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ENDOTRIVIAL MODULES FOR THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract In this paper we determine the group of endotrivial modules for certain symmetric and alternating groups in characteristic p. If $p = 2$, then the group is generated by the class of $\Omega^n(k)$ except in a few low degrees. If $p > 2$, then the group is only determined for degrees less than p^2. In these cases we show that there are several Young modules which are endotrivial.

Keywords: endotrivial modules; Young modules; symmetric groups; alternating groups

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1. Introduction

Endotrivial modules play an important role in the modular representation theory of finite groups. They are the building blocks of the endo-permutation modules and are an essential part of the Picard group of self equivalences of the stable category of kG-modules in the case when G is a finite group and k is a field of characteristic p. A few years ago, Carlson and Thévenaz completed a program to classify the endotrivial kG-modules over p-groups [**9**]. Building on this result, Bouc [**7**] has completed a similar program to classify the endo-permutation modules over p-groups.

More recently, using the classification for p-groups, the authors of this paper started a project to classify the endotrivial modules over families of finite simple groups and related groups (see [**10**]). For a finite group of Lie type in the defining characteristic, we have found the structure of the group of endotrivial modules and explicit generators for the group in many cases. The group of endotrivial modules is cyclic, generated by the class of the syzygy module $\Omega(k)$, except in the cases in which the group has Lie rank 1, or that the Lie rank is 2 and the field of definition of the group is small.

The natural next case, which we introduce with this paper, is the groups of endotrivial modules for the symmetric and alternating groups. In spite of the advanced nature of

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the representation theory of symmetric groups, determining the torsion in the endotrivial group is considerably more difficult than the case for groups of Lie type. To this point, we have obtained only partial results. In particular, we give complete classifications with generators for the symmetric and the alternating groups, Σ_n and A_n, on n letters, in the cases when $p = 2$, or $p > 2$ and $n < p^2$.

Let k be a field of characteristic p which is a splitting field for a given finite group G and all its subgroups, and let $T(G)$ denote the group of endotrivial kG-modules. In this paper we prove the following.

Theorem 1.1. *Let Σ_n be the symmetric group on n letters.*

(a) *If $p = 2$, then*

$$T(\Sigma_n) \cong \begin{cases} \{0\} & \text{if } n \leqslant 3, \\ \mathbb{Z}^2 & \text{if } n = 4, 5, \\ \mathbb{Z} & \text{if } n \geqslant 6. \end{cases}$$

(b) *If $p \geqslant 3$ and $1 \leqslant n < 2p$, then*

$$T(\Sigma_n) \cong \begin{cases} \{0\} & \text{if } n < p, \\ \mathbb{Z}/2(p-1)\mathbb{Z} & \text{if } n = p, p+1, \\ \mathbb{Z}/2(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } p+2 \leqslant n < 2p. \end{cases}$$

(c) *If $p \geqslant 3$ and $2p \leqslant n < p^2$, then*

$$T(\Sigma_n) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } 2p \leqslant n < 3p, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } 3p \leqslant n < p^2. \end{cases}$$

(d) *If $p \geqslant 3$ and $p^2 \leqslant n$, then*

$$\text{the torsion-free rank of } T(\Sigma_n) \text{ is } \begin{cases} 2 & \text{if } p^2 \leqslant n < p^2 + p, \\ 1 & \text{if } p^2 + p \leqslant n. \end{cases}$$

Theorem 1.2. *Let A_n be the alternating group on n letters.*

(a) *If $p = 2$, then*

$$T(A_n) \cong \begin{cases} \{0\} & \text{if } n \leqslant 3, \\ \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) & \text{if } n = 4, 5, \\ \mathbb{Z}^2 & \text{if } n = 6, 7, \\ \mathbb{Z} & \text{if } n \geqslant 8. \end{cases}$$

(b) *If $p \geqslant 3$ and $1 \leqslant n < 2p$, then*

$$T(A_n) \cong \begin{cases} \{0\} & \text{if } n < p, \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } n = p, p+1, \\ \mathbb{Z}/2(p-1)\mathbb{Z} & \text{if } p+2 \leqslant n < 2p. \end{cases}$$

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(c) *If* $p \geqslant 3$ *and* $2p \leqslant n < p^2$*, then*

$$T(A_n) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z}) & \text{if } p = 3 \text{ and } n = 6, 7, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } p > 3 \text{ and } n = 2p, 2p+1, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } 2p+2 \leqslant n < 3p, \\ \mathbb{Z} & \text{if } 3p \leqslant n < p^2. \end{cases}$$

(d) *If* $p \geqslant 3$ *and* $p^2 \leqslant n$*, then*

$$\text{the torsion-free rank of } T(A_n) \text{ is } \begin{cases} 2 & \text{if } p^2 \leqslant n < p^2 + p, \\ 1 & \text{if } p^2 + p \leqslant n. \end{cases}$$

In the case when $p = 2$, the problem is reasonably straightforward because the Sylow 2-subgroups are self-normalizing except in a couple of well-known cases. These results are found in § 4. A surprise in the odd characteristic case for the symmetric groups is that there are several Young modules which are endotrivial.

We strongly suspect that, for $p > 2$ and $n \geqslant p^2$, the groups $T(\Sigma_n)$ and $T(A_n)$ have no torsion beyond that coming from the sign representation. The torsion-free ranks of these groups are known from general principles that give us the results stated above. That is, the torsion-free rank of $T(G)$ depends only on the number of G-conjugacy classes of maximal elementary abelian p-subgroups of order p^2. However, we do not know (yet) how to determine generators for the torsion-free part of $T(G)$ unless this rank is 1, or if $p = 2$ and the Sylow 2-subgroups are dihedral. For our purposes, this concerns the groups $T(\Sigma_n)$ and $T(A_n)$ for $p \geqslant 3$ and $p^2 \leqslant n < p^2 + p$. In addition, finding the torsion elements of $T(\Sigma_n)$ and $T(A_n)$ for $p \geqslant 3$ and $p^2 \leqslant n$ has not been completed at the time of writing. We plan to investigate these issues in future work.*

Several of our calculations for small groups relied on the algebra software MAGMA (cf. [**6**]). The computations turned out to be very effective in revealing to us that there are non-trivial endotrivial modules whose class is a torsion element in the group of endotrivial modules. This led us to understand and prove the underlying theoretical facts.

2. Setting

Throughout the paper, let G be a finite group, usually the symmetric group Σ_n or the alternating group A_n acting on the set $\{1, \dots, n\}$, for some n, and let k be a field of characteristic p which is a splitting field for G and all of its subgroups. For this, it is sufficient that k contains all mth roots of unity, where m is the order of G. When defining subgroups of the symmetric or alternating groups we assume the natural ordering on the letters unless otherwise indicated. For example, we write $H = \Sigma_a \times \Sigma_b \subseteq \Sigma_n$ to mean the subgroup where Σ_a is the collection of all permutations on $\{1, \dots, a\}$ and Σ_b is the set of all permutations in Σ_n on $\{a+1, \dots, a+b\}$ $(a + b \leqslant n)$.

* Note added in proof: these problems have now been solved. Proofs will appear in [**11**].

For two subgroups H and K of G let $[G/H]$ denote a complete set of representatives for the left H-cosets in G and let $[H \setminus G/K]$ be a complete set of representatives for the H–K double cosets in G. Normally, we assume that the identity element 1 of G is one of the representatives, so that $[G/H] \setminus \{1\}$ means a set of representatives of the nonidentity cosets.

We consider only finitely generated left modules over group algebras. If M is a kG-module, we write $\operatorname{Res}^G_H M$, or $M\downarrow^G_H$, for the restriction of M to kH where H is a subgroup of G. If N is another kG-module, we denote by $\operatorname{Hom}_k(M, N)$ the kG-module of all k-linear maps $M \longrightarrow N$. We let $\operatorname{End}_k M = \operatorname{Hom}_k(M, M)$ denote the k-endomorphism ring of M. The k-dual of M is $M^* = \operatorname{Hom}_k(M, k)$, where k also denotes the trivial kG-module. Let $M \otimes N$ be the tensor product of two modules M and N over the base field k with diagonal action of the group G. We write $M \mid N$ to mean that the module M is a direct summand of N.

For M a kG-module, we denote the kernel of the projective cover $P \longrightarrow M$ by $\Omega(M)$ and the cokernel of the injective hull $M \longrightarrow Q$ by $\Omega^{-1}(M)$. Inductively, for any $n > 1$, we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{1-n}(M))$. The combinatorics are that $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$ and that, for any module N, there exists a projective module P such that $\Omega^n(M) \otimes N \cong \Omega^n(M \otimes N) \oplus P$.

We write $\operatorname{mod}(kG)$ for the category of all finitely generated kG-modules and $\operatorname{stmod}(kG)$ for the stable module category, namely the quotient of $\operatorname{mod}(kG)$ by the subcategory of projective modules. That is, the stable category has the same objects as $\operatorname{mod}(kG)$. For two finitely generated kG-modules M and N, the morphisms from M to N in the stable category are given by

$$\underline{\operatorname{Hom}}_{kG}(M, N) = \operatorname{Hom}_{kG}(M, N)/\operatorname{PHom}_{kG}(M, N),$$

where $\operatorname{PHom}_{kG}(M, N)$ is the subspace of homomorphisms that factor through a projective module.

Definition 2.1. A kG-module M is endotrivial provided that $\operatorname{End}_k M \cong k \oplus (\text{proj})$ or, equivalently, $\operatorname{End}_k M \cong k$ in $\operatorname{stmod}(kG)$.

Recall that $\operatorname{Hom}_k(M, N) \cong M^* \otimes N$ as kG-modules. Consequently, the tensor product of two endotrivial modules is endotrivial. This allows us to define the group of endotrivial modules whose elements are classes $[M]$ as follows.

Definition 2.2. Two endotrivial kG-modules are *equivalent* if they are isomorphic in $\operatorname{stmod}(kG)$. That is, $[M] = [N]$ if $M \oplus P \cong N \oplus Q$ for projective modules P and Q. The group of endotrivial kG-modules is the set $T(G)$ of equivalence classes $[M]$ of endotrivial kG-modules M, with the operation given by the rule $[M] + [N] = [M \otimes N]$.

Clearly, $T(G)$ is abelian, and we have that $0 = [k]$ and $-[M] = [M^*]$. Furthermore, if p does not divide the order of G, then every module is projective. In this case, the definition of an endotrivial module does not have much meaning, as every object in the stable category is equivalent to the zero object, and also every module is an endotrivial module, by a strict interpretation of the definition. In that case, we set $T(G) = \{0\}$.

The next theorem collects some useful properties of the group $T(G)$.

Theorem 2.3. *Let G be any finite group. The group $T(G)$ is finitely generated. Thus, there is a torsion-free subgroup $TF(G)$ of finite rank, such that $T(G) \cong TT(G) \oplus TF(G)$, where $TT(G)$ denotes the torsion subgroup of $T(G)$.*

(a) *The modules $\Omega^n(k)$ are endotrivial and their classes generate a cyclic direct summand of $T(G)$ [**1**, **9**, **12**, **13**].*

(b) *Let n denote the number of conjugacy classes of maximal elementary abelian p-subgroups of p-rank 2 in G. Then the rank of $TF(G)$ is n if G has p-rank at most 2 and is $n+1$ if the p-rank of G is greater than 2 [**1**, **10**].*

(c) *If E is an abelian p-group having p-rank at least 2, then $T(E) \cong \mathbb{Z}$ and is generated by the class of $\Omega(k)$ [**12**, **13**].*

  (i) *If G has p-rank at least 2, then the product of the restriction maps*

$$\mathrm{Res} : TF(G) \longrightarrow \prod_E T(E)$$

  *from $TF(G)$ to all of the elementary abelian p-subgroups E of p-rank 2 of G, is injective [**1**, **10**].*

  (ii) *A kG-module is endotrivial if and only if its restriction to every elementary abelian p-subgroup of G is endotrivial [**9**].*

(d) *Let P be a Sylow p-subgroup of G.*

  (i) *The torsion subgroup $TT(P)$ is trivial except in the case when P is cyclic, quaternion or semidihedral [**9**].*

  (ii) *If $TT(P)$ is trivial, then $TT(G)$ is generated by the classes $[M]$ of indecomposable endotrivial kG-modules M such that $M\downarrow_P^G \cong k \oplus (\mathrm{proj})$, for a projective kP-module (proj) [**10**]. (Note that, in general, a module with vertex P and trivial source is not endotrivial.)*

(e) *The restriction map $\mathrm{Res}_H^G : T(G) \longrightarrow T(H)$ is injective, provided the subgroup H of G contains the normalizer of a Sylow p-subgroup of G [**10**].*

It follows from (d) in the above theorem that the torsion subgroup $TT(P)$ of $T(P)$ is often trivial. We will show in the next sections that, as a consequence of this, $TT(G)$ is generated by the equivalence classes of the indecomposable trivial source modules that are endotrivial, for an arbitrary finite group G having a Sylow p-subgroup isomorphic to P. If G is the symmetric group, it turns out that some of these modules can be found among the Young modules. We refer the reader to [**19**] for the basic background on the representation theory of the symmetric groups, and to [**15**, **16**, **18**] for deeper investigations of the Young modules. Some computer computations of Young module can be found on the first author's web page (www.math.uga.edu/~jfc/hecke.html).

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Definition 2.4. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be the a partition of n and let $G = \Sigma_n$.

(a) A *Young subgroup* Σ_λ *associated with* λ is a subgroup of G that is conjugate to $\Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$.

(b) The *Specht module* S^λ *associated with* λ is the kG-module with k-basis the set of standards polytabloids (cf. [**19**]). It is a submodule of the *permutation module* $M^\lambda = k\uparrow_{\Sigma_\lambda}^G$.

(c) The *Young module* Y^λ *associated with* λ is the unique (up-to-isomorphism) indecomposable direct summand of M^λ that contains S^λ.

It should also be noted that all of the indecomposable direct summands of the permutation modules M^λ are Young modules Y^μ for some partitions μ which are greater than or equal to λ in the dominance ordering. In particular, we will call upon the following observation.

Lemma 2.5. *Suppose that S is a Young subgroup of $G = \Sigma_n$ that contains a Sylow p-subgroup of G. Suppose that N is an indecomposable kG-module such that $N\downarrow_S^G \cong k \oplus (\text{proj})$. Then N is a Young module with vertex a Sylow p-subgroup of G.*

Proof. Because S contains a Sylow p-subgroup, N is relatively S-projective. Consequently, N is a direct summand of $N\downarrow_S^G\uparrow_S^G$. Since k is a direct summand of $N\downarrow_S^G$, a vertex of N is a Sylow p-subgroup of G, and N is a direct summand of $k\uparrow_S^G$. $\square$

Remark 2.6. Note that, in the proof of the above lemma, N is a Young module simply because every indecomposable summand of $k\uparrow_S^G$ is a Young module. We cannot conclude that N is the Young module corresponding to the same partition as the Young subgroup S. Indeed, the trivial module $N = k$ satisfies the hypotheses of the lemma, even when the subgroup S corresponds to a non-trivial partition.

3. Some generalities

In this section we present a few general results that will be used in the course of this paper.

Proposition 3.1. *Suppose that H is a normal subgroup of G and that p does not divide the index of H in G. Let M be an indecomposable endotrivial kG-module. Then $M\downarrow_H^G$ is endotrivial and indecomposable.*

Proof. Assume that H is normal in G and that M is an endotrivial indecomposable kG-module. Then $M\downarrow_H^G \cong M_0 \oplus Q$, with M_0 an indecomposable endotrivial module and Q a projective module. Since p does not divide $|G : H|$, M is projective relative to H and we deduce that M is a direct summand of $M_0\uparrow_H^G$. It follows that $M\downarrow_H^G$ is a direct summand of

$$M_0\uparrow_H^G\downarrow_H^G \cong \bigoplus_{g\in[G/H]} {}^gM_0,$$

since $H \lhd G$. The only way that this can happen is if Q is the zero module. $\square$

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Lemma 3.2. *Let H be a subgroup of G that contains a Sylow p-subgroup P of G. If $|P| > |G:H|$, then the kernel of the restriction map $\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$ is generated by the classes of the one-dimensional kG-modules M such that $M\downarrow^G_H = k$. In particular, if $G = \Sigma_n$, $H = \Sigma_{n-1}$, $n > 2p$, and if $p \nmid n$, then the restriction map $\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$ is injective. The same applies if $G = A_n$ and $H = A_{n-1}$. Similarly, if $p > 2$, then the kernel of the restriction map $T(\Sigma_n) \longrightarrow T(A_n)$ has order 2 and is generated by the class of the sign representation.*

Proof. Let M be an indecomposable endotrivial kG-module such that $M\downarrow^G_H \cong k \oplus$ (proj). Then, by relative projectivity and a vertex argument, we have that $M \mid k\uparrow^G_H$. Since $\mathrm{Dim}(k\uparrow^G_H) = |G:H| < |P|$, the kH-module $M\downarrow^G_H$ has no non-zero projective summand. Therefore, $\mathrm{Dim}(M) = 1$.

Assume that $G = \Sigma_n$ and $H = \Sigma_{n-1}$ (respectively, $G = A_n$ and $H = A_{n-1}$), with $n > 2p$. Assume also that $p \nmid n$. Then, the index $|G:H| = n < |P|$, where P is a common Sylow p-subgroup of G and H. Hence, $\mathrm{Ker}(\mathrm{Res}^G_H)$ consists of the isomorphism classes (in $\mathrm{stmod}(kH)$) of the one-dimensional kG-modules that restrict trivially to H. If $G = \Sigma_n$ or $G = A_n$, then there is exactly one such module: the trivial module. Likewise, $\mathrm{Ker}(\mathrm{Res}^{\Sigma_n}_{A_n})$ is generated by the class of the sign representation if $p > 2$. □

We will now provide some general information about the situation when the field has characteristic 2. The following proposition is recorded in [**3**, Lemma 5.4, Theorem 5.5].

Proposition 3.3. *Suppose that G is a finite group whose Sylow 2-subgroups are dihedral. Then G has two conjugacy classes of elementary abelian 2-subgroups of rank 2, represented by subgroups E_1 and E_2. Let P_0 be the projective cover of the trivial kG-module k. Then, taking the radical modulo the socle of P_0 (i.e. the 'heart' of P_0), we get*

$$\mathrm{Rad}(P_0)/\,\mathrm{Soc}(P_0) \cong M \oplus M^*,$$

where $M\downarrow^G_{E_1} \cong \Omega(k)$ and $M\downarrow^G_{E_2} \cong \Omega^{-1}(k)$. Hence, M is an endotrivial module and

$$TF(G) = \langle [\Omega(k)], [M] \rangle \cong \mathbb{Z}^2.$$

Moreover, if a dihedral group D is a Sylow 2-subgroup of G, then the restriction map $\mathrm{Res}^G_H : TF(G) \longrightarrow T(D)$ is an isomorphism.

Proposition 3.3 provides the answer to a question that was left open in [**10**]. Namely, in the case when $G = \mathrm{PSL}_3(\mathbb{F}_2)$ (i.e. the finite group of Lie type $A_2(2)$), we have that the Sylow 2-subgroups are dihedral of order 8. In [**10**, Theorem 8.1], we demonstrated that $T(G) \cong \mathbb{Z}^2$. Now, the diagram of the projective cover P_0 of the trivial module is (cf. [**5**, § 11])

$$\begin{matrix} & k & \\ M & & M^*, \\ & k & \end{matrix} \quad \text{with } M \text{ of dimension 3.}$$

Thus, we have $\mathrm{Rad}(P_0)/\,\mathrm{Soc}(P_0) \cong M \oplus M^*$. By Proposition 3.3, M is endotrivial and we have $T(G) = \langle [\Omega(k)], [M] \rangle$.

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We end this section with a known result about the normalizers of the Sylow 2-subgroups of the symmetric and alternating groups.

Proposition 3.4. *Let n be a positive integer.*

(a) *The Sylow 2-subgroups of Σ_n are self-normalizing for all n.*

(b) *The Sylow 2-subgroups of A_n are self-normalizing for all $n \geqslant 6$.*

Proof. This is a result due to Weisner (see [**22**, Corollary 2, Theorem, p. 124]). □

4. The case $p = 2$

In this section, we give a complete characterization of the group $T(G)$ in the case when $G = \Sigma_n$ and $G = A_n$ for $p = 2$. For each family of groups, we first determine the isomorphism type of $T(G)$, and then describe the generators in detail.

Theorem 4.1. *Suppose that $G = \Sigma_n$ and $p = 2$. Then $TT(G) = \{0\}$, for all positive integers n. We have that*

$$T(G) \cong TF(G) \cong \begin{cases} \{0\} & \text{if } n \leqslant 3, \\ \mathbb{Z}^2 & \text{if } n = 4, 5, \\ \mathbb{Z} & \text{if } n \geqslant 6. \end{cases}$$

Proof. Let P be a Sylow p-subgroup of G. In this case, $N := N_G(P) = P$, by Proposition 3.4. By the classification of endotrivial modules of a p-group (cf. [**9**]), we know that $TT(P) = \{0\}$. Indeed, P is either cyclic of order 2, dihedral of order 8, or has all maximal elementary abelian 2-subgroups of 2-rank at least 3, in the three cases. The last statement is an exercise for the reader. If $n = 6$ or 7, then P is a direct product of a cyclic group of order 2 with a dihedral group of order 8, and the conclusion about the ranks of the maximal elementary abelian subgroups is obvious. If $n = 8$ or 9, then P is a wreath product of a dihedral group of order 8 and a cyclic group of order 2, and again the conclusion holds. For larger n, we always have that P is a direct product of wreath products of Sylow subgroups of smaller symmetric groups.

Because $N = P$, we have that $TT(N) = \{0\}$, and so $TT(G) = \{0\}$. The result for $T(G)$ is then a direct consequence of Theorem 2.3 and of [**10**, Corollary 3.2]. □

Let us now explore the group $T(G)$ further. First, note that, in the theorem, if $T(G)$ is cyclic, then $T(G) = \langle[\Omega(k)]\rangle$, and this tells us all about $T(G)$, for $n \geqslant 6$ or $n \leqslant 3$. When $n = 4$ or 5, the class $[\Omega(k)]$ generates one summand of $T(G)$ and, because P is dihedral, there are at least two ways of finding a generator for the other summand. For one we know the structure of the projective cover P_0 of the trivial module k, as in Proposition 3.3. The structure of these modules can be seen from the diagrams in [**5**]. The other method is to draw on the classical representation theory of the symmetric group, which turns out to be a matter of linear algebra.

Say $P = \langle x, y \mid x^2 = y^2 = (xy)^4 = 1\rangle$, so that the centre of P is generated by the element $(xy)^2$. By [**8**, Theorem 5.4], we have that $T(P) = \langle[\Omega(k)], [\Omega_{P/\langle x\rangle}(k)]\rangle \cong \mathbb{Z}^2$,

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where $\Omega_{P/\langle x\rangle}(k)$ is the kernel of the map $k[P/\langle x\rangle] \longrightarrow k$ sending each coset $u\langle x\rangle$ to 1, and $k[P/\langle x\rangle]$ is the permutation kP-module with k-basis the cosets of $P/\langle x\rangle$, on which P acts by left multiplication.

On the other hand, the Specht module $S^{(3,1)}$ has dimension 3 and we can take as k-basis the set of $(3,1)$-polytabloids, i.e. the row equivalence classes of the standard $(3,1)$-tableaux. We refer the reader to [**19**] for the details.

A direct computation shows that the kP-modules $S^{(3,1)}\downarrow_P^G$ and $\Omega_{P/\langle x\rangle}(k)$ are isomorphic, and hence it proves that

$$T(G) = \langle [\Omega(k)], [S^{(3,1)}]\rangle.$$

Let us now consider $G = \Sigma_5$ and let $G' = \Sigma_4 < G$. Then, by [**21**, Lemma 1.5], we have that $S^{(3,1)}\uparrow_{G'}^G \cong S^{(4,1)} \oplus M$, where M is an extension

$$0 \longrightarrow S^{(3,1,1)} \longrightarrow M \longrightarrow S^{(3,2)} \longrightarrow 0.$$

Thus, M has dimension 11, and is the Green correspondent of $S^{(3,1)}$ over kG. It is also the Green correspondent of $\Omega_{P/\langle x\rangle}(k)$. Thus, by [**3**, § 6], we have that M is endotrivial and $M\downarrow_P^G \cong \Omega_{P/\langle x\rangle}(k) \oplus kP$. This shows that

$$T(G) = \langle [\Omega(k)], [M]\rangle,$$

where M is the above module.

For the alternating groups we have the following. Again, the case of $n \leqslant 3$ is trivial, since then 2 does not divide the order of A_n.

Theorem 4.2. *Suppose that $G = A_n$, the alternating group on n letters, and $p = 2$. The group of endotrivial modules has the form*

$$T(G) \cong \begin{cases} \{0\} & \text{if } n \leqslant 3, \\ \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) & \text{if } n = 4, 5, \\ \mathbb{Z}^2 & \text{if } n = 6, 7, \\ \mathbb{Z} & \text{if } n \geqslant 8. \end{cases}$$

Proof. Let P be a Sylow 2-subgroup of G. If $n = 4$ or 5, then P is isomorphic to a Klein four group and the normalizer N of P has the form $P \rtimes C_3$, where C_3 is a cyclic group of order 3. Hence, $TF(G) = TF(N) \cong \mathbb{Z}$ and so

$$T(N) = TF(N) \oplus TT(N) \cong \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}).$$

Indeed, since k is a splitting field for N, the group $TT(N)$ identifies with the character group of N, which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Now, in both cases, P is a trivial intersection (TI) subgroup of G. That is, if $x \in G$ and $x \notin N_G(P)$, then $xPx^{-1} \cap P = \{1\}$. In such a situation, the stable module categories stmod(kG) and stmod(kN) are equivalent, by the induction and restriction functors, and these functors induce isomorphisms $T(G) \cong T(N)$ (cf. [**10**, Proposition 2.8]). Hence, we have that $T(G) \cong \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})$.

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For $n > 5$, the Sylow 2-subgroups P of G are self-normalizing (cf. Proposition 3.4). Consequently, there are no non-trivial one-dimensional representations of the normalizer $N = P$ of P. It follows that $TT(G) = TT(N) = \{0\}$. In the cases in which $n = 6$ or 7, the Sylow 2-subgroups are dihedral of order 8 and hence we have that $T(G) = TF(G) \cong \mathbb{Z}^2$, by Theorem 2.3. In all other cases, G has no maximal elementary abelian 2-subgroup of rank 2. □

As for the symmetric group, let us look for the additional generator of $TF(A_n)$ needed when $n = 6, 7$. For this, we use Proposition 3.3, since P is dihedral, and we consider the diagrams of the projective cover of the trivial module given in [**4**, Appendix, pp. 206, 213]. It turns out that there are endotrivial modules of dimension 19 if $n = 6$, and of dimension 35 if $n = 7$ that are the kA_n-Green correspondents of the kP-module $\Omega_{P/\langle x\rangle}(k)$, as defined above. In particular, we observe that the restriction map $\mathrm{Res}^G_P : T(G) \longrightarrow T(P)$ is an isomorphism.

5. The case $n < 2p$, for $p \geqslant 3$

From this point on we assume that $p \geqslant 3$ (i.e. p is an odd prime). Note first that if $n < p$, then p does not divide the order of $G = \Sigma_n$, or of A_n, and every module is projective. Thus, $T(G) = \{0\}$ if $n < p$, as mentioned in § 2. We now consider the case when $p \leqslant n < 2p$.

Proposition 5.1. *Let $G = \Sigma_n$, and let $N = N_G(P)$, where P is a Sylow p-subgroup of G.*

(a) *If $n = p$ or $n = p + 1$, then*

$$T(G) = TT(G) \cong TT(N) \cong \mathbb{Z}/2(p-1)\mathbb{Z}.$$

(b) *If $n = p + b$, with $1 < b < p$, then*

$$T(G) = TT(G) \cong TT(N) \cong \mathbb{Z}/2(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Proof. Suppose that $p \leqslant n < 2p$. Then P is cyclic of order p, and so $T(P) = TT(P) = \langle[\Omega(k)]\rangle \cong \mathbb{Z}/2\mathbb{Z}$. The subgroup P is a TI subgroup of G. Hence, the restriction functor induces an isomorphism $T(G) \cong T(N)$ (cf. [**10**, Proposition 2.8]).

In the case when $n = p$ or $n = p+1$, we have $N \cong C_p \rtimes C_{p-1}$, and $TT(N) \cong \mathbb{Z}/2(p-1)\mathbb{Z}$, since we assume that k is a splitting field for N. That is, by a routine calculation it can be shown that $\Omega^2(k)$ is a one-dimensional kN-module which is a generator for the multiplicative group of one-dimensional kN-modules (one-dimensional characters of N). Consequently, the class of $\Omega(k)$ generates a cyclic subgroup of order $2(p-1)$ in $T(N)$. But this must be all of $T(N)$ since we know that the kernel of the restriction map from $T(N)$ to $T(P)$ is the group generated by the one-dimensional kN-modules and that $T(P)$ has order 2.

When $n = p + b$, for an integer $1 < b < p$, we have that $N \cong (C_p \rtimes C_{p-1}) \times \Sigma_b$, and so $TT(N) \cong \mathbb{Z}/2(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, where the $\mathbb{Z}/2\mathbb{Z}$ factor is the subgroup generated by the class of the inflation of the sign representation of Σ_b to N. □

Similar arguments apply to the alternating groups, and they lead us to the following result.

Proposition 5.2. *Let $A = A_n$, and let $N_A = N_A(P)$, where P is a Sylow p-subgroup of A.*

(a) *If $n = p$ or $n = p + 1$, then*

$$T(A) = TT(A) \cong TT(N_A) \cong \mathbb{Z}/(p-1)\mathbb{Z}.$$

(b) *If $n = p + b$, with $1 < b < p$, then*

$$T(A) = TT(A) \cong TT(N_A) \cong \mathbb{Z}/2(p-1)\mathbb{Z}.$$

Proof. We have, as above, $T(P) = TT(P) = \langle[\Omega(k)]\rangle \cong \mathbb{Z}/2\mathbb{Z}$. Because P is a TI subgroup of A, we obtain $T(A) \cong T(N_A)$. Now, embed A into $G = \Sigma_{p+b}$ and consider $N = N_G(P)$ as in the previous proposition. By Lemma 3.2, the restriction map $T(N) \longrightarrow T(N_A)$ has kernel equal to $\mathbb{Z}/2\mathbb{Z}$. That is, the class of the sign representation is the only non-trivial element of the kernel. In the case when $b = 0, 1$, we find immediately that $T(N_A) \cong \mathbb{Z}/(p-1)\mathbb{Z}$, as asserted. Now suppose that $1 < b < p$. Then, $N \cong (C_p \rtimes C_{p-1}) \times \Sigma_b$. Let $J = A_n \cap \Sigma_b$, where here the Σ_b means the Σ_b factor in N as we have expressed it. Note first that J is in the kernel of every one-dimensional representation of N_A, since it is in the commutator subgroup of Σ_b. Hence, $T(N_A/J) \cong T(N_A)$ by the inflation map. But now $N_A/J \cong C_p \rtimes C_{p-1}$. This proves the last statement. □

6. The case $n = ap$, for an integer $2 \leqslant a < p$

We begin this section with the description of the p-local group structure of the groups $G = \Sigma_n$ and $A = A_n$, in the case when $2p \leqslant n < p^2$. Write $n = ap + b$, with $2 \leqslant a < p$ and $0 \leqslant b < p$. Note that p must be an odd prime. We assume that G acts on the set $\{1, \dots, n\}$. Let P be a common Sylow p-subgroup of G and of A having the form

$$P = \langle(1, \dots, p), (p+1, \dots, 2p), \dots, ((a-1)p+1, \dots, ap)\rangle$$

and let $N = N_G(P)$ be the normalizer of P in G and $N_A = N_A(P) = N \cap A$. For simplicity, if $X \subseteq G$, we denote by X_A the intersection $X \cap A$.

Let us describe certain subgroups of G. We leave it to the reader to determine the structure of the corresponding subgroups for the alternating groups. The group P is elementary abelian of p-rank a, and $N \cong N_p \wr \Sigma_a \times \Sigma_b$, where $N_p \cong C_p \rtimes C_{p-1}$ is the normalizer in Σ_p of a Sylow p-subgroup of Σ_p. Here, '$\wr$' denotes the wreath product and, for a positive integer ℓ, we denote by C_ℓ a (multiplicative) cyclic group of order ℓ. We will also use the notation X^a for the direct product of a copies of a group X. Let S denote the Young subgroup containing P and corresponding to the partition $(p, \dots, p, b)$ of n. Then $H = NS$ is the normalizer of S in G and it is isomorphic to $\Sigma_p \wr \Sigma_a \times \Sigma_b$.

Hence, in Figure 1 we show the (three-dimensional) diagram of the inclusions of these subgroups (with obvious identifications of the subgroups of G).

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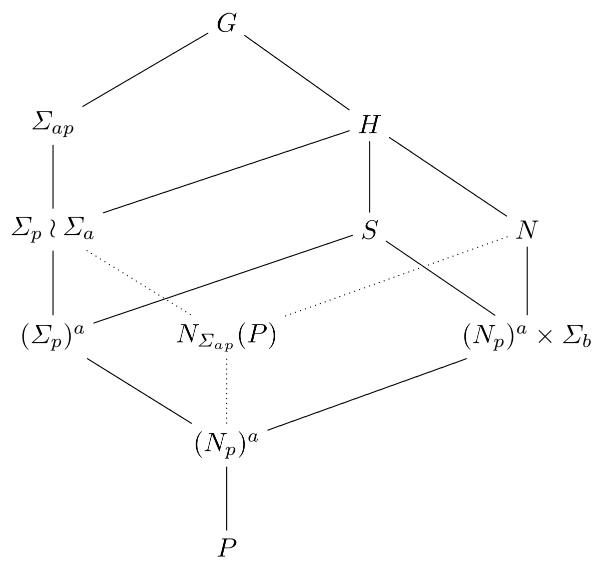

Figure 1. Subgroups of Σ_{ap+b}.

In this section, we consider the case in which $n = ap$, for an integer $2 \leqslant a < p$. For convenience, we will denote by $N_S = N_S(P) \cong (N_p)^a$.

Because P is elementary abelian of p-rank at least 2, $TT(P)$ is trivial. Thus, $TT(N)$ and $TT(N_A)$ are generated by the isomorphism classes of one-dimensional kN-modules and kN_A-modules, respectively, by Theorem 2.3. Since the arguments for A and for G are alike, we will work through only the case of the symmetric group, and state the corresponding results for the alternating groups without rewriting the reasoning.

From $N \cong N_p \wr \Sigma_a$, we deduce that $TT(N) \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, where the $\mathbb{Z}/(p-1)\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ factors correspond to the subgroups generated by the k-linear characters of N_p and by the sign representation of Σ_a, respectively.

Let X be a kN-module of dimension 1, and let Y be the kH-Green correspondent of X. This makes sense because $N \subseteq H$. Then let $X_{N_S} = X\downarrow^N_{N_S}$, and, since $N_S = N_S(P)$, it makes sense to consider the kS-Green correspondent Y_S of X_{N_S}. Matching the modules with the corresponding groups, we get Figure 2 (where the double-headed dotted arrows denote the Green correspondence between the indicated modules and groups).

Our aim now is to limit the choices for the module Y_S.

Lemma 6.1. *If Y is an endotrivial kH-module, then Y_S is an endotrivial kS-module and $Y\downarrow^H_S \cong Y_S$.*

Proof. By a property of the Green correspondence, Y is a direct summand of the induced module $X\uparrow^H_N$. Thus, using the Mackey formula and the equality $H = NS$, we obtain that $Y\downarrow^H_S$ is a direct summand of

$$X\uparrow^H_N\downarrow^H_S \cong \bigoplus_{g\in[S\backslash H/N]} {}^gX\downarrow^{{}^gN}_{{}^gN\cap S}\uparrow^S_{{}^gN\cap S} = X_{N_S}\uparrow^S_{N_S} \cong Y_S \oplus Z,$$

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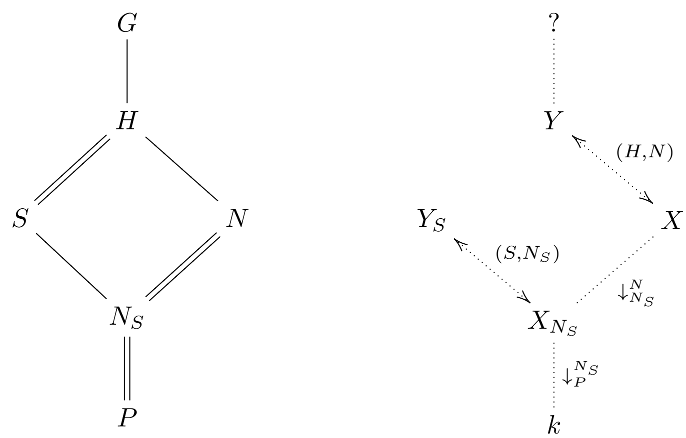

Figure 2. Green correspondents for Σ_{ap+b}.

where Z is a direct sum of indecomposable kS-modules each having vertex strictly contained in P. We remind the reader that $[S\backslash H/N]$ is a complete set of representative of the S–N double cosets in H. Since Y is indecomposable with vertex P, its restriction $Y\downarrow^H_S$ to S must have Y_S as direct summand. Consequently, if we assume that Y is endotrivial, then Y_S must also be endotrivial and the other direct summands appearing in the decomposition of $Y\downarrow^H_S$ are projective. That is, because Y is endotrivial, its restriction to S is endotrivial and can have only one non-projective indecomposable component. Now, the facts that $S \triangleleft H$ and $p \nmid |H:S|$ imply, by Proposition 3.1, that $Y\downarrow^H_S \cong Y_S$. □

By the contrapositive of the lemma, if Y_S is not endotrivial, then neither is Y. So we are down to the question of finding criteria for Y_S to be endotrivial. This is the main result of the section.

Proposition 6.2. *The module Y_S is endotrivial if and only if* $\mathrm{Dim}(Y_S) = 1$.

Proof. The ‘if’ part is clear. So now assume that Y_S is endotrivial. Let $K = N_p \times {\Sigma_p}^{a-1} \subseteq G$, where the ith factor Σ_p acts on $\{ip+1,\dots,(i+1)p\}$, for $1 \leqslant i \leqslant a-1$ and where N_p is the normalizer of $\langle(1\cdots p)\rangle$ in the subgroup Σ_p of S that acts on $\{1,\dots,p\}$. Thus, we have $N_S < K < S$. Because Y_S is an endotrivial module, we may write $Y_S\downarrow^S_K \cong U \oplus V$ for kK-modules U and V with U indecomposable endotrivial and V projective. Moreover, the Green correspondence implies that $U\downarrow^K_{N_S} \cong X_{N_S} \oplus W$ for some projective kN_S-module W (since U is endotrivial). We also have that U is a direct summand of $X_{N_S}\uparrow^K_{N_S}$. Moreover,

$$X_{N_S}\uparrow^K_{N_S}\downarrow^K_{N_S} \cong X_{N_S} \oplus \bigoplus_{g\in[N_S\backslash K/N_S]\backslash\{1\}} {}^gX_{N_S}\downarrow^{{}^gN_S}_{{}^gN_S\cap N_S}\uparrow^{N_S}_{{}^gN_S\cap N_S}.$$

Since the largest normal p-subgroup $O_p(K)$ of K is non-trivial and is contained in N_S, none of the indecomposable factors of the right-hand side is projective. From the fact that U is endotrivial, we deduce that $U\downarrow^K_{N_S} \cong X_{N_S}$. Therefore, $\mathrm{Dim}(U) = 1$.

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Now, Y_S is a direct summand of $U\uparrow_K^S$, by the Green correspondence, which applies since $N_S \subseteq K$. Therefore, $Y_S\downarrow_K^S$ is a direct summand of

$$U\uparrow_K^S\downarrow_K^S \cong U \oplus \bigoplus_{g\in[K\backslash S/K]\backslash\{1\}} {}^gU\downarrow^{{}^gK}_{{}^gK\cap K}\uparrow^K_{{}^gK\cap K}.$$

Again, g runs over a set of representatives of the K–K-double cosets in S, such that $g \notin K$. Now, note that for K-coset representatives $[S/K]$ in S we may take the N_p-coset representatives $[\Sigma_p/N_p]$ in Σ_p of the corresponding normal subgroups of S and K, respectively. Thus, we may assume that every coset representative centralizes the subgroup $(\Sigma_p)^{a-1}$ of K. It follows that every intersection ${}^gK \cap K$, for $g \in [K \setminus S/K]$, contains a non-trivial p-subgroup. Hence, the right-hand side in the displayed isomorphism does not contain any projective summands. Since Y_S is endotrivial, we deduce that $Y_S\downarrow_K^S \cong U$. Hence, $\mathrm{Dim}(Y_S) = 1$, as claimed. □

The last two results have an immediate consequence.

Corollary 6.3. *Let M be an indecomposable endotrivial kG-module (where $G = \Sigma_{ap}$, $2 \leqslant a < p$) whose class belongs to the torsion subgroup $TT(G)$. Then, $M\downarrow_H^G \cong Y \oplus (\mathrm{proj})$, where Y is an endotrivial kH-module of dimension* 1.

By very similar arguments for the alternating group $A = A_{ap}$, with $2 \leqslant a < p$, we prove the following.

Corollary 6.4. *Let M be an indecomposable endotrivial kA-module whose class belongs to the torsion subgroup $TT(A)$. Then $M\downarrow_{H_A}^A \cong Y \oplus (\mathrm{proj})$, where Y is an endotrivial kH_A-module of dimension* 1.

7. The case $n = 2p$

In this section, we apply the results of § 6 in the case when $G = \Sigma_{2p}$ and $A = A_{2p}$. As noted, $T(P) = \langle[\Omega(k)]\rangle \cong \mathbb{Z}$. We assume the notation of § 6 for G and for A, and, since the reasonings for A and for G are alike, we give the details for G and leave the alternating case as an exercise for the reader.

By [**2**, 6.1, 6.2] the subgroup H is strongly p-embedded in G, that is ${}^gH \cap H$ is a p'-group for any $g \in G \setminus H$. Therefore, for any kH-module Y of dimension 1, we have that

$$\begin{aligned} Y\uparrow_H^G\downarrow_P^G \cong Y\downarrow_P^H \oplus \bigoplus_{g\in[P\backslash G/H]\backslash\{1\}} {}^gY\downarrow^{{}^gH}_{{}^gH\cap P}\uparrow^P_{{}^gH\cap P} \\ = k \oplus \bigoplus_{g\in[P\backslash G/H]\backslash\{1\}} k\uparrow^P_{{}^gH\cap P} = k \oplus (\mathrm{proj}), \end{aligned}$$

where the sums are over representatives of the nonidentity P–H-double cosets. The isomorphism holds because for $g \notin H$ the intersection ${}^gH \cap P$ is trivial. Moreover, the one-dimensional modules for $H \cong \Sigma_p \wr \Sigma_2$ form a Klein four group. We may take for generators

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the kH-modules ε_H and $Y = k_{\Sigma_p} \wr \varepsilon_{\Sigma_2}$, where ε_K denotes the sign representation of the group K.

Observe that the isomorphism $Y\downarrow_S^H \cong k$ forces the kG-Green correspondent M of Y to be the Young module $Y^{(p,p)}$, associated with the partition (p,p) of $2p$. Indeed, M is an indecomposable non-trivial direct summand with vertex P of the permutation module $k\uparrow_S^G$ (cf. [**19**] and [**15**]).

Proposition 7.1. *Let $G = \Sigma_{2p}$ for p an odd prime. Then any one-dimensional kH-module induced to G is endotrivial. The group $T(G)$ is given by*

$$T(G) = \langle [\Omega(k)], [\varepsilon_G], [Y^{(p,p)}] \rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2.$$

Using this result, we obtain the following.

Proposition 7.2. *Let $A = A_{2p}$ for p an odd prime. Then any one-dimensional kH_A-module, when induced to A, is endotrivial. The group $T(A)$ is given by*

$$T(A) = \langle [\Omega(k)], [Y^{(p,p)}\downarrow_A^G], [M] \rangle \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } p > 3, \\ \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z}) & \text{if } p = 3, \end{cases}$$

where M is an indecomposable endotrivial kA-module such that $M\uparrow_A^G$ is indecomposable.

The last claim above comes from the nature of the one-dimensional kH_A-modules. Namely, if $E \cong A_p \times A_p$, then $E \subseteq H$ and is also a subgroup of H_A. It is easy to see that the restriction of any one-dimensional kH_A-module Y to E is trivial. Thus, Y is a $k[H_A/E]$-module. Since the quotient H/E is a dihedral group of order 8 and H_A has index 2 in H, we have either that Y extends to H, or that the induction $Y\uparrow_{H_A}^H$ is indecomposable. In the first case, this means that Y is the restriction of a kH-module, and so we get $Y^{(p,p)}\downarrow_A^G$ as kA-Green correspondent. Otherwise, $Y\uparrow_{H_A}^H$ is indecomposable, which implies that the kA-Green correspondent M of Y has the property that $M\uparrow_A^G$ is indecomposable. We refer the reader to § 8 for further details on the structure of M.

Note that, if $p = 3$, then H_A/E is cyclic of order 4, and so $TT(A) \cong TT(H_A) \cong \mathbb{Z}/4\mathbb{Z}$, since we assume that k is a splitting field for H_A. In particular, this shows that $Y^{(p,p)}\downarrow_A^G \cong M \otimes M$ in $\mathrm{stmod}(kA)$.

Before leaving this case, we record the following fact for later application.

Proposition 7.3. *For the group $H = \Sigma_p \wr \Sigma_2$, we have that*

$$TT(H) = \langle [\varepsilon_H], [Y] \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

where ε_H is the sign representation on H and Y is the one-dimensional module on which $\Sigma_p \times \Sigma_p$ acts trivially and Σ_2 acts by the sign representation.

Proof. It is clear that ε_H and Y are endotrivial modules and, moreover, they generate the group of one-dimensional kH-modules. The only question is whether or not there are other endotrivial modules generating $TT(H)$. Therefore, suppose that U is an indecomposable endotrivial kH-module whose restriction to P is isomorphic to $k \oplus (\text{proj})$. By Lemma 6.1 and Proposition 6.2, the kS-module $U\downarrow_S^H$ is an indecomposable endotrivial module and has dimension 1. Hence, the class $[U]$ is in the subgroup of $T(G)$ generated by ε_H and $[Y]$. □

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8. Case $n = 2p + b$, for $0 < b < p$

In this section, we consider the cases $G = \Sigma_{2p+b}$ and $A = A_{2p+b}$, for an integer b with $0 < b < p$. Throughout this section we use the notation for G and for A given at the beginning of § 6.

Again, $TF(G) = \langle[\Omega(k)]\rangle$ and $TF(A) = \langle[\Omega(k)]\rangle$ are infinite cyclic, because the Sylow p-subgroup P is elementary abelian of order p^2. By Theorem 2.3, since $T(P)$ is torsion-free, the groups $TT(G)$ and $TT(A)$ are generated by the classes of indecomposable modules whose restriction to $N = N_G(P)$ and N_A, respectively, are isomorphic to the direct sum of a one-dimensional module and a projective module. In the case of the symmetric group, we show that these modules are Young modules, or the sign representation. Then, we will easily deduce a set of generators for $T(G)$. The situation for the alternating groups is more complicated and hence it is handled separately. Before splitting the question into subcases, we point out the following fact, which shows that $TT(G)$ and $TT(A)$ have at most order 4.

Lemma 8.1. *In the above notation, the restriction maps* $\mathrm{Res}^G_{\Sigma_{2p}} : T(G) \longrightarrow T(\Sigma_{2p})$ *and* $\mathrm{Res}^A_{A_{2p}} : T(A) \longrightarrow T(A_{2p})$ *are injective, for all* $1 \leqslant b \leqslant p-1$.

Proof. The proof is immediate, by the transitivity of the restriction map and by Lemma 3.2. □

At this point, we handle separately the groups G and A, and for simplicity, we start with G. Namely, we exhibit a non-trivial endotrivial module whose class is in $TT(G)$, and this will prove that $TT(G)$ is a Klein four group.

For convenience, let us set $G_b = \Sigma_{2p+b}$, H_b for the subgroup of the form $(\Sigma_p \wr \Sigma_2) \times \Sigma_b$, and $S_b \cong \Sigma_p^2 \times \Sigma_b$, for $0 \leqslant b < p$ (cf. § 6).

Let H and Y be as in Proposition 7.3. We identify H with the quotient H_b/Σ_b, and let $\tilde{Y}$ and $\tilde{\varepsilon}_H$ be the inflations of Y and ε_H to H_b, respectively. Since $\tilde{Y}\downarrow^{H_b}_{S_b} = k$, the kG_b-Green correspondent V of $\tilde{Y}$ is isomorphic to a direct summand of the permutation module $M^{(p,p,b)} = k\uparrow^{G_b}_{S_b}$ (cf. Lemma 2.5). Hence, V is a Young module of the form Y^λ for some partition λ with $\lambda \trianglerighteq (p,p,b)$. By [**15**, Theorem 2], the Young module Y^λ associated with a partition λ of n is projective if and only if λ is p-restricted. A partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of a positive integer n is *p-restricted* (or *p-column regular*) if $\lambda_i - \lambda_{i+1} < p$, for all $1 \leqslant i < s$, and if $\lambda_s < p$. This restricts the number of possible endotrivial modules among all the Young modules.

Proposition 8.2. *Let* $G_b = \Sigma_{2p+b}$ *for an integer* b *such that* $0 < b < p$. *Consider the same notation as above and as in Proposition 7.3. Let* μ *be the partition* $(p+b, p)$ *of* n. *Then we have the following.*

(a) *The Young module* Y^μ *is endotrivial.*

(b) *The* kG*-Green correspondent of* $\tilde{\varepsilon}_H$ *is not endotrivial, and so*

$$T(G) = \langle[\Omega(k)], [\varepsilon_G], [Y^\mu]\rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \quad \forall 0 \leqslant b < p.$$

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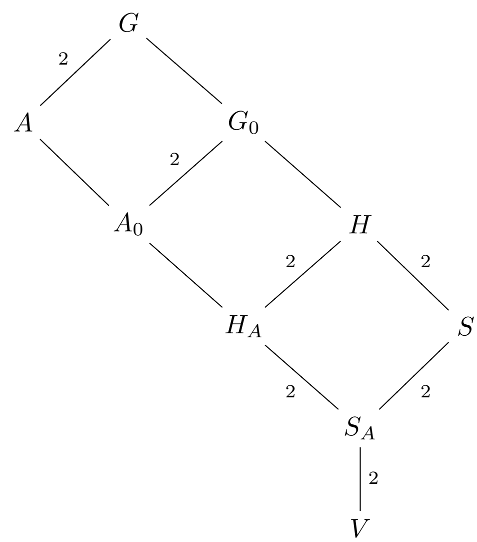

Figure 3. Subgroups of Σ_{2p+1}.

Proof. Let us first observe that (b) follows immediately from (a), as noted above in Lemma 8.1, and by Proposition 7.1. Hence, the only thing we need to show is that Y^{μ} is endotrivial.

We proceed by induction on b. We know that the result holds for $b = 0$, by Proposition 7.1. Assume $b \geqslant 1$. We apply [**18**, Theorem 5.1]. In the notation of Henke's article (where k is an index, and not the underlying field), we have $r = 2p + b$, $k = p$ and $t(k) = 0$, for all $0 < b < p$. Thus, again in Henke's notation, $V_p = 0$, implying that

$$\operatorname{Res}^{G_b}_{G_{b-1}} Y^{\mu} \cong Y^{(p+b-1,p)} \oplus \begin{cases} Y^{(p+b,p-1)} & \text{if } 0 < b < p-1, \\ 0 & \text{if } b = p-1. \end{cases}$$

Now, the partition $(p+b, p-1)$ of $2p+b-1$ is p-restricted for all integers b such that $0 < b < p-1$. In these cases, the kG_{b-1}-module $Y^{(p+b,p-1)}$ is projective (cf. [**15**, Theorem 2]), and $Y^{(p+b-1,p)}$ is endotrivial by the induction hypothesis and by Proposition 7.1. If $b = p-1$, then the kG_{b-1}-module $Y^{(2p-2,p)}$ is not projective, since the partition $(2p-2, p)$ is not p-restricted. However, it is isomorphic to the Specht module $S^{(2p-2,p)}$, by the computation of [**14**, (2.6)]. Then, applying the modular branching rules (cf. [**21**]), the restricted kG_{b-2}-module $\operatorname{Res}^{G_{b-1}}_{G_{b-2}} S^{(2p-2,p)}$ is still indecomposable, and hence, it is isomorphic to the Young module $Y^{(2p-3,p)}$, which is endotrivial by the induction hypothesis. This shows that the Young kG_b-module we started with, namely $Y^{(2p-1,p)}$ is endotrivial. This completes the case for the symmetric groups. □

We now turn to the alternating groups. Let us first consider the group $A = A_{2p+1}$. Write $G = \Sigma_{2p+1}$, $G_0 = \Sigma_{2p}$, $A_0 = A_{2p}$, $H = \Sigma_p \wr \Sigma_2$, $H_A = H \cap A$, $S = \Sigma_p \times \Sigma_p$, $S_A = S \cap A$ and $V = A_p \times A_p$. So we have the diagram in Figure 3, where a number on

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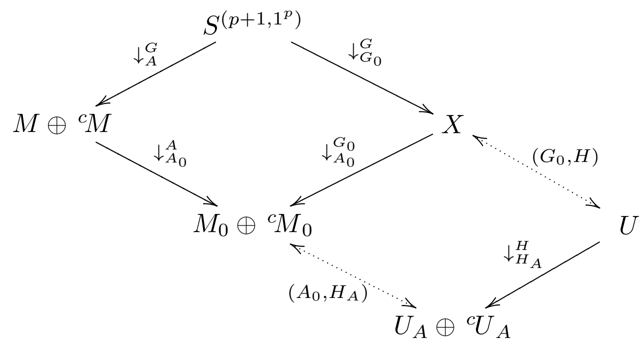

Figure 4. Restrictions of modules for Σ_{2p+1}.

an edge indicates the index of the subgroup. Since the quotient H/V is a dihedral group of order 8, there is a simple kH-module U of dimension 2, whose restrictions to H_A and S split as a direct sum of two non-isomorphic modules. That is, a direct summand U_A of $U\downarrow^H_{H_A}$ is a one-dimensional module whose kA-Green correspondent might be endotrivial. We prove that this is indeed the case.

Observe that, by Proposition 7.2, the kA_0-Green correspondent of U_A is endotrivial and, hence, the kG_0-Green correspondent of U restricts to A_0 as a direct sum of two non-isomorphic endotrivial modules. On the other hand, a direct summand U_S of $U\downarrow^H_S$ is isomorphic to the outer tensor product $k_{\Sigma_p} \otimes \varepsilon_{\Sigma_p}$ as a $k\Sigma_p \otimes k\Sigma_p$-module (recall that $S = \Sigma_p \times \Sigma_p$). Consequently, the induced module $U_S\uparrow^G_S$ has a direct summand which is the signed Young module $Y^{(p+1|p)}$ in the notation of [**17**]. Roughly, the signed Young $k\Sigma_n$-module $Y^{(a|b)}$ is defined as a certain uniquely determined indecomposable direct summand of the signed permutation module $M^{(a|b)} = (1_{\Sigma_a} \otimes \varepsilon_{\Sigma_b})\uparrow^{\Sigma_n}_{\Sigma_a\times\Sigma_b}$, for a 2-part partition (a,b) of n. Now, by [**17**, Proposition 3.6.1], we have that $Y^{(p+1|p)} \cong S^{(p+1,1^p)}$, for the (ordinary) Specht module. By property of the Green correspondence (or by [**20**, 7.6]), we have that $S^{(p+1,1^p)}\downarrow^G_A \cong M \oplus {}^cM$, for two non-isomorphic kA-modules M and ${}^cM$, and where we may assume that M is the kA-Green correspondent of U_A. Upon restriction to G_0 and by the modular branching rules (cf. [**21**, Lemma 1.5]), we have that $S^{(p+1,1^p)}\downarrow^G_{G_0}$ is an indecomposable module isomorphic to a nonsplit extension X given by the exact sequence

$$0 \longrightarrow S^{(p,1^p)} \longrightarrow X \longrightarrow S^{(p+1,1^{p-1})} \longrightarrow 0$$

and X is isomorphic to the kG_0-Green correspondent of U. Also, X is isomorphic to the signed Young module $Y^{(p|p)}$, and $X\downarrow^{G_0}_{A_0} \cong M_0 \oplus {}^cM_0$, with M_0 and ${}^cM_0$ endotrivial and non-isomorphic. Relating to the above picture of inclusion of subgroups of G, we can summarize the situation into the diagram in Figure 4 (where the double-headed dotted arrows denote the Green correspondence between the indicated modules and groups).

In particular, since $S^{(p+1,1^p)}\downarrow^G_{G_0}$ is indecomposable, the restriction $M\downarrow^A_{A_0}$ is also indecomposable, and we may assume (without loss of generality) that $M\downarrow^A_{A_0} \cong M_0$. The fact that M_0 is endotrivial implies that M is also endotrivial (since M extends M_0 to A). This proves the following.

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Proposition 8.3. *Let $A = A_{2p+1}$ and $G = \Sigma_{2p+1}$. Then,*

$$T(A) = \langle [\Omega(k)], [Y^{(p+1,p)}\downarrow^G_A], [M]\rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2,$$

where M is an indecomposable direct summand of the restriction to A of the Specht kG-module $S^{(p+1,1^p)}$. Moreover, $M\downarrow^A_{A_{2p}}$ is indecomposable.

Let us now consider the group $A = A_{2p+2}$. Then A contains a group G' isomorphic to Σ_{2p}, and A is contained in $G = \Sigma_{2p+2}$. We show the following.

Lemma 8.4. *The restriction map $\mathrm{Res}^A_{G'} : T(A) \longrightarrow T(G')$ is injective and the cokernel has order 2, generated by the class of the sign representation $\varepsilon_{G'}$ of G'.*

Proof. Let M be an indecomposable endotrivial kA-module such that $M\downarrow^A_{G'} \cong k \oplus (\mathrm{proj})$. By relative projectivity and a vertex argument, we deduce that M is a direct summand of $k\uparrow^A_{G'}$. Thus, M is a direct summand of

$$k\uparrow^G_{G'}\downarrow^G_A \cong M^{(2p,1,1)}\downarrow^G_A, \quad \text{since } G' \cong \Sigma_{2p} \subseteq G.$$

Now, $M^{(2p,1,1)} \cong \oplus_\mu Y^\mu$ for some partitions μ of $2p+2$ such that $\mu \trianglerighteq (2p,1,1)$. Thus, M is a direct summand of Y^μ for some such μ. Since M has vertex a Sylow p-subgroup and is a trivial source module, it must be that Y^μ has complexity 2. By [**16**, Theorem 3.3.2], this leaves us with only two candidates, namely $Y^{(2p+2)} = k$, whose Green correspondent is the trivial module, and $Y^{(2p+1,1)} \cong S^{(2p+1,1)}$ (the Specht module corresponding to the natural permutation representation of G). In the second case, it is known that the restriction of $S^{(2p+1,1)}$ to A is indecomposable (cf. [**20**, 7.6]) and has dimension $2p+1$ (cf. [**19**]). Therefore, the latter has no endotrivial summand. This proves the injectivity of the restriction map $\mathrm{Res}^A_{G'}$.

The statement for the cokernel can be proved by using a argument similar to the one used to verify the injectivity of the restriction. Indeed, let M be an indecomposable endotrivial kA-module such that $M\downarrow^A_{G'} \cong \varepsilon_{G'} \oplus (\mathrm{proj})$. We then note that M does not extend to G. Otherwise, we would have that $M = N\downarrow^G_A$ for N an indecomposable endotrivial kG-module, which would be one of the modules ε_G, $Y^{(p+2,p)}$ or $\varepsilon_G \otimes Y^{(p+2,p)}$. But

$$\varepsilon_G\downarrow^G_{G'} \cong \varepsilon_G\downarrow^G_A\downarrow^A_{G'} = k \quad \text{and} \quad Y^{(p+2,p)}\downarrow^G_{G'} \cong Y^{(p,p)} \oplus (\mathrm{proj}),$$

by transitivity of the restriction. Thus, $M\uparrow^G_A\downarrow^G_A \cong M \oplus {}^cM$, with $M \ncong {}^cM$. Thus, ${}^cM\downarrow^A_{G'} \cong k \oplus (\mathrm{proj})$ and so, by the above argument, we conclude that if such an M exists, then M has dimension 1. Hence, M does not exist, as was to be shown. □

Now Lemmas 3.2 and 8.4 imply the following.

Proposition 8.5. *Let $A = A_{2p+b}$ and $G = \Sigma_{2p+b}$, for an integer b with $1 < b < p$. Then $T(A)$ is the image of the restriction map $\mathrm{Res}^G_A : T(G) \longrightarrow T(A)$, that is*

$$T(A) = \langle [\Omega(k)], [Y^{(p+b,p)}\downarrow^G_A]\rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

9. Case $n = ap + b$, for $3 \leqslant a < p$ and $0 \leqslant b < p$

In this section we determine $T(G)$ and $T(A)$ for all groups $G = \Sigma_{ap+b}$ and $A = A_{ap+b}$, such that $3 \leqslant a < p$ and $0 \leqslant b < p$. As in most of the work, the two can be handled similarly and, in order to avoid repetitions, we state the results for both groups and we only make explicit the reasoning for the symmetric groups, leaving the translation to the alternating group as an exercise for the reader.

Again we have that $T(P) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}$, since P is an elementary abelian group of p-rank a. We begin with the case when $b = 0$.

Proposition 9.1. *Let $G = \Sigma_{ap}$ and $A = A_{ap}$, for an integer a such that $3 \leqslant a < p$. Then*

$$T(G) = \langle [\Omega(k)], [\varepsilon_G] \rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \quad \textit{and} \quad T(A) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}.$$

Proof. By Proposition 7.3, we have that $TT(H) = \langle [\varepsilon_H], [Y] \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$, where $Y = k_{\Sigma_p} \wr \varepsilon_{\Sigma_a}$. We also know from Theorem 2.3 that the map $\mathrm{Res}^G_H : TT(G) \longrightarrow TT(H)$ is injective. Note that the kG-Green correspondent of ε_H is ε_G, which is endotrivial. Let M be the kG-Green correspondent of Y. The only question is whether or not M is endotrivial. Note also that $Y \downarrow^H_S \cong k$ implies M is a Young module that is a non-trivial direct summand with vertex P of the permutation module $M^\lambda = k \uparrow^G_S$, where $\lambda = (p, \dots, p)$.

Let $U = \Sigma_p \times \Sigma_{(a-1)p}$ be the subgroup of G containing S, with Σ_p acting on $\{1, \dots, p\}$ and $\Sigma_{(a-1)p}$ on $\{p+1, \dots, ap\}$. We set $K = U \cap H \cong \Sigma_p \times (\Sigma_p \wr \Sigma_{(a-1)})$ and $L = K \cap N$. Hence, $L = N_U(P)$ and $S \subseteq K$.

Assume that M is endotrivial. Then, $M \downarrow^G_U = M_U \oplus (\mathrm{proj})$ for an indecomposable endotrivial kU-module M_U, and $M \downarrow^G_K \cong Y_K \oplus (\mathrm{proj})$, where $Y_K = Y \downarrow^H_K$. By transitivity of the restriction, we deduce then that $M_U \downarrow^U_K \cong Y_K \oplus (\mathrm{proj})$. More precisely, Y_K and M_U are Green correspondents, since $K \supseteq L = N_U(P)$. In particular, M_U is a direct summand of $Y_K \uparrow^U_K$, and, by the Mackey formula,

$$Y_K \uparrow^U_K \downarrow^U_K \cong Y_K \oplus \bigoplus_{g \in [K \backslash U / K] \backslash \{1\}} {}^g Y_K \downarrow^{{}^g K}_{{}^g K \cap K} \uparrow^K_{{}^g K \cap K}.$$

By choice of the subgroups U and K, we have that p divides $|{}^g K \cap K|$, for any $g \in U$, and so the right-hand side has no projective summand. Therefore, $M_U \downarrow^U_K \cong Y_K$. In particular, M_U has dimension 1.

In addition, since p does not divide $|G : U|$, the kG-module M is a non-projective module that is relatively projective to U. Hence, $M \downarrow^G_U$ is a direct summand of

$$M_U \uparrow^G_U \downarrow^G_U \cong M_U \oplus \bigoplus_{g \in [U \backslash G / U] \backslash \{1\}} {}^g M_U \downarrow^{{}^g U}_{{}^g U \cap U} \uparrow^U_{{}^g U \cap U}.$$

Note that p divides $|{}^g U \cap U|$, for all $g \in G$, since the intersection ${}^g U \cap U$ contains a subgroup $\Sigma_{(a-2)p}$ for all $g \in G$. Therefore, the right-hand side has no projective summand. Hence, the assumption $M \downarrow^G_U = M_U \oplus (\mathrm{proj})$ implies that $M \downarrow^G_U = M_U$, and so M has

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dimension 1. Since the only one-dimensional kG-modules are isomorphic to k or ε_G, we have proved the claim for G.

Now, for A, we observe that the same arguments as for G work. This implies that $TT(A)$ is trivial, since there is no non-trivial one-dimensional kA-module. □

We now apply Lemma 8.1 in order to determine the endotrivial modules in the remaining cases.

Corollary 9.2. *Let $G = \Sigma_{ap+b}$ and $A = A_{ap+b}$ for $3 \leqslant a < p$ and $0 \leqslant b < p$. Then*

$$T(G) = \langle [\Omega(k)], [\varepsilon_G] \rangle \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \quad \textit{and} \quad T(A) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}.$$

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