $S_{\mathrm{ren}}[\phi^{1,2}]$ is the renormalized action of the $\phi^{1,2}$ field (with physical mass $m^2_{\phi r}$ and physical coupling constant $\lambda_{\phi r}$),

$$S_{\mathrm{ren}}[\phi^a] = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2_{\phi r}(\phi^a)^2 - \frac{1}{4!}\lambda_{\phi r}(\phi^a)^4 \right) \tag{5.34}$$

where $a = 1, 2$. The kernel for the nonlocal potential in equation (5.33)

$$V_{\phi\psi}(x-y) = U(x-y) - \mathrm{sgn}(x^0-y^0)\mu(x-y) \tag{5.35}$$

is symmetric.

For the biquadratic interaction case analyzed here, the potential renormalization is thus

$$\Delta V^{(2)}(x-y) = U^{(2)}(x-y) - \mathrm{sgn}(x^0-y^0)\mu^{(2)}(x-y) \tag{5.36}$$

which is symmetric; and $\mu^{(2)}, \nu^{(2)}$ and $U^{(2)}$ are real nonlocal kernels

$$\mu^{(2)}(x-y) = \frac{1}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\; e^{ip(x-y)}\; \pi\sqrt{1-\frac{4m^2_\psi}{(-p^2)}}\;\theta(-p^2-4m^2_\psi)\;\mathrm{sgn}(p_0) \tag{5.37}$$

$$\nu^{(2)}(x-y) = \frac{2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\; e^{ip(x-y)}\; \pi\;\sqrt{1-\frac{4m^2_\psi}{(-p^2)}}\;\theta(-p^2-4m^2_\psi) \tag{5.38}$$

$$U^{(2)}(x-y) = -\frac{2}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\; e^{ip(x-y)} \int_0^1 d\alpha\;\ln\left|1-i\epsilon-\alpha(1-\alpha)\frac{(-p^2)}{m^2_\psi}\right| \tag{5.39}$$

For a general polynomial-type coupling with S_{int} given by equation (5.4), the renormalized full effective action has the same form as that derived above for biquadratic coupling, except that $(\phi^a(x))^2$ would be replaced by $f[\phi^a(x)]$, etc. (and the kernels would carry superscripts indicating the proper order k instead of (2)). To second order in λ the renormalized full effective action is given by [Zha90, HuPaZh93a]

$$\begin{aligned}
&S_{\mathrm{eff}}[\phi^1,\phi^2] \\
&\quad = S_{\mathrm{ren}}[\phi^1] + \hbar^{k-1}\int d^4x\int d^4y\;\frac{1}{2}\lambda^2_{\phi\psi} f[\phi^1(x)]\Delta V^{(k)}(x-y)f[\phi^1(y)] \\
&\qquad - S_{\mathrm{ren}}[\phi^2] - \hbar^{k-1}\int d^4x\int d^4y\;\frac{1}{2}\lambda^2_{\phi\psi} f[\phi^2(x)]\Delta V^{(k)}(x-y)f[\phi^2(y)] \\
&\qquad - \hbar^{k-1}\int d^4x\int^{x^0} d^4y\;\lambda^2_{\phi\psi} \\
&\qquad \times \Big\{ \left(f[\phi^1(x)] - f[\phi^2(x)]\right)\mu^{(k)}(x-y)\left(f[\phi^1(y)] + f[\phi^2(y)]\right) \\
&\qquad - i\hbar^{k-1}\left(f[\phi^1(x)] - f[\phi^2(x)]\right)\nu^{(k)}(x-y)\left(f[\phi^1(y)] - f[\phi^2(y)]\right)\Big\}
\end{aligned} \tag{5.40}$$

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Renormalization of the potential which arises from the contribution of the environment appears only for even order k couplings. This is a generalization of the result obtained in [Zha90, HuPaZh93a] where it was shown that the nonlocal kernel $\mu^{(k)}(s_1 - s_2)$ is associated with the nonlocal dissipation (or the generalized viscosity) function that appears in the corresponding Langevin equation and $\nu^{(k)}(s_1 - s_2)$ is associated with the time–time autocorrelation function of the stochastic forcing (noise) term. In general ν is nonlocal, which gives rise to colored noises. Only at high temperatures would the noise kernel become a delta function, which corresponds to a white noise source. Let us examine more closely the meaning of the noise kernel.

5.1.2 Noise and fluctuations

The real part of the influence functional comes from the imaginary part of the influence action which contains the noise kernel. This term can be rewritten using a functional Gaussian identity introduced by Feynman and Vernon [FeyVer63] and discussed in Chapter 3. Thus introducing a stochastic forcing term $\xi^{(k)}$ coupled to the field:

$$-\int d^4x \; \xi^{(k)}(x) \; \{f[\phi^1(x)] - f[\phi(x^2)] \;\}/\hbar \tag{5.41}$$

we can view $\xi^{(k)}(x)$ as a classical nonlinear noise source external to the system arising from the environment. The reduced density matrix is calculated by taking a stochastic average over the distribution $\mathcal{P}[\xi^{(k)}]$ of this source. Since the expansion of the action is to quadratic order, the associated noise is Gaussian. It is completely characterized by

$$\langle \xi^{(k)}(x) \rangle_\xi = 0 \tag{5.42}$$

$$\langle \xi^{(k)}(x)\xi^{(k)}(y) \rangle_\xi = \hbar^k \nu^{(k)}(x-y) \tag{5.43}$$

where $\nu^{(k)}$ is redefined by absorbing the $\lambda^2_{\phi\psi}$. We see that the nonlocal kernel $\hbar^k\nu^{(k)}(x-y)$ is just the two-point autocorrelation function of the external stochastic source $\xi^{(k)}(x)$ called colored noise.

In this framework, the expectation value of any functional operator $\mathcal{Q}[\phi]$ of the field ϕ is then given by

$$\langle \mathcal{Q}[\phi] \rangle = \int D\xi^{(k)}(x)\, \mathcal{P}[\xi^{(k)}] \int d\phi\, \rho_r(\phi, \phi, [\xi^{(k)}])\, \mathcal{Q}[\phi] \tag{5.44}$$

$$= \Big\langle \langle \mathcal{Q}[\phi] \rangle_{\text{quantum}} \Big\rangle_{\text{noise}} \tag{5.45}$$

This provides the physical interpretation of $\nu^{(k)}(x-y)$ as a noise or fluctuation kernel of the quantum field.

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5.1.3 Langevin equation and fluctuation–dissipation relation

We will now derive the semiclassical equation of motion generated by the influence action S_{IF}. Define a "center-of-mass" function ϕ_+ and a "relative" function ϕ_- as follows

$$\phi_+(x) = \frac{1}{2}[\phi^1(x) + \phi^2(x)] \tag{5.46}$$

$$\phi_-(x) = \phi^1(x) - \phi^2(x) \tag{5.47}$$

The equation of motion for ϕ is derived by demanding (cf. [CalHu87])

$$\left.\frac{\delta S_{\mathrm{eff}}}{\delta \phi_-}\right|_{\phi_-=0} = 0 \tag{5.48}$$

which gives

$$-\frac{\partial L_r}{\partial \phi} + \frac{d}{dt}\frac{\partial L_r}{\partial \dot{\phi}} + 2\frac{\partial f(\phi)}{\partial \phi}\int_0^x d^4y\ \gamma^{(k)}(x-y)\frac{\partial f(\phi(y))}{\partial y^0} = F_\xi^{(k)}(x) \tag{5.49}$$

We see that this is in the form of a Langevin equation with a nonlinear stochastic force

$$F_\xi^{(k)}(x) = \xi^{(k)}(s)\frac{\partial f(\phi)}{\partial \phi} \tag{5.50}$$

This corresponds to a multiplicative noise arising from a nonlinear field coupling (additive if $f(\phi) = \phi$). L_r is the renormalized effective Lagrangian of the system action S_{eff}. The nonlocal kernel $\gamma^{(k)}(t-s)$ defined by

$$\frac{\partial}{\partial(x^0 - y^0)}\gamma^{(k)}(x-y) = \hbar^{k-1}\mu^{(k)}(x-y) \tag{5.51}$$

is responsible for nonlocal dissipation. Interaction with the environment field imparts a dissipative force in the effective dynamics of the system field given by

$$F_\gamma^{(k)}(x) = 2\int d^4y\ \gamma^{(k)}(x-y)\frac{\partial f(\phi(y))}{\partial y^0}\ \frac{\partial f(\phi(x))}{\partial \phi} \tag{5.52}$$

Only in special cases like a high temperature ohmic environment will the dissipation become local.

In the biquadratic coupling example the corresponding stochastic force is

$$F_\xi^{(2)}(x) \sim \xi^{(2)}(x)\phi(x) \tag{5.53}$$

The $\gamma^{(2)}$ kernel is

$$\gamma^{(2)}(x-y) = \frac{\hbar}{16\pi^2}\int \frac{d^4p}{(2\pi)^4}\ e^{ip(x-y)}\pi\sqrt{1 - \frac{4m_\psi^2}{(-p^2)}}\ \theta(-p^2 - 4m_\psi^2)\ \frac{1}{|p_0|} \tag{5.54}$$

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and the dissipative force is

$$F_{\gamma}^{(2)}(x) \sim \hbar \int d^4y\ \mu(x-y)\phi^2(y)\phi(x) \tag{5.55}$$

As discussed in [HuPaZh93a], we can show that a general fluctuation–dissipation relation exists between the dissipation and the noise kernels in the form

$$\hbar^{k-1}\nu^{(k)}(x) = \int d^4y\ K^{(k)}(x-y)\gamma^{(k)}(y) \tag{5.56}$$

Apart from a delta function $\delta^3(\mathbf{x}-\mathbf{x}')$, the fluctuation–dissipation kernel K^k for quantum fields has exactly the same form as for the quantum Brownian harmonic oscillator. In general it is a rather complicated expression [HuPaZh93a], but simplifies at high and zero temperatures. At high temperatures,

$$K^{(k)}(s) = \frac{2k_BT}{\hbar}\delta(s) \qquad (s \equiv x-y) \tag{5.57}$$

which gives back the famous Einstein relation. At zero temperature,

$$K^{(k)}(s) = \int_0^{+\infty} \frac{d\omega}{\pi}\omega\cos\omega s \tag{5.58}$$

which is the same as in the linear coupling case. Both limiting forms are independent of k. In other words, at both high and zero temperatures, the FDT is insensitive to the way the system is coupled to the environment.

Our derivation of the fluctuation–dissipation relation shows that it has a more general meaning than the more restrictive conditions where it is usually presented, e.g. in the near-equilibrium or the linear response regimes. It should be viewed as a categorical relation depicting the stochastic stimulation of the system and the averaged response of the environment.

5.2 Quantum functional master equation

We now turn to a derivation of the functional master equation for the system field with the interaction described in the last section. The full equation is quite involved because it contains nonlinear nonlocal dissipation and multiplicative colored noise plus a nonlocal potential term. Just to see the qualitative features let us first examine the simplified case under a local truncation to the dissipation kernel and the noise kernel (i.e. white noise) and omitting the nonlocal potential term. Namely, we set

$$\gamma(x-x') = \gamma_0\delta^4(x-x') \tag{5.59}$$

$$\hbar^{k-1}\nu(x-x') = \nu_0\delta^4(x-x') \tag{5.60}$$

$$V_{\phi\psi}(x-x') = 0 \tag{5.61}$$

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Under this approximation the quantum master equation derived from the influence functional is much simpler. However, we need to emphasize that this approximation violates the fluctuation–dissipation relation at zero temperature. The effective action equation (5.33) simplifies to

$$\begin{aligned}
S_{\text{eff}}[\phi^1,\phi^2] = \int d^4x \Big\{ & -\frac{1}{2}\partial_\mu\phi^1\partial^\mu\phi^1 - \frac{1}{2}m_{\phi r}^2(\phi^1)^2 - \frac{1}{4!}\lambda_{\phi r}(\phi^1)^4 \\
& + \frac{1}{2}\partial_\mu\phi^2\partial^\mu\phi^2 + \frac{1}{2}m_{\phi r}^2(\phi^2)^2 + \frac{1}{4!}\lambda_{\phi r}(\phi^2)^4 \\
& - 2\lambda_{\phi\psi}^2\gamma_0\left((\phi^1)^2-(\phi^2)^2\right)\left(\phi^1\frac{\partial\phi^1}{\partial s}+\phi^2\frac{\partial\phi^2}{\partial s}\right) \\
& + (i/2)\,\lambda_{\phi\psi}^2\nu_0\left((\phi^1)^2-(\phi^2)^2\right)^2 \Big\}
\end{aligned} \tag{5.62}$$

We can now write down a "Hamiltonian" which corresponds to equation (5.62) as

$$\begin{aligned}
&\hat{H}_\rho[\phi^1,\phi^2,t] \\
&= \int d^3\mathbf{x}\Big\{-\frac{\hbar^2}{2}\frac{\delta^2}{\delta(\phi^1(\mathbf{x}))^2} + \frac{1}{2}(\nabla\phi^1(\mathbf{x}))^2 + \frac{1}{2}m_{\phi r}^2(\phi^1(\mathbf{x}))^2 + \frac{1}{4!}\lambda_{\phi r}(\phi^1(\mathbf{x}))^4 \\
&\quad + \frac{\hbar^2}{2}\frac{\delta^2}{\delta(\phi^2(\mathbf{x}))^2} - \frac{1}{2}(\nabla\phi^2(\mathbf{x}))^2 - \frac{1}{2}m_{\phi r}^2(\phi^2(\mathbf{x}))^2 - \frac{1}{4!}\lambda_{\phi r}(\phi^2(\mathbf{x}))^4 \\
&\quad - 2i\hbar\lambda_{\phi\psi}^2\gamma_0\left[(\phi^1(\mathbf{x}))^2-(\phi^2(\mathbf{x}))^2\right]\left[\phi^1(\mathbf{x})\frac{\delta}{\delta\phi^1(\mathbf{x})} - \phi^2(\mathbf{x})\frac{\delta}{\delta\phi^2(\mathbf{x})}\right] \\
&\quad - (i/2)\,\lambda_{\phi\psi}^2\nu_0\left[(\phi^1(\mathbf{x}))^2-(\phi^2(\mathbf{x}))^2\right]^2\Big\}
\end{aligned} \tag{5.63}$$

which also is correct up to order of $\lambda_{\phi\psi}^2$. Therefore the quantum functional master equation is given by the following functional "Schrödinger" equation

$$i\hbar\frac{\partial}{\partial t}\;\rho_r[\phi^1,\phi^2,t] = \hat{H}_\rho[\phi^1,\phi^2,t]\;\rho_r[\phi^1,\phi^2,t] \tag{5.64}$$

This quantum functional master equation for the system field is very similar to the quantum master equation for the anharmonic oscillator with nonlinear dissipation and nonlinear coupled noise in the Brownian particle model treated in [HuPaZh93a].

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Let us define the Wigner functional for the quantum field as follows

$$W[\phi,\pi,t] = \int d\psi(\mathbf{x})\, \exp\left\{ i\hbar^{-1} \int d^3\mathbf{x}\, \pi(\mathbf{x})\psi(\mathbf{x}) \right\} \times \rho_r \left[\phi - \frac{1}{2}\psi, \phi + \frac{1}{2}\psi, t \right] \tag{5.65}$$

Applying equation (5.65) to both sides of the above functional master equation, we can obtain the following Wigner functional equation

$$\begin{aligned} &\frac{\partial}{\partial t} W[\phi,\pi,t] \\ &= \int d^3\mathbf{x} \left\{ -\pi(\mathbf{x}) \frac{\delta}{\delta\phi(\mathbf{x})} - [\nabla\phi(\mathbf{x})\cdot\nabla + m_{\phi r}^2 \phi(\mathbf{x}) + \frac{1}{6}\lambda_{\phi r}\phi^3(\mathbf{x})] \frac{\delta}{\delta\pi(\mathbf{x})} \right. \\ &\quad + 4\lambda_{\phi\psi}^2 \gamma_0 \phi^2(\mathbf{x}) \frac{\delta}{\delta\pi(\mathbf{x})}\pi(\mathbf{x}) + 2\hbar\lambda_{\phi\psi}^2 \nu_0 \phi^2(\mathbf{x}) \frac{\delta^2}{\delta\pi^2(\mathbf{x})} \\ &\quad \left. + \hbar^2 \lambda_{\phi r}\phi(\mathbf{x}) \frac{\delta^3}{\delta\phi^3(\mathbf{x})} + 2\lambda_{\phi\psi}^2 \hbar^2 \gamma_0 \phi(\mathbf{x}) \frac{\delta^3}{\delta\phi(\mathbf{x})\delta\pi^2(\mathbf{x})} \right\} W[\phi,\pi,t] \end{aligned} \tag{5.66}$$

It is clear that the last two terms on the right-hand side of equation (5.66) which contain third-order derivatives are the quantum corrections. In the classical limit they go to zero, and equation (5.66) becomes the functional Fokker–Planck equation. We also know that the Wigner functional (5.65) becomes the classical phase space distribution functional in the classical limit.

The quantum Wigner function contains just as much information as the wavefunction so it oscillates and can assume negative values. In particular it does not exhibit a peak along the classical trajectory in phase space except at high temperature or for harmonic oscillators. Thus viewing the quantum Wigner function as possessing the equivalent traits of a classical one-particle phase space distribution function is untenable except under special conditions. This has special significance in quantum–classical correspondence issues. See discussions in [Hab90, HabLaf90].

One can also show that the following “equilibrium” state distribution

$$W[\phi,\pi] \sim \exp -\bar{\beta} \int d^3\mathbf{x} \left\{ \frac{1}{2}\pi^2(\mathbf{x}) + \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m_{\phi r}^2\phi^2(\mathbf{x}) + \frac{1}{4!}\lambda_{\phi r}\phi^4(\mathbf{x}) \right\} \tag{5.67}$$

is the asymptotic solution of the above functional Wigner equation in the classical limit, provided that $\bar{\beta}^{-1} = \hbar\nu_0/\gamma_0$. Thus the above functional Wigner equation can describe the process of relaxation to equilibrium state.

We are now ready to present the functional master equation. The calculation closely parallels that of the QBM case studied in Chapter 3 [HuPaZh92, Paz94]. The quantum functional master equation for the case of nonlocal dissipation,

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colored noise and nonlocal potential is given by

$$\begin{aligned}
& i\hbar\frac{\partial}{\partial t}\rho_r[\phi^1,\phi^2,t] \\
&= \int d^3\mathbf{x}\Big\{-\frac{\hbar^2}{2}\frac{\delta^2}{\delta(\phi^1(\mathbf{x}))^2}+\frac{1}{2}(\nabla\phi^1(\mathbf{x}))^2+\frac{1}{2}m_{\phi r}^2(\phi^1(\mathbf{x}))^2+\frac{1}{4!}\lambda_{\phi r}(\phi^1(\mathbf{x}))^4 \\
&\quad +\frac{\hbar^2}{2}\frac{\delta^2}{\delta(\phi^2(\mathbf{x}))^2}-\frac{1}{2}(\nabla\phi^2(\mathbf{x}))^2-\frac{1}{2}m_{\phi r}^2(\phi^2(\mathbf{x}))^2-\frac{1}{4!}\lambda_{\phi r}(\phi^2(\mathbf{x}))^4 \\
&\quad -i\lambda_{\phi\psi}^2 a_1(\mathbf{x},s)\left[(\phi^1(\mathbf{x}))^2+(\phi^2(\mathbf{x}))^2\right]+2\lambda_{\phi\psi}^2 a_2(\mathbf{x},t)\left[(\phi^1(\mathbf{x}))^2+(\phi^2(\mathbf{x}))^2\right] \\
&\quad -\lambda_{\phi\psi}^2\,(\phi^1(\mathbf{x}))^2\,\{v*\hat{O}_+^2\}(\mathbf{x},t)+\lambda_{\phi\psi}^2(\phi^2(\mathbf{x}))^2\,\{v*\hat{O}_-^2\}(\mathbf{x},t) \\
&\quad -\lambda_{\phi\psi}^2\left[(\phi^1(\mathbf{x}))^2-(\phi^2(\mathbf{x}))^2\right]\,\{\mu*(\hat{O}_+^2+\hat{O}_-^2)\}(\mathbf{x},t) \\
&\quad -(i/2)\,\lambda_{\phi\psi}^2\left[(\phi^1(\mathbf{x}))^2-(\phi^2(\mathbf{x}))^2\right]\,\{\nu*(\hat{O}_+^2-\hat{O}_-^2)\}(\mathbf{x},t)\Big\}\;\rho_r[\phi,\phi',t]
\end{aligned} \tag{5.68}$$

where $*$ denotes convolution, namely

$$\{v*\phi^2\}(\mathbf{x},t)=\int_{t_0}^{t}ds\int d^3\mathbf{x}'\;v(\mathbf{x}-\mathbf{x}',t-s)\phi^2(\mathbf{x}',s) \tag{5.69}$$

The time-dependent coefficients in equation (5.68) are as follows

$$a_1(\mathbf{x},t)=\int_{t_0}^{t}ds\int d^3\mathbf{x}'\;v(\mathbf{x}-\mathbf{x}',t-s)Q(s)=\{v*Q\}(\mathbf{x},t) \tag{5.70}$$

$$a_2(\mathbf{x},t)=\int_{t_0}^{t}ds\int d^3\mathbf{x}'\;\nu(\mathbf{x}-\mathbf{x}',t-s)Q(s)=\{\nu*Q\}(\mathbf{x},t) \tag{5.71}$$

where

$$Q(s)=\int\frac{d^3\mathbf{k}}{(2\pi)}\;\frac{\sin\omega(\mathbf{k})(s-t_0)\sin\omega(\mathbf{k})(t-s)}{\omega(\mathbf{k})\sin\omega(\mathbf{k})(t-t_0)} \tag{5.72}$$

and the operators

$$\hat{O}_+(\mathbf{x},s)\equiv\{\alpha(t-s)*\phi_f^1\}(\mathbf{x})-\left\{\beta(t-s)*i\frac{\hbar\delta}{\delta\phi_f^1}\right\}(\mathbf{x}) \tag{5.73}$$

$$\hat{O}_-(\mathbf{x},s)\equiv\{\alpha(t-s)*\phi_f^2\}(\mathbf{x})+\left\{\beta(t-s)*i\frac{\hbar\delta}{\delta\phi_f^2}\right\}(\mathbf{x}) \tag{5.74}$$

with

$$\alpha(\mathbf{x},s)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\;e^{i\mathbf{k}\cdot\mathbf{x}}\;\cos\omega(\mathbf{k})s \tag{5.75}$$

$$\beta(\mathbf{x},s)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\;e^{i\mathbf{k}\cdot\mathbf{x}}\;\frac{\sin\omega(\mathbf{k})s}{\omega(\mathbf{k})} \tag{5.76}$$

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This is a nonstationary quantum functional master equation. In spite of its complicated appearance (convolution products appearing in the equation), it is not difficult to see that the structure of the equation is similar to the nonstationary quantum master equation for a Brownian anharmonic oscillator with nonlinear dissipation and multiplicative colored noise. Actually, in momentum space, the convolution product becomes a direct product, so the above equation in momentum space will become the quantum master equation for one particular mode (harmonic oscillator). However, the different modes will still be coupled together in the quantum master equation because of mode–mode coupling in the system field via the nonlinear potential. We now turn to the case of one interacting field divided into two sectors.

5.3 The closed time path coarse-grained effective action

To add some physical flavor to our derivation and in anticipation of applications to problems in cosmology, we consider the action of a massless scalar field with $\lambda\phi^4$ self-interaction coupled conformally to a spatially-flat Friedmann–Lemaitre–Robertson–Walker universe. The conformal-related field χ (introduced in Chapter 4) is related to ϕ by $\chi = a(t)\phi$ and the conformal time η is related to the cosmic time t by $\eta = \int dt/a(t)$. We shall use d^4x to denote $d^3x\, d\eta$ in the remainder of this chapter. Since our purpose here is more to illustrate the coarse-graining idea in the construction of a CGEA than to discuss cosmological applications (see Chapter 15), we can just view the scale factor $a(t) = e^{\alpha}$ as a scaling parameter rather than a dynamical function determined from Einstein's equations. The content of this section can thus be used without reference to cosmology by treating a as a constant, e.g. setting $a = 1$ would keep us in a Minkowski spacetime with the conformal time η acting as the global time t. However we wish to tag along the scale factor a so that later we can view the inflationary cosmology in the light of scaling [Hu91] without added effort.

We begin by separating the quantum field $\chi(x, \eta)$ into two parts, $\chi = \chi_< + \chi_>$, where $\chi_<$ contains the lower k wave modes and $\chi_>$ the higher k modes. We can refer to these two sectors as the system and the environment respectively. Two useful physical examples are

Case A (critical phenomena)

$$\chi_< :|\, \mathbf{k} \,|< \Lambda/s, \qquad \chi_> : \Lambda/s <|\, \mathbf{k} \,|< \Lambda \tag{5.77}$$

where Λ is the ultraviolet cut-off and $s > 1$ is the coarse-graining parameter which gives the fraction of total k modes counted in the environment.

Case B (stochastic inflation)

$$\chi_< :|\, \mathbf{k} \,|< \epsilon Ha, \qquad \chi_> :|\, \mathbf{k} \,|> \epsilon\, Ha, \qquad \epsilon \approx 1 \tag{5.78}$$

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where the Hubble constant $H(t) \equiv \dot{a}/a$ (the event horizon in the de Sitter universe) serves to divide the physical wavelength $\mathbf{p} \equiv \mathbf{k}/a$ into two sectors, with a window function measuring how sharp the division is. We will have more to say about this point in Chapter 15. For now, we can build up our intuition for the coarse-graining ideas using Case A as an illustrative example. The separation of χ can also be made in other manners, depending on the physical set-up of the problem and the questions one asks. The formalism we present here is quite general. Our presentation for this model follows [LomMaz96].

Explicitly, we define the system by

$$\chi_<(\mathbf{x}, \eta) = \int_{|\mathbf{k}|<\Lambda_c} \frac{d^3\mathbf{k}}{(2\pi)^3} \chi(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.79}$$

and the environment by

$$\chi_>(\mathbf{x}, \eta) = \int_{|\mathbf{k}|>\Lambda_c} \frac{d^3\mathbf{k}}{(2\pi)^3} \chi(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.80}$$

The system field contains the modes with wavelengths longer than the critical value Λ_c^{-1}, while the environment field contains wavelengths shorter than Λ_c^{-1}. Λ_c corresponds to $s^{-1}\Lambda$. After the splitting, the total action can be written as

$$S[a, \chi] = S[\chi_<] + S_0[\chi_>] + S_{\rm int}[a, \chi_<, \chi_>] \tag{5.81}$$

where S_0 denotes the kinetic term

$$S_0[\chi] = -\frac{1}{2} \int d\eta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ \chi(\mathbf{k}, \eta) \left[\frac{\partial^2}{\partial \eta^2} + k^2 \right] \chi(\mathbf{k}, \eta) \right\} \tag{5.82}$$

$S[\chi_<]$ is the system action,

$$S[\chi_<] = S_0[\chi_<] - \int d^4x \left\{ \frac{1}{2} M^2 \chi_<^2 + \frac{\lambda}{4!} \chi_<^4 \right\} \tag{5.83}$$

and the interaction part is given by

$$S_{int}[a, \chi_<, \chi_>] = -\int d^4x \left\{ \left[\frac{1}{2} M^2 + \frac{\lambda}{4} \chi_<^2(x) \right] \chi_>^2 + \frac{\lambda}{4!} \chi_>^4 + \frac{\lambda}{6} \chi_<^3 \chi_> + \frac{\lambda}{6} \chi_< \chi_>^3 \right\} \tag{5.84}$$

with

$$M^2 = \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] a^2 \tag{5.85}$$

We are interested in the influence of the environment on the evolution of the system. To this end, we seek to construct the Feynman–Vernon influence functional, following the methods of Chapter 3. This is obtained by integrating over the environment field configurations between an initial time $\eta = -\eta_i$ and a final time $\eta = \eta_f$. When η_i, η_f are larger than any other characteristic time and the environment field is initially in the vacuum state, the Feynman–Vernon

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influence action turns into the so-called closed time path (CTP) coarse-grained effective action (CGEA) $S_\Lambda[a^1,\chi^1_<,a^2,\chi^2_<]$, which is defined by

$$\begin{aligned}&\exp\left\{i\hbar^{-1}S_{\Lambda_c}[a^1,\chi^1_<,a^2,\chi^2_<]\right\}\\&\quad=\exp i\hbar^{-1}\left\{S[\chi^1_<]-S[\chi^2_<]\right\}\int d\chi_{>f}\int^{\chi_{>f}}D\chi^1_>\int^{\chi_{>f}}D\chi^2_>\exp i\hbar^{-1}\\&\qquad\times\left\{S_0[\chi^1_>]+S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]-S_0[\chi^2_>]-S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]\right\}\end{aligned}\tag{5.86}$$

The integration here is performed over all fields $\chi^1_>$ ($\chi^2_>$) with positive (negative) frequency modes in the remote past that coincide at the final time $\chi^1_>=\chi^2_>=\chi_{>f}$. More general initial conditions will be discussed in later chapters.

We now derive the CTP CGEA perturbatively in λ and M^2, up to quadratic order in both quantities. A simple calculation leads to

$$\begin{aligned}S_{\Lambda_c}[a^1,\chi^1_<,a^2,\chi^2_<]&=S[\chi^1_<]-S[\chi^2_<]+\langle S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]\rangle_0-\langle S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]\rangle_0\\&\quad+\frac{i}{2\hbar}\left\{\langle S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]^2\rangle_0-\langle S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]\rangle_0{}^2\right\}\\&\quad-i\hbar^{-1}\left\{\langle S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]\rangle_0\right.\\&\quad\left.-\langle S_{\text{int}}[a^1,\chi^1_<,\chi^1_>]\rangle_0\langle S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]\rangle_0\right\}\\&\quad+\frac{i}{2\hbar}\left\{\langle S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]^2\rangle_0-\langle S_{\text{int}}[a^2,\chi^2_<,\chi^2_>]\rangle_0{}^2\right\}\end{aligned}\tag{5.87}$$

where the quantum average of a functional of the fields $\mathcal{Q}$ is defined with respect to the kinetic action S_0

$$\langle\mathcal{Q}[\chi^1_>,\chi^2_>]\rangle_0=\int d\chi_{>f}\int^{\chi_{>f}}\mathcal{D}\chi^1_>\int^{\chi_{>f}}\mathcal{D}\chi^2_>\exp i\hbar^{-1}\left\{S_0[\chi^1_>]-S_0[\chi^2_>]\right\}\mathcal{Q}\tag{5.88}$$

Equation (5.87) is the in–in version of the Dyson–Feynman series.

We define the propagators of the environment field as

$$\langle T\chi^1_>(x)\chi^1_>(y)\rangle_0=G^{\Lambda_c}_F(x-y),\tag{5.89}$$

$$\langle\chi^1_>(x)\chi^2_>(y)\rangle_0=G^{\Lambda_c}_-(x-y),\tag{5.90}$$

$$\langle\tilde{T}\chi^2_>(x)\chi^2_>(y)\rangle_0=G^{\Lambda_c}_D(x-y).\tag{5.91}$$

where $T,\tilde{T}$ denote time- and reversed-time ordering respectively.

Despite their appearance these propagators are not the usual Feynman, negative-frequency Wightman and Dyson propagators of the scalar field since, in this case, the momentum integration is restricted by the presence of the (infrared) cut-off Λ_c. The explicit expressions are

$$G^{\Lambda_c}_F(x-y)=-i\hbar\int_{|\mathbf{p}|>\Lambda_c}\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{1}{p^2-i\varepsilon}\tag{5.92}$$

$$G^{\Lambda_c}_-(x-y)=\int_{|\mathbf{p}|>\Lambda_c}\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}2\pi\hbar\delta(p^2)\Theta(-p^0)\tag{5.93}$$

$$G^{\Lambda_c}_D(x-y)=i\hbar\int_{|\mathbf{p}|>\Lambda_c}\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{1}{p^2+i\varepsilon}\tag{5.94}$$

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As an example, we show the expression for the propagator $G_F^{\Lambda_c}$. The usual massless Feynman propagator is

$$\hbar^{-1}\Delta_F(x) = \frac{1}{8\pi^2}\frac{1}{\sigma} + \frac{i}{8\pi}\delta(\sigma) \tag{5.95}$$

while

$$\begin{aligned}\hbar^{-1}G_F^{\Lambda_c}(x) &= \frac{(-1)}{8\pi^2}\left[\frac{\cos[\Lambda_c(r-x^0)]}{r(r-x^0)} + \frac{\cos[\Lambda_c(r+x^0)]}{r(r+x^0)}\right] \\ &\quad + \frac{i}{8\pi^2}\left[\frac{\sin[\Lambda_c(r-x^0)]}{r(r-x^0)} - \frac{\sin[\Lambda_c(r+x^0)]}{r(r+x^0)}\right] \\ G_F^{\Lambda_c}(x) &\equiv \Delta_F(x) - G_F^{|\mathbf{p}|<\Lambda_c}(x)\end{aligned} \tag{5.96}$$

where $\sigma = -\frac{1}{2}x^2$ and $r = |\mathbf{x}|$.

The CTP CGEA can be computed from equations (5.87)–(5.91) using standard techniques [GreMul97]. Defining

$$\left(\tilde{M}^{1,2}\right)^2 = M^2 + \frac{\lambda}{2}\left(\chi_<^{1,2}\right)^2 \tag{5.97}$$

$$\begin{aligned}\chi_-^{(n)} &= (\chi_<^1)^n - (\chi_<^2)^n, & \chi_+^{(n)} &= \frac{1}{2}\left[(\chi_<^1)^n + (\chi_<^2)^n\right] \\ \lambda Q_- &= (\tilde{M}^1)^2 - (\tilde{M}^2)^2, & \lambda Q_+ &= \frac{1}{2}\left[(\tilde{M}^1)^2 + (\tilde{M}^2)^2\right]\end{aligned} \tag{5.98}$$

and using simple identities for the propagators, the CTP CGEA can be written as

$$\begin{aligned}S_{\Lambda_c} =& S(\chi_<^1) - S(\chi_<^2) + \frac{\lambda}{4}\int d^4x\, G_F^{\Lambda_c}(0)Q_-(x) \\ &+\hbar^{-1}\lambda^2\int d^4x\int d^4y\,\Theta(y^0-x^0)\left\{\frac{1}{18}\chi_+^{(3)}(x)\,\mathrm{Im}G_F^{\Lambda_c}(x-y)\,\chi_-^{(3)}(y)\right. \\ &\quad\left.-\frac{1}{4}Q_+(x)\,\mathrm{Im}\left[G_F^{\Lambda_c}(x-y)\right]^2 Q_-(y) - \frac{1}{3}\chi_+^{(1)}(x)\,\mathrm{Im}\left[G_F^{\Lambda_c}(x-y)\right]^3\chi_-^{(1)}(y)\right\} \\ &+\frac{i\lambda^2}{4\hbar}\int d^4x\int d^4y\,\Theta(y^0-x^0)\left\{\frac{1}{18}\chi_-^{(3)}(x)\,\mathrm{Re}G_F^{\Lambda_c}(x-y)\,\chi_-^{(3)}(y)\right. \\ &\quad+\frac{1}{4}Q_-(x)\,\mathrm{Re}\left[G_F^{\Lambda_c}(x-y)\right]^2 Q_-(y) \\ &\quad\left.-\frac{1}{3}\chi_-^{(1)}(x)\,\mathrm{Re}\left[G_F^{\Lambda_c}(x-y)\right]^3\chi_-^{(1)}(y)\right\}\end{aligned} \tag{5.99}$$

The real part of the CTP CGEA in equation (5.99) contains divergences and must be renormalized. As the propagators in equations (5.89)–(5.91) differ from the usual ones only by the presence of the infrared cut-off, the ultraviolet divergences coincide with those of the usual $\lambda\chi^4$ theory. The effective action can therefore be renormalized using the standard procedure. (For the renormalization of quantum fields in curved spacetimes, it is necessary to add to the Einstein–Hilbert term in the gravitational action a cosmological constant, and terms quadratic in the curvature tensor. See, e.g. [BirDav82].)

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Consider the square of the Feynman propagator. Using dimensional regularization we find

$$[G_F^{\Lambda_c}(x)]^2 = [\Delta_F(x)]^2 + [G_F^{(|\mathbf{p}|<\Lambda_c)}(x)]^2 - 2\Delta_F(x)G_F^{(|\mathbf{p}|<\Lambda_c)}(x) \quad (5.100)$$

where

$$\hbar^{-2}\Delta_F^2(x) = \frac{1}{16\pi^2}\left[\frac{i}{n-4} + i\psi(1) - 4\pi i + \ln(4\pi\mu^2)\right]\delta^4(x) + iR_1(x) + R_2(x) \quad (5.101)$$

$$R_1(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}\ln|p^2|$$

$$R_2(x) = \frac{\pi}{(2\pi)^4}\int d^4p\, e^{ipx}\Theta(-p^2)$$

Note that the divergence is the usual one, i.e. proportional to $\delta^4(x-y)$ and independent of Λ_c. Consequently, the term $\mathrm{Re}[G_F^{\Lambda}(x-y)]^2 Q_+(x)Q_-(y)$ in equation (5.99) is divergent and renormalizes the coupling constant λ and the constants that appear in the gravitational action. The other divergences can be treated in a similar way. One can also check that the imaginary part of the effective action does not contain divergences. Of course, a successful ultraviolet renormalization does not guarantee that an approximation scheme such as RG-improved perturbation theory will be well behaved. An example is in the RG equations for $\lambda\phi^4$ fields [Hu91] where loops depend on a factor $e^{H(t-t_0)}$ which would invalidate perturbation theory. Further "infrared" H-dependent (environmentally friendly) renormalization of λ is needed [OCoSte94a, OCoSte94b, EiOCSt95, Ste98, FEOS96].

As we have seen in Chapter 3, the (nonlocal) real and imaginary parts of S_{Λ_c} can be associated with the dissipation and noise respectively, which are related by an integral equation known as the fluctuation–dissipation relation.

5.3.1 Stochastic equations

The Langevin equation

We now show how to derive a Langevin equation for the system field from the CTP CGEA. This equation takes into account the three fundamental effects of the environment on the system: renormalization, dissipation and noise.

The CTP CGEA for our model is given in equation (5.99). Since the imaginary part is quadratic in the system field, we can invoke the Gaussian identity used by Feynman and Vernon [FeyVer63], as discussed in Chapter 3. The CTP CGEA can thus be rewritten as

$$S_{\Lambda_c}[\chi_<^1, \chi_<^2] = -i\hbar\ln\int\mathcal{D}\xi_1 P[\xi_1]\int\mathcal{D}\xi_2 P[\xi_2] \times\int\mathcal{D}\xi_3 P[\xi_3]\exp\left\{i\hbar^{-1}S_{\mathrm{eff}}[\chi_<^1,\chi_<^2,\xi_1,\xi_2,\xi_3]\right\} \quad (5.102)$$

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where

$$
\begin{aligned}
&S_{\text{eff}}[\chi^1_<, \chi^2_<, \xi_1, \xi_2, \xi_3] \\
&\quad = \mathrm{Re} S_{\Lambda_c}[\chi^1_<, \chi^2_<] - \int d^4x \left[\chi^{(3)}_-(x)\xi_1(x) + Q_-(x)\xi_2(x) + \chi^{(1)}_-(x)\xi_3(x) \right]
\end{aligned} \tag{5.103}
$$

and $\xi_1(x)$, $\xi_2(x)$, and $\xi_3(x)$ are Gaussian stochastic sources with zero mean and auto-correlations

$$
\langle \xi_1(x)\xi_1(y) \rangle = \frac{\lambda^2}{9} \mathrm{Re} G^{\Lambda_c}_F(x-y) \tag{5.104}
$$

$$
\langle \xi_2(x)\xi_2(y) \rangle = \frac{\lambda^2}{2} \mathrm{Re} \left[G^{\Lambda_c}_F(x-y) \right]^2 \tag{5.105}
$$

$$
\langle \xi_3(x)\xi_3(y) \rangle = \frac{2\lambda^2}{3} \mathrm{Re} \left[G^{\Lambda_c}_F(x-y) \right]^3 \tag{5.106}
$$

From this effective action it is easy to derive the stochastic field equation for the system

$$
\left. \frac{\partial S_{\text{eff}}[\chi^1_<, \chi^2_<, \xi_1, \xi_2, \xi_3]}{\partial \chi^1_<} \right|_{\chi^1_< = \chi^2_<} = 0 \tag{5.107}
$$

It is given by

$$
\begin{aligned}
&\left(\frac{\partial^2}{\partial \xi_3^2} - \nabla^2 \right) \chi_< + \left[M^2 + \frac{\lambda}{2} G^{\Lambda_c}_F(0) \right] \chi_< + \frac{\lambda}{6} \chi^3_< \\
&\quad + \frac{\lambda^2}{6} \chi^2_<(x) \int d^4y\, \theta(x_0 - y_0) \mathrm{Im} G^{\Lambda_c}_F(x,y) \chi^3_<(y) \\
&\quad - \frac{\lambda^2}{2} \chi_<(x) \int d^4y\, \theta(x_0 - y_0) \mathrm{Re}[G^{\Lambda_c}_F(x,y)]^2 \chi^2_<(y) \\
&\quad - \frac{\lambda^2}{3} \int d^4y\, \theta(x_0 - y_0) \mathrm{Im}[G^{\Lambda_c}_F(x,y)]^3 \chi_<(y) \\
&= 3\xi_1(x)\chi^2_<(x) + 2\xi_2(x)\chi_<(x) + \xi_3(x)
\end{aligned} \tag{5.108}
$$

This is the functional Langevin equation derived from the variation of the CTP CGEA. By construction, it is real and causal. We see that it contains multiplicative and additive colored noise. The nonlinear coupling between modes adds complexity to the Langevin equation. This class of equations was first derived by Sinha and Hu [SinHu91] for considerations of the validity of minisuperspace approximations in quantum cosmology, and by Lombardo and Mazzitelli in [LomMaz96], whose treatment we follow here. Greiner and Müller [GreMul97] obtained a similar stochastic equation in flat spacetime, for a thermal environment. They found explicit expressions for the momentum-dependent dissipation function in the Langevin equation using a Markovian approximation for the soft modes.

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The master equation (for one-mode system)

As we have seen in Chapter 3, an equivalent depiction of the dynamics of the open system is obtained from the master equation for the reduced density matrix. A functional master equation for the long-wavelength modes may be derived along similar lines [LomMaz96], but in general it is very complicated. A tractable result can be obtained when the system field contains only one mode $\mathbf{k} = \mathbf{k}_0$. This is a sort of "minisuperspace" (the space of modes in this case) approximation. Also we keep terms only up to $O\left(\lambda^2\right)$. Under these approximations the general form of the master equation is given by

$$\begin{aligned}
i\hbar\partial_h \rho_r[\chi^1_{<f}, \chi^2_{<f}, \eta] &= \langle \chi^1_{<f}| \left[\hat{H}_{\text{ren}}, \hat{\rho}_r\right] |\chi^2_{<f}\rangle \\
&- i\lambda^2 \left[-\frac{[(\chi^1_{<f})^3 - (\chi^2_{<f})^3]^2 V}{1152} D_3(\mathbf{k}_0;\eta) \right. \\
&+ \frac{[(\chi^1_{<f})^2 - (\chi^2_{<f})^2]^2 V}{32} D_2(\mathbf{k}_0;\eta) \\
&\left. - \frac{(\chi^1_{<f} - \chi^2_{<f})^2 V}{6} D_1(\mathbf{k}_0;\eta) \right] \rho_r[\chi^1_{<f}, \chi^2_{<f}, \eta] + \dots
\end{aligned} \tag{5.109}$$

Due to the complexity of the equation, we only show the correction to the usual unitary evolution term coming from the noise kernels. The full expression can be found in [LomMaz96]. This equation contains three time-dependent diffusion coefficients $D_i(\eta)$. (The subscripts 3, 2, 1 refer to the order of the system field $\phi_{<f}$.) Up to one loop, only D_3 and D_2 survive and are given by

$$\begin{aligned}
D_3(\mathbf{k}_0;\eta) &= \int_0^t ds\, \cos^3(\mathbf{k}_0 s)\, \mathrm{Im}G_F^{\Lambda_c}(3\mathbf{k}_0;\eta - s) \\
&\approx \frac{1}{6k_0} \int_0^t ds\, \cos^3(k_0 s) \cos(3k_0 s)\, \theta(3k_0 - \Lambda_c) \\
&= \frac{2k_0\eta + 3\sin(2k_0\eta) + \frac{3}{2}\sin(4k_0\eta) + \frac{1}{3}\sin(6k_0\eta)}{576\, k_0^2} \\
&\qquad \text{for} \quad \frac{\Lambda_c}{3} < k_0 < \Lambda_c \qquad (k_0 \equiv |\mathbf{k}_0|)
\end{aligned} \tag{5.110}$$

$$D_2(\mathbf{k}_0;\eta) = \int_0^h ds\, \cos^2(k_0 s) \left\{ \mathrm{Re}[G_F^{\Lambda_c}(2\mathbf{k}_0;\eta - s)]^2 + 2\mathrm{Re}[G_F^{\Lambda_c}(0;\eta - s)]^2 \right\} \tag{5.111}$$

Using the expressions

$$\begin{aligned}
\mathrm{Re}[G_F^{\Lambda_c}(2\mathbf{k}_0;\eta - s)]^2 = \frac{\pi\hbar^2}{k_0} &\left\{ \int_{\Lambda_c}^{2k_0+\Lambda_c} dp \int_{\Lambda_c}^{2k_0+p} dz\, \cos[(p+z)s] \right. \\
&\left. + \int_{2k_0+\Lambda_c}^{\infty} dp \int_{p-2k_0}^{p+2k_0} dz\, \cos[(p+z)s] \right\}
\end{aligned} \tag{5.112}$$

$$\mathrm{Re}[G_F^{\Lambda_c}(0;\eta - s)]^2 = \pi\hbar^2 \left\{ 2\pi\delta(s) - 2\frac{\sin(2\Lambda_c s)}{s} \right\} \tag{5.113}$$

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the D_2 diffusion coefficient can be written as

$$\begin{aligned}
\hbar^{-2} D_2(\mathbf{k}_0;\eta) = \frac{\pi}{4}\Bigg\{ & 3\pi - \left(\frac{3}{2} - \frac{\Lambda_c}{2k_0}\right)\mathrm{Si}[2\eta(\Lambda_c - k_0)] \\
& - \left(2 - \frac{\Lambda_c}{2k_0}\right)\mathrm{Si}[2\Lambda_c\eta] - \left(\frac{3}{2} + \frac{\Lambda_c}{2k_0}\right)\mathrm{Si}[2\eta(\Lambda_c + k_0)] \\
& - (1 + \frac{\Lambda_c}{2k_0})\mathrm{Si}[2\eta(2k_0 + \Lambda_c)] \\
& + \frac{1}{4k_0\eta}\,(\cos[2\Lambda_c\eta] - \cos[2\eta(\Lambda_c + k_0)] \\
& + \cos[2\eta(\Lambda_c - k_0)] - \cos[2\eta(2k_0 + \Lambda_c)])\Bigg\}
\end{aligned} \tag{5.114}$$

where Si$[z]$ denotes the sine-integral function [AbrSte72].

Equation (5.109) is the field-theoretical version of the QBM master equation we were looking for, except that the system here has nonlinear coupling. Owing to the existence of three interaction terms ($\chi_<^3\chi_>$, $\chi_<^2\chi_>^2$, and $\chi_<\chi_>^3$) there are three diffusion coefficients in the master equation. The form of the coefficients is fixed by these couplings and by the particular choice of the quantum state of the environment.

Note that these results are valid in the single-mode approximation. In this approximation one obtains a reduced density matrix for each mode $\mathbf{k}_0$, and neglects the interaction between different system modes. Due to this interaction, ρ_r will be different from $\prod_{\mathbf{k}_0}\rho_r(\mathbf{k}_0)$ in the general case. These results will be applied to the consideration of decoherence of quantum fields in Chapter 9 and cosmological structure formation in Chapter 15.

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