SMOOTH DERIVATIONS ON
ABELIAN C*-DYNAMICAL SYSTEMS

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Abstract

Let \((A, \mathbb{R}, \sigma)\) be an abelian C*-dynamical system. Denote the generator of \(\sigma\) by \(\delta_0\) and define \(A_\infty = \bigcap_{n \geq 1} D(\delta_0^n)\). Further define the Lipschitz algebra

\[ A_{1/2} = \left\{ f \in A; \sup_{|t| > 0} \| (\sigma_t f - f) / t \| < +\infty \right\}. \]

If \(\delta\) is a \(*\)-derivation from \(A_\infty\) into \(A_{1/2}\), then it follows that \(\delta\) is closable, and its closure generates a strongly continuous one-parameter group of \(*\)-automorphisms of \(A\). Related results for local dissipations are also discussed.


1. Introduction

Let \(A = C_0(X)\) be an abelian C*-algebra, let \(t \in \mathbb{R} \rightarrow \sigma_t\) be a strongly continuous one-parameter group of \(*\)-automorphisms of \(A\) with generator \(\delta_0\), and set \(A_n = D(\delta_0^n), A_\infty = \bigcap_{n \geq 1} D(\delta_0^n)\). Our aim is to continue the investigation [1], [2], [3] of the structure of \(*\)-derivations from \(A_\infty\) into \(A\). To describe the elements of this investigation, it is necessary to introduce a number of additional concepts.

First let \(S\) denote the group of homeomorphisms of \(X\) associated with \(\sigma\) by

\[ (\sigma_t f)(\omega) = f(S_t \omega), \]

where \(f \in C_0(X), t \in \mathbb{R}\), and \(\omega \in X\). Next define the fixed point set \(X_0\) by

\[ X_0 = \{ \omega \in X; S_t \omega = \omega \text{ for all } t \in \mathbb{R} \}. \]
Further, introduce the period $p$ of each $\omega \in X$ by

$$p(\omega) = \inf\{ t > 0; S_t \omega = \omega \},$$

and the frequency $\nu$ by $\nu(\omega) = 1/p(\omega)$. Thus $X_0$ corresponds to the set of points with period zero.

Now in [1] it was established that $\delta$ is a $*$-derivation from $A_\infty$ into $A$ if, and only if, $\delta = \lambda \delta_0$, where $\lambda$ denotes multiplication by a real function which vanishes on $X_0$, is continuous over $X \setminus X_0$, and is polynomially bounded in the frequency, i.e.

$$|\lambda(\omega)| \leq c \left(1 + \nu(\omega)^k \right)$$

for some $c > 0$ and $k \geq 0$, and for all $\omega \in X \setminus X_0$. (For an earlier partial result see [4], and for an alternative derivation of the polynomial bounds see [2].) Note that the representation $\delta = \lambda \delta_0$ implies that both $\pm \delta$ are dissipative, and in particular that $\delta$ is closable. Let $\overline{\delta}$ denote its closure.

It was also established in [1] that $\delta$ maps $A_\infty$ into $A_n$ if, and only if, $\lambda \in D(\delta_0^n)$ and $\delta_0^n \lambda$ is polynomially bounded in the frequency for all $0 \leq m \leq n$. Note that here $\delta_0$ is defined as the derivative at the origin of $\sigma$ extended to $C(X)$ by the definition $(\sigma f)(\omega) = f(S_\omega \omega)$.

Finally, in [3] it was proved that if $\sigma$ maps $A_\infty$ into $A_1$, then $\overline{\delta}$ is automatically the generator of a strongly continuous one-parameter group of $*$-automorphisms. The principal aim of this paper is to generalize and optimize this last statement. A key feature of this generalization is the Lipschitz algebra

$$A_{1/2} = \left\{ f \in A; \sup_{|t| > 0} \| (\sigma f - f)/t \| < +\infty \right\}.$$ 

It is easily established that $A_{1/2}$ is a $*$-algebra and that $A_1 \subseteq A_{1/2}$. In particular, $A_{1/2}$ is norm dense. Our first result, in Section 2, shows that the $*$-derivation $\delta$ maps $A_\infty$ into $A_{1/2}$ if, and only if, $\delta = \lambda \delta_0$, where $\lambda$ and $(\sigma \lambda - \lambda)/t$ are polynomially bounded in the frequency, with the latter bound uniform in $t$. In Section 3 we prove that these conditions are sufficient to ensure that $\overline{\delta}$ is a generator. The proof of this result is based upon a version of the Trotter-Kato theorem on semigroup convergence which is given in an appendix. In Section 4 related results for dissipations are discussed.

2. Smooth derivations

In this section we derive a characterization of derivations from $A_\infty$ into $A_{1/2}$. But first note that since $\|(\sigma f - f)/t\| \leq 2\|f\|/|t|$, one can equally well define $A_{1/2}$ by

$$A_{1/2} = \left\{ f \in A; \sup_{0 < |t| \leq 1} \| (\sigma f - f)/t \| < +\infty \right\}.$$
In fact since
\[(\sigma_t - 1)f = \sum_{m=0}^{n-1} \sigma_{mt/n}(\sigma_{t/n} - 1)f,\]
one has
\[\|(\sigma_t - 1)f/t\| \leq \|(\sigma_{t/n} - 1)f/(t/n)\|,\]
and hence
\[A_{1/2} = \left\{ f \in A; \limsup_{t \to 0} \|(\sigma_t f - f)/t\| < +\infty \right\}.\]

**Theorem 2.1.** Let \(\sigma\) be a strongly continuous one-parameter group of \(*\)-automorphisms of an abelian \(C^*\)-algebra \(A = C_0(X)\) with generator \(\delta_0\) and associated flow \(S\) on \(X\). Let \(X_0 \subseteq X\) denote the fixed points of \(S\) and \(\nu(\omega)\) the frequency of the point \(\omega \in X\) under the group \(S\). Define \(A_\infty = \bigcap_{n \geq 1} D(\delta_0^n)\) and
\[\chi = \left\{ f \in A; \sup_{0 < |r| \leq 1} \|(\sigma_r f - f)/r\| < +\infty \right\}.\]

If \(\delta\) is a \(*\)-derivation from \(A_\infty\) into \(A\), then the following conditions are equivalent:

1. \(\delta(A_\infty) \subseteq A_{1/2}\).
2. \(\delta = \lambda \delta_0|_{A_{\infty}}\), where \(\lambda\) vanishes on \(X_0\), \(\lambda\) is continuous on \(X \setminus X_0\), and there exist positive constants \(c_1, c_2\) and non-negative integers \(k_1, k_2\) such that

\[|\lambda(\omega)| \leq c_1(1 + \nu(\omega)^{k_1}),\]
\[|\lambda(S_t \omega) - \lambda(\omega)| \leq c_2(1 + \nu(\omega)^{k_2})|t|\]

for all \(\omega \in X \setminus X_0\) and \(t \in \mathbb{R}\).

**Proof.** The proof is an elaboration of arguments given in [1].

2 \implies 1. Observation 6 in Section 3 of [1] establishes that if \(f \in A_\infty\) and \(k\) is a positive integer, then the function
\[\omega \in X \setminus X_0 \rightarrow \nu(\omega)^k(\delta_0 f)(\omega)\]
vanishes at infinity on \(X \setminus X_0\). Consequently, \(\delta f = \lambda \delta_0 f\) is continuous on \(X\) and vanishes at infinity on \(X \setminus X_0\). Thus \(\delta(A_\infty) \subseteq A\). But
\[t^{-1}(\sigma_t - 1)\delta f = (\sigma_t \lambda) t^{-1}(\sigma_t - 1)\delta_0 f + (t^{-1}(\sigma_t - 1)\lambda) \delta_0 f\]
\[= (\sigma_t \lambda) t^{-1} \int_0^t ds \frac{d}{ds} \sigma_t \delta_0 f + (t^{-1}(\sigma_t - 1)\lambda) \delta_0 f,\]
and hence
\[
|t^{-1}(\sigma f - \delta f)(\omega)| \leq c_1|t|^{-1} \int_0^{|t|} ds \left| \sigma_1 \left((1 + \nu)^k \delta^2 f\right)(\omega) \right|
\]
\[+ c_2 \left|(1 + \nu)^k \delta f f(\omega)\right|.
\]
Consequently the above observation implies that \( \delta \) maps \( A_{\infty} \) into \( A_{1/2} \).

By the above observation implies that \( \delta \) maps \( A_{\infty} \) into \( A_{1/2} \).

1 \( \Rightarrow \) 2. Since \( \delta(A_{\infty}) \subset A \), it follows from [1], Theorem 1.2 that Condition 2 is verified with the possible exception of the uniform polynomial bound on \(|(\lambda(S, \omega) - \lambda(\omega))/t| \). But this bound follows from the hypothesis \( \delta(A_{\infty}) \subset A_{1/2} \). We will prove this in two stages. First we prove that \( \omega \rightarrow (\lambda(S, \omega) - \lambda(\omega))/t \) is uniformly bounded on sets of bounded frequencies.

Suppose \( \nu(\omega) \leq N/2 \). Then the map \( t \in (-1/N, 1/N) \mapsto S_t \omega \) is injective. Now let \( F_N \in C^\infty_c(-1/N, 1/N) \) be an infinitely differentiable function with compact support in \((-1/N, 1/N)\) such that \( F_N = 1 \) on \([-1/2N, 1/2N]\) and define \( G_N \) by setting \( G_N(t) = tF_N(t) \). Hence \( G_N(t) = 1 \) for \( t \in [-1/2N, 1/2N] \). But it then follows that there exists a \( g_N \in A_\infty \), with compact support, such that \( g_N(S_t \omega) = G_N(t) \) for \( t \in (-1/N, 1/N) \) (see, for example, the argument used in the proof of Observation 5.2 of [2]). Consequently,
\[
(\sigma, \delta g_N - \delta g_N)(\omega)/t = (\lambda(S, \omega)G_N'(t) - \lambda(\omega)G_N(0))/t
\]
\[= (\lambda(S, \omega) - \lambda(\omega))/t.
\]
for \( t \in [-1/2N, 1/2N] \). Combining this with the estimate given at the beginning of the section, one concludes that
\[
\sup_{0 < |t| \leq 1} \left| (\lambda(S, \omega) - \lambda(\omega))/t \right| \leq \sup_{0 < |t| \leq 1/2N} \left| (\lambda(S, \omega) - \lambda(\omega))/t \right|
\]
\[\leq \sup_{0 < |t| \leq 1} \left| (\sigma, \delta g_N - \delta g_N)/t \right|,\]
i.e. one has boundedness on sets of bounded frequency.

Now consider polynomial boundedness on sets of large frequency. We establish this property by adapting the argument used to prove Observation 5 in Section 3 of [1]. We argue by contradiction.

Assume there exist sequences \( \omega_i \in X \setminus X_0 \) and \( 0 < |t_i| \leq 1 \) such that \((\lambda(S, \omega_i) - \lambda(\omega_i))/t_i \) is not polynomially bounded in the frequency. One may assume that \( \nu(\omega_i) \geq 1/2 \) because of the boundedness property proved above. Proceeding as in Section 3 of [1], one constructs functions \( f_i \in A_\infty \) with compact support \( O_i \) such that
\[
\|f_i\| \leq 2(2\pi \nu(\omega_i))^{j}, \quad j = 1, 2, \ldots, i,
\]
\( S_{[-1,1]} \omega_i \subseteq O_i \),
\( f_i(S_t \omega) = \exp\{2\pi i \nu(\omega_i) t\} \), \( t \in [-1, 1] \).
and the $O_i$ are mutually disjoint. Then if $\rho_i$ is any sequence in $C$ which is rapidly decreasing in the sense that
\[
\lim_{i \to \infty} \nu(\omega_i)^j \rho_i = 0
\]
for $j = 1, 2, \ldots$, it follows that
\[
f = \sum_{i \geq 1} \rho_i f_i
\]
converges with respect to the $C_\nu$-seminorms to an $f \in A_\infty$, and $f = \rho_i f_i$ on $O_i$. Hence $(\delta f)(\omega) = \rho_i (\delta f_i)(\omega)$ for all $\omega \in O_i$. Consequently,
\[
(\sigma_i \delta f - \delta f)(\omega_i)/t_i = \rho_i (\sigma_i \delta f_i - \delta f_i)(\omega_i)/t_i
\]
\[
= \rho_i \left( \lambda(S_i \omega_i) e^{2\pi i \nu(\omega_i)/t_i} - \lambda(\omega_i) \right) \nu(\omega_i) 2\pi i / t_i.
\]
Since $\| (\sigma_i \delta f - \delta f) / t \|$ is bounded uniformly in $t$, it follows that the coefficient of $\rho_i$ must be bounded by a polynomial in the frequencies $\nu(\omega_i)$. But
\[
\left( \lambda(S_i \omega_i) e^{2\pi i \nu(\omega_i)/t_i} - \lambda(\omega_i) \right) / t_i = \left( \lambda(S_i \omega_i) - \lambda(\omega_i) \right) / t_i + \lambda(S_i \omega_i) (e^{2\pi i \nu(\omega_i)/t_i} - 1) / t_i,
\]
and the second term on the right hand side is bounded by a polynomial in the frequencies $\nu(\omega_i)$. Hence the first term is also polynomially bounded, which is inconsistent with the initial hypothesis. This completes the proof that
\[
| (\lambda(S_i \omega) - \lambda(\omega)) / t | \leq c_2 \left( 1 + \nu(\omega)^k \right)
\]
for all $\omega \in X \setminus X_0$ and $t \in \mathbb{R}$, and completes the proof of the theorem.

3. Generators

In this section we establish that the condition $\delta(A_\infty) \subseteq A_{1/2}$ is sufficient to ensure that $\delta$ is a generator. The proof uses the Lipschitz criterion derived in Theorem 2.1 and semigroup convergence techniques.

The major part of the proof consists of a generator result for derivations $\delta = \lambda \delta_0$, where $\lambda$ is a real continuous function over $X \setminus X_0$ which satisfies slightly more general bounds than those derived in Theorem 2.1. Note that the Lipschitz bounds automatically imply that $\lambda$ is continuous along orbits, and the continuity across orbits plays practically no part in our proof. It is only used to ensure that $\delta$ is densely defined. (The domain $D(\delta)$ of $\delta = \lambda \delta_0$ is, by definition, those $f \in D(\delta_0)$ such that $\lambda \delta_0(f) \in A$.)

Although the following result could be partly deduced from Theorems 2.6 and 2.10 of [3], we give an almost independent proof based on resolvent convergence arguments. But to apply this technique it is convenient to use a density result from [3] which essentially allows one to avoid high frequencies.
First, for each $N \geq 0$, introduce the closed set $X^{(N)} = \{ \omega \in X; \nu(\omega) \geq N \}$. Second define $A^{(N)} \subseteq A$ by

$$A^{(N)} = \{ f \in A; f(S_t \omega) = f(\omega) \text{ for all } t \in \mathbb{R}, \omega \in X^{(N)} \}.$$ 

It follows immediately that each $A^{(N)}$ is a $C^*$-subalgebra of $A$, and if $N \leq M$, then $A^{(N)} \subseteq A^{(M)}$. But one deduces from Lemma 2.7 of [3] that

$$A = \bigcup_{N \geq 0} A^{(N)},$$

where the bar denotes norm closure. Finally each $A^{(N)}$ is $\sigma$-invariant, and hence one can deduce information about the system $(A, R, \sigma)$ by examining the subsystems $(A^{(N)}, R, \sigma)$.

**Theorem 3.1.** Let $\sigma$ be a strongly continuous one-parameter group of $*$-automorphisms of an abelian $C^*$-algebra $A = C_0(X)$ with generator $\delta_0$ and associated flow $S$ on $X$. Let $X_0 \subseteq X$ denote the fixed points of $S$ and $\nu(\omega)$ the frequency of the point $\omega \in X$ under the group $S$. Define $A_\infty = \bigcap_{n \geq 1} D(\delta_0^n)$.

If $\lambda$ is a real continuous function over $X \setminus X_0$ satisfying

$$C_{1/2}: \begin{cases} |\lambda(\omega)| \leq K_1(\nu(\omega)), \\ |\lambda(S_t \omega) - \lambda(\omega)| \leq |t|K_2(\nu(\omega)), \end{cases} \quad t \in \mathbb{R},$$

where $K_1: \mathbb{R}^+ \to \mathbb{R}^+$ are positive non-decreasing functions, then the closure $\overline{\delta}$ of the derivation $\delta = \lambda \delta_0$ is the generator of a strongly continuous one-parameter group of $*$-automorphisms of $A$. Moreover $D(\delta) \cap A_\infty$ is a core of $\overline{\delta}$.

**Remark.** This result differs from similar statements in [3] in two respects. First the boundedness assumption on $\lambda$ is much stronger than that of Theorem 2.6 of [3] which established the existence of a unique generator extension of $\lambda \delta_0$. Second the differentiability assumption, $\lambda \in D(\delta_0)$, which was necessary in Theorem 2.12 of [3] in order to identify the unique generator extension of $\lambda \delta_0$ with its closure, is not necessary in the present context. The advantage of the present result is that it suffices for the discussion of derivations from $A_\infty \to A_{1/2}$, it is considerably easier to prove than the analogous results of [3], and it has potential extensions to non-abelian systems.

**Proof.** Fix $N \geq 0$ and consider the dynamical system $(A^{(N)}, R, \sigma)$. Here $A^{(N)}$ is the $C^*$-subalgebra of $A$ introduced before the proposition, and we identify $\sigma$ with its restriction to $A^{(N)}$. We also identify $\delta_0$ and its restriction but explicitly indicate its domain $A^{(N)} \cap D(\delta_0)$.

Now if $f \in A^{(N)} \cap D(\delta_0)$, then $(\delta_0 f)(\omega) = 0$ whenever $\nu(\omega) \geq N$. Hence $A^{(N)} \cap D(\delta_0) \subseteq A^{(N)} \cap D(\delta)$, and one has

$$|((\lambda \delta_0 f)(\omega)| \leq K_1(N)|((\delta_0 f)(\omega)|,$$

$$|((\sigma \lambda - \lambda) \delta_0 f)(\omega)| \leq |t|K_2(N)|((\delta_0 f)(\omega)|.$$
Thus on the range $R(\delta_0)$ of $\delta_0$, restricted to $A^{(N)}$, the function $\lambda$ is uniformly bounded and satisfies a uniform Lipschitz condition. Alternatively stated, one has
\[
\|\lambda\|_N \leq K_1(N),
\]
\[
\|\sigma_\lambda - \lambda\|_N \leq K_2(N)|\tau|,
\]
where $\|\cdot\|_N$ denotes the usual operator norm calculated on the range $R(\delta_0)$ of $\delta_0$ restricted to $A^{(N)}$. This reduces the proof of the propositions on $A^{(N)}$ to the case that $K_1$ and $K_2$ are uniformly bounded. We will handle this by regularizing $\lambda$ and then using a convergence argument.

First, for $\alpha > 0$, define the regularization $\lambda_\alpha$ by
\[
\lambda_\alpha = \frac{1}{\alpha^2} \int_0^\alpha ds \int_0^s dt \sigma_{s+\lambda}.\]

Then $\lambda_\alpha \in D(\delta_0^2)$, and
\[
\delta_0 \lambda_\alpha = \frac{1}{\alpha^2} \int_0^\alpha ds \int_0^s dt \sigma_{s+\alpha} \lambda - \sigma_s \lambda.
\]

Therefore,
\[
\|\lambda_\alpha\|_N \leq K_1(N), \quad \|\delta_0 \lambda_\alpha\|_N \leq K_2(N),
\]
and
\[
\|\lambda_\alpha - \lambda\|_N \leq \frac{1}{\alpha^2} \int_0^\alpha ds \int_0^s dt \|\sigma_{s+\lambda} - \lambda\|_N \leq \alpha K_2(N).
\]

In particular, $\lambda_\alpha \rightarrow \lambda$ uniformly on the range of $\delta_0$, as $\alpha \rightarrow 0$.

Second, for $\beta > 0$, define $H_{\alpha,\beta}$ on $A^{(N)} \cap D(\delta_0^2)$ by
\[
H_{\alpha,\beta} = \lambda_\alpha \delta_0 - \beta \delta_0^2.
\]

It follows easily that $H_{\alpha,\beta}$ is the generator of a $C_0$-semigroup $\tau^{\alpha,\beta}$ of contractions on $A^{(N)}$. To establish this, note that $-\delta_0^2$ is the generator of a contraction semigroup, the Gaussian semigroup associated with $\sigma$. Moreover,
\[
\|\delta_0 f\| \leq b\|\delta_0^2 f\| + \|f\|/b
\]
for all $f \in D(\delta_0^2)$ and $b > 0$, by application of Taylor’s theorem and the triangle inequality to the function $t \mapsto \sigma_t f$. Hence
\[
\|\lambda_\alpha \delta_0 f\| \leq b\beta\|\delta_0^2 f\| + \left( K_1(N)^2/\beta b \right) \|f\|
\]
for all $f \in D(\delta_0^2)$ and $b > 0$. But $H_{\alpha,\beta}$ is dissipative (see, for example, Lemma 4.1 of [3]), and hence $H_{\alpha,\beta}$ generates a $C_0$-semigroup of contractions $\tau^{\alpha,\beta}$ by perturbation theory.

It follows from general semigroup theory that $\|(I + \epsilon H_{\alpha,\beta})^{-1}\| \leq 1$ for all $\epsilon > 0$, and we use this to prove the strong convergence of $(I + \epsilon H_{\alpha,\beta})^{-1}$ in the limit $\beta \rightarrow 0$, then $\alpha \rightarrow 0$. For this proof, note that
\[
\left\{(I + \epsilon H_{\alpha,\beta_1})^{-1} - (I + \epsilon H_{\alpha,\beta_2})^{-1}\right\} f
\]
\[
= \epsilon(\beta_1 - \beta_2)(I + \epsilon H_{\alpha,\beta_2})^{-1}\delta_0^2(I + \epsilon H_{\alpha,\beta_1})^{-1} f
\]
and
\[
\left\{(I + \varepsilon H_{\alpha_1, \beta})^{-1} - (I + \varepsilon H_{\alpha_2, \beta})^{-1}\right\}f = \varepsilon \left(I + \varepsilon H_{\alpha_2, \beta}\right)^{-1}(\lambda_{\alpha_1} - \lambda_{\alpha_2}) \delta_0 \left(I + \varepsilon H_{\alpha_1, \beta}\right)^{-1}f
\]
for all \(f \in \mathbb{A}(N)\). Hence
\[
\left\|\left\{(I + \varepsilon H_{\alpha_1, \beta})^{-1} - (I + \varepsilon H_{\alpha_2, \beta})^{-1}\right\}f\right\| \leq \varepsilon |\beta_1 - \beta_2| \left\|\delta_0^2 \left(I + \varepsilon H_{\alpha_1, \beta}\right)^{-1}f\right\|
\]
and
\[
\left\|\left\{(I + \varepsilon H_{\alpha_1, \beta})^{-1} - (I + \varepsilon H_{\alpha_2, \beta})^{-1}\right\}f\right\| \leq \varepsilon \|\lambda_{\alpha_1} - \lambda_{\alpha_2}\|_N \left\|\delta_0 \left(I + \varepsilon H_{\alpha_1, \beta}\right)^{-1}f\right\|
\]
Now since all the resolvents \((I + \varepsilon H_{\alpha, \beta})^{-1}\) are contractions, it suffices to prove the strong convergence on a dense subspace of \(\mathbb{A}(N)\) such as \(\mathbb{A}(N) \cap D(\delta_0^2)\). But it follows from the first of these estimates that one has convergence as \(\beta \to 0\) if
\[
\left\|\delta_0^2 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\| \text{ is bounded uniformly in } \beta \text{ for } f \in \mathbb{A}(N) \cap D(\delta_0^2).
\]
Then it follows from the second estimate that one has convergence as \(\beta \to 0\) and then \(\alpha \to 0\) if, in addition, \(\|\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\|\) is bounded uniformly in \(\alpha\) and \(\beta\) for \(f \in \mathbb{A}(N) \cap D(\delta_0^2)\). Hence we next examine these boundedness properties.

First, if \(f \in \mathbb{A}(N) \cap D(\delta_0)\), then
\[
\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f = \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}\{\delta_0 f + \left[I + \varepsilon H_{\alpha, \beta}, \delta_0\right] \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\}
\]
\[
= \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}\{\delta_0 f - \varepsilon (\delta_0 \lambda_{\alpha}) \delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\}.
\]
Hence, setting \(K_2 = K_2(N)\), we obtain
\[
\left\|\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\| \leq \|\delta_0 f\| + \varepsilon K_2 \left\|\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\|,
\]
and for \(\varepsilon K_2 < 1\), one has the bound
\[
\left\|\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\| \leq \|\delta_0 f\| (1 - \varepsilon K_2)^{-1},
\]
which is uniform in \(\alpha\) and \(\beta\). Similarly, if \(f \in \mathbb{A}(N) \cap D(\delta_0^2)\), one finds that
\[
\left\|\delta_0^2 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\| \leq \|\delta_0^2 f\| + 2\varepsilon \|\delta_0 \lambda_{\alpha}\|_N \left\|\delta_0^2 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\|
\]
\[
+ \varepsilon \|\delta_0^2 \lambda_{\alpha}\|_N \left\|\delta_0 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\|.
\]
Thus, if \(2\varepsilon K_2 < 1\), one has the bound
\[
\left\|\delta_0^2 \left(I + \varepsilon H_{\alpha, \beta}\right)^{-1}f\right\| \leq \left[\|\delta_0^2 f\| + \varepsilon \|\delta_0^2 \lambda_{\alpha}\|_N \|\delta_0 f\| (1 - \varepsilon K_2)^{-1}\right] (1 - 2\varepsilon K_2)^{-1},
\]
which is uniform in \(\beta\).
Therefore, if \( 0 < \varepsilon < (2K_2)^{-1} \), we have established the existence of the strong limit

\[
R_\varepsilon = \lim_{\alpha \to 0} \lim_{\beta \to 0} (I + \varepsilon H_{\alpha, \beta})^{-1}.
\]

Next consider the convergence of \((I + \varepsilon H_{\alpha, \beta})^{-1}\) as \(\varepsilon \to 0\). Given \(f \in A^{(N)}\) and \(\kappa > 0\), one can choose \(g \in A^{(N)} \cap D(\delta_0^2)\) such that \(\|f - g\| < \kappa\|f\|/2\). Hence, setting \(K_1 = K_1(N)\), we have

\[
\left\| \left( \left( I + \varepsilon H_{\alpha, \beta} \right)^{-1} - I \right) f \right\| \leq \kappa\|f\| + \varepsilon\|H_{\alpha, \beta}g\|
\]

\[
\leq \kappa\|f\| + \varepsilon\|\lambda_\alpha\|_\infty\|\delta_0g\| + \varepsilon\beta\|\delta_0^2g\|
\]

\[
\leq \kappa\|f\| + \varepsilon K_1\|\delta_0g\| + \varepsilon\beta\|\delta_0^2g\|.
\]

Consequently,

\[
\lim_{\varepsilon \to 0} \sup_{0 < \alpha, \beta < 1} \left\| \left( \left( I + \varepsilon H_{\alpha, \beta} \right)^{-1} - I \right) f \right\| \leq \kappa\|f\|.
\]

Since \(\kappa > 0\) was arbitrary, this proves that \((I + \varepsilon H_{\alpha, \beta})^{-1}\) converges strongly to the identity as \(\varepsilon \to 0\), uniformly in \(\alpha\) and \(\beta\).

Therefore it follows from the version of the Trotter-Kato convergence theorem given in the appendix that there exists a \(C_0\)-contraction semigroup \(T\) on \(A^{(N)}\) with generator \(H\) such that \(\tau_{t}^{\alpha, \beta} \to \tau_t\) in the limit \(\beta \to 0\), then \(\alpha \to 0\), uniformly for \(t\) in finite intervals of \(\mathbb{R}^+\), and \((I + \varepsilon H_{\alpha, \beta})^{-1} \to (I + \varepsilon H)^{-1}\) as \(\beta \to 0\) then \(\alpha \to 0\), uniformly for \(\varepsilon > 0\). We will argue that \(H = \lambda \delta_0^{N} A^{(N)} \cap D(\delta_0^2)\).

Let \(f \in D(H)\) and set \(g = (I + \varepsilon H)f\). Now choose \(g_\gamma \in A^{(N)}\) such that \(\|g_\gamma - g\| \to 0\) as \(\gamma \to 0\). Next set \(f_{\alpha, \beta, \gamma} = (I + \varepsilon H_{\alpha, \beta})^{-1}g_\gamma\). Since \(D(H_{\alpha, \beta}) = D(\delta_0^2)\), one has \(f_{\alpha, \beta, \gamma} \in A^{(N)} \cap D(\delta_0^2)\). But

\[
\lim_{\gamma \to 0} \lim_{\alpha \to 0} \lim_{\beta \to 0} f_{\alpha, \beta, \gamma} = f.
\]

Moreover,

\[
(I + \varepsilon \lambda \delta_0)f_{\alpha, \beta, \gamma} - g_\gamma = \{ e(\lambda - \lambda_\alpha)\delta_0 - \varepsilon \beta \delta_0^2 \} f_{\alpha, \beta, \gamma}
\]

and

\[
\left\| \left\{ \left( \lambda - \lambda_\alpha \right)\delta_0 - \beta \delta_0^2 \right\} f_{\alpha, \beta, \gamma} \right\| \leq \left\| \lambda - \lambda_\alpha \right\|_\infty \left\| \delta_0 \left( I + \varepsilon H_{\alpha, \beta} \right)^{-1} g_\gamma \right\|
\]

\[
+ \beta \left\| \delta_0^2 \left( I + \varepsilon H_{\alpha, \beta} \right)^{-1} g_\gamma \right\|.
\]

Therefore, using the estimates derived above, one finds that

\[
\lim_{\gamma \to 0} \lim_{\alpha \to 0} \lim_{\beta \to 0} (I + \varepsilon \lambda \delta_0)f_{\alpha, \beta, \gamma} = g = (I + \varepsilon H)f.
\]
This establishes that the closure \( \tilde{\delta} \) of \( \lambda \delta_0 \mid_{A^{(N)} \cap D(\delta_0^2)} \) is an extension of \( H \). But since \( H \) is a generator, it has no proper dissipative extensions, and hence \( H = \tilde{\delta} \).

The foregoing argument applies equally well to \( -\lambda \), and consequently both \( \pm \delta \) are generators of \( C_0 \)-contraction semigroups. Hence, by a simple standard argument, \( \tilde{\delta} \) in fact generates a \( C_0 \)-group of isometries \( \tau \). Finally, as \( \lambda \) is real, \( \delta \) is a \( * \)-derivation, and \( \tau \) must be a group of \( * \)-automorphisms of \( A^{(N)} \). Next we extend \( \tau \) from \( A^{(N)} \) to \( A \).

For each \( N \geq 0 \), we have constructed a group of \( * \)-automorphisms, which we now denote by \( \tau^{(N)} \), of \( A^{(N)} \) and the generator of \( \tau^{(N)} \) is \( H^{(N)} = \lambda \delta_0 \mid_{A^{(N)} \cap D(\delta_0^2)} \).

But if \( N \leq M \), then \( A^{(N)} \subseteq A^{(M)} \), and \( \tau^{(N)} \subseteq \tau^{(M)} \) because \( H^{(N)} \subseteq H^{(M)} \). Thus, defining \( \tau \) on \( \bigcup_{N \geq 0} A^{(N)} \) by setting \( \tau = \tau^{(N)} \) on \( A^{(N)} \), one can then extend \( \tau \) to a \( C_0 \)-group of \( * \)-automorphisms of \( A \) by continuity, because

\[
A = \bigcup_{N \geq 0} A^{(N)}. \]

The generator \( H \) of \( \tau \) is by construction a closed extension of \( \lambda \delta_0 \) restricted to \( D = \bigcup_{N \geq 0} A^{(N)} \cap D(\delta_0^2) \), but in fact \( H = \lambda \delta_0 \mid_D \). To prove this, let \( f \in D(H) \) and set \( g = (1 + \varepsilon H) f \).

By density there exists a sequence \( g_\varepsilon \in A^{(N)} \) such that \( \| g_\varepsilon - g \| \to 0 \), and, since \( H^{(N)} = H \mid_{A^{(N)}} \) is a generator, there exists a sequence \( f_\varepsilon \in D(H^{(N)}) \) such that \( g_\varepsilon = (1 + \varepsilon H^{(N)}) f_\varepsilon \). Then it follows that \( f_\varepsilon = (1 + \varepsilon H)^{-1} g_\varepsilon \to f \).

But as \( A^{(N)} \cap D(\delta_0^2) \) is a core of \( H^{(N)} \), one concludes that \( \bigcup_{N \geq 0} A^{(N)} \cap D(\delta_0^2) \) is a core of \( H \).

Finally, if \( f \in A^{(N)} \cap D(\delta_0^2) \), and if \( h \in C_0(R) \) is a positive, infinitely differentiable function with support in \([-1, 1]\) and total integral one, then, defining

\[
f_n = n \int dt h(nt) \sigma_t f,
\]

one has \( f_n \in A^{(N)} \cap A_\infty \). But \( f_n \rightarrow f \) and \( \delta_0 f_n \rightarrow \delta_0 f \) by strong continuity of \( \sigma \). Moreover, \( \lambda \) is bounded on the range of \( \delta_0 \) restricted to \( A^{(N)} \), so \( \delta f_n = \lambda \delta_0 f_n \rightarrow \lambda \delta_0 f = \delta f \). Therefore \( \bigcup_{N \geq 0} A^{(N)} \cap A_\infty \) is a core of \( H \).

Combining Theorems 2.1 and 3.1, one obtains the result stated in the abstract.

**Corollary 3.2.** Let \( \sigma \) be a strongly continuous one-parameter group of \( * \)-automorphisms of an abelian \( C^* \)-algebra \( A \) with generator \( \delta_0 \). Define \( A_\infty = \bigcap_{n \geq 1} D(\delta_0^n) \) and

\[
A_{1/2} = \left\{ f \in A; \sup_{0 < |t| < 1} \| (\sigma_t f - f) / t \| < +\infty \right\}.
\]

If \( \delta \) is a \( * \)-derivation from \( A_\infty \) into \( A_{1/2} \), then \( \delta \) is closable, and its closure \( \tilde{\delta} \) is the generator of a strongly continuous one-parameter group of \( * \)-automorphisms of \( A \).
PROOF. It follows from Theorem 2.1 that \( \delta = \lambda \delta_0 \), where \( \lambda \) satisfies the condition \( C_{1/2} \) of Theorem 3.1 with \( K_1 \) and \( K_2 \) polynomials, thereby ensuring that \( A_\infty \subseteq D(\delta) \). The corollary is then a direct consequence of Theorem 3.1.

4. Local dissipations

An operator \( H: A_\infty \to A \) is defined to be local if \( \text{supp}(Hf) \subseteq \text{supp}(f) \) for all \( f \in A_\infty \), and to be a dissipation if

\[
H(\hat{ff}) \leq H(\hat{f})f + \hat{f}H(f)
\]

for all \( f \in A_\infty \). In [1] it was demonstrated that \( H \) is a local dissipation if, and only if, it has the form

\[
Hf = \lambda_0 f + \lambda_1 \delta_0 f - \lambda_2 \delta_0^2 f,
\]

where \( \lambda_0 \) is bounded and continuous on \( X \), where \( \lambda_1, \lambda_2 \) vanish on \( X_0 \) and are polynomially bounded and continuous on \( X \setminus X_0 \), and where \( \delta_0, \delta_1, \delta_2 \geq 0 \). Moreover, if \( H \) maps \( A_\infty \) into \( A_n \), then the \( \lambda_i \in A_n \), \( \delta_0 \lambda_0 \) is bounded, and \( \delta_0 \lambda_1, \delta_0 \lambda_2 \) are polynomially bounded, for \( j \leq n \).

In [3] it was conjectured, in analogy with results for derivations, that if \( H: A_\infty \to A_2 \) is a local dissipation, then its closure generates a \( C_0 \)-semigroup of positive contractions. This conjecture was verified in the special case that \( \lambda_1 \) is bounded by \( \lambda_2^{1/2} \). The general conjecture is, however, false, as we next demonstrate with a specific example. Subsequently we extend the positive results on dissipations obtained in [3] to local dissipations \( H: A_\infty \to A_{3/2} \), where \( A_{3/2} \) is defined by

\[
A_{3/2} = \{ f \in A_1; \delta_0 f \in A_{1/2} \}.
\]

Then we discuss some other possible characterizations of \( A_{3/2} \).

First consider the example \( A = C_0(\mathbb{R}) \) and \( \sigma \) the group of translations. (I am indebted to Charles Batty for help in constructing this example.) Thus \( \delta_0 = d/dx \) and \( A_\infty = C_0^\infty(\mathbb{R}) \). Now let \( C_0^\infty(\mathbb{R}) \) denote the infinitely differentiable functions with compact support and \( H: C_c^\infty(\mathbb{R}) \to C_0(\mathbb{R}) \) the operator defined by

\[
H = -x^2 \frac{d^2}{dx^2} - (1 - x^2) \frac{d}{dx}.
\]

Then \( H \) is a dissipation, but if \( g \) is defined by

\[
g(x) = x^{-2} \exp\{-x^{-1}\}, \quad x > 0,
\]

\[
= 0, \quad x \leq 0,
\]

one readily computes that

\[
\int dx \, g(x) ((1 + H)f)(x) = 0
\]
for all \( f \in C_c^\infty(\mathbb{R}) \). Thus the closure of \( H \) is definitely not the generator of a contraction semigroup on \( C_0(\mathbb{R}) \).

This example is almost a counterexample to the conjecture in [3]. It fails only because the coefficients of \( H \) are unbounded at infinity, and hence \( H \) is not defined on all of \( A_\infty \). One can, however, convert this example into a genuine counterexample on \( C(\mathbb{T}) \) by the change of variable \( x \in \mathbb{R} \mapsto y \in \mathbb{T} \), where \( y = \tan^{-1} x \). One then obtains the more complicated expression

\[
H = \frac{1}{4} \left( -\sin^2 2y \frac{d^2}{dy^2} + 2(\sin 2y(1 - \cos 2y) - 2 \cos 2y) \frac{d}{dy} \right),
\]

which has bounded continuous coefficients and can be defined on all of \( C^\infty(\mathbb{T}) \). But by the above calculation \( R(1 + H) \neq C(\mathbb{T}) \), and hence \( H \) is not a generator.

Next we turn to the examination of local dissipations \( H : A_\infty \to A_{3/2} \), and we begin by remarking that, by an extension of the proof of Theorem 2.1, one can establish that the coefficients \( \lambda_0, \lambda_1, \lambda_2 \in D(\delta_0) \) satisfy the condition

\[
C_{3/2} : \left\{ \begin{array}{l}
|\lambda(\omega)| \leq L_1(\nu(\omega)) \\
|\delta_0 \lambda(\omega)| \leq L_2(\nu(\omega)) \\
|\delta_0 \lambda(\sigma_\omega) - (\delta_0 \lambda)(\omega)| \leq t|L_3(\nu(\omega))|
\end{array} \right.
\]

where the \( L_i \) are polynomials. Hence the basic problem is to show that the closure of \( H = \lambda_1 \delta_0 - \lambda_2 \delta_0^2 \) on \( A_\infty \), where \( \lambda_2 \geq 0 \), and where \( \lambda_1, \lambda_2 \) satisfy \( C_{3/2} \) on \( X \setminus X_0 \), is a generator. (The term \( \lambda_0 \) causes no problems, since it is positive and bounded, and \( H \) is closable because it is automatically dissipative [3].)

Now the closure of \( \lambda_1 \delta_0 \) generates a group of \(*\)-automorphisms by Theorem 3.1, or Theorem 3.1 of [3], and we next argue that the closure of \( -\lambda_2 \delta_0^2 \) generates a positive contraction semigroup. Then if \( \lambda_1 \) is bounded by \( \lambda_2^{1/2} \), the generator result follows for the sum \( H = \lambda_1 \delta_0 - \lambda_2 \delta_0^2 \) by perturbation theory.

**Proposition 4.1.** Let \((A, \mathcal{R}, \sigma)\) be an abelian \(C^*\)-dynamical system. Denote the generator of \( \sigma \) by \( \sigma_0 \) and set \( A_\infty = \cap_{n \geq 1} D(\delta_0^n) \).

If \( \lambda \) is a non-negative continuous function over the spectrum \( X \) of \( A \) which satisfies condition \( C_{3/2} \) above, then the closure \( \overline{H} \) of \( H = -\lambda \delta_0 \) generates a positive \( C^0\)-contraction semigroup. Moreover, \( A_\infty \cap D(H) \) is a core of \( \overline{H} \).

**Remarks.** 1. For this result it suffices that the functions \( L_i \) which occur in Condition \( C_{3/2} \) are positive, and finite-valued. If the \( L_i \) are polynomials, then \( A_\infty \subseteq D(H) \); and hence \( A_\infty \) is a core of \( \overline{H} \).

2. It follows from the first part of the proof that, since \( \lambda \) is non-negative, the bound on \( \delta_0 \lambda \) in Condition \( C_{3/2} \) follows from the other two bounds, and in fact one can assume that \( L_2^2 \leq 2 L_1 L_3 \).
**Proof.** The key to the proof is the observation that

\[ 0 \leq \sigma \lambda = \lambda + t(\delta_0 \lambda) + \int_0^t ds (\sigma - 1)(\delta_0 \lambda). \]

Therefore

\[ 0 \leq \lambda (\omega) + t(\delta_0 \lambda)(\omega) + (t^2/2) L_3(\nu(\omega)) \]

for all \( t \in \mathbb{R} \), and hence

\[ |(\delta_0 \lambda)(\omega)|^2 \leq 2 \lambda (\omega) L_3(\nu(\omega)). \]

But this implies that \( \lambda^{1/2} \in D(\delta_0) \), and so

\[ |(\delta_0 \lambda^{1/2})(\omega)|^2 = |(\delta_0 \lambda)(\omega)|^2/2 \lambda (\omega) \leq L_3(\nu(\omega))/2. \]

Hence by Theorem 3.1 of [3], or Theorem 3.1 in the previous section, the closure \( \bar{\delta} \) of \( \delta = \lambda^{1/2} \delta_0 \) generates a \( C_0 \)-group of \(*\)-automorphisms \( \tau \) of \( A \). Consequently, \( -\bar{\delta}^2 \) generates a positive \( C_0 \)-contraction semigroup \( \rho \), the convolution semigroup associated with \( \tau \), defined by

\[ \rho_t f = (\pi t)^{-1/2} \int_{-\infty}^{\infty} ds e^{-s^2/\tau_s} f. \]

Next let \( A^{(N)} \) denote the \( C^* \)-subalgebra of \( A \) spanned by those \( f \in A \) which satisfy \( f(S_t \omega) = f(\omega) \) for all \( t \in \mathbb{R} \) and all \( \omega \) in the closed set \( X^{(N)} = \{ \omega : \nu(\omega) \geq N \} \). The groups \( \sigma \) and \( \tau \) leave each \( A^{(N)} \) invariant, and hence \( \rho \) also leaves the \( A^{(N)} \) invariant. Next, following [3], we observe that

\[ H = -\lambda \delta_0^2 = -(\lambda^{1/2} \delta_0)^2 + (\delta_0 \lambda^{1/2})(\lambda^{1/2} \delta_0), \]

and if \( f \in A^{(N)} \cap D((\lambda^{1/2} \delta_0)^2) \), then

\[ \| (\delta_0 \lambda^{1/2}) \lambda^{1/2} \delta_0 f \| \leq L_3(N)^{1/2} \| \lambda^{1/2} \delta_0 f \|^{2^{1/2}} \]

\[ \leq b \| (\lambda^{1/2} \delta_0)^2 f \| + (L_3(N)/2) \| f \|/b \]

for all \( b > 0 \). Now \( H \) is dissipative, and hence \( H \) generates a \( C_0 \)-semigroup \( \kappa \) of contractions by perturbation theory. Since \( H \) is a dissipation, \( \kappa \) is also positive by [3], Proposition 4.3. Now \( \kappa \) is defined on each \( A^{(N)} \) consistently, i.e. if \( N \leq M \), then the restriction of \( \kappa \) from \( A^{(M)} \) to \( A^{(N)} \) agrees with the direct definition of \( \kappa \) on \( A^{(N)} \). Finally, \( A = \bigcup_{N \geq 1} A^{(N)} \), and hence \( \kappa \) extends to a positive \( C_0 \)-contraction semigroup on \( A \) by continuity.

The core property follows by the same arguments used to derive the analogous property in Theorem 3.1.

We conclude with some comments on the definition of \( A^{3/2} \). There are various alternatives to the choice that we have used. But the following propositions show that the obvious ones coincide, even for non-abelian \( A \).
PROPOSITION 4.2. Let \((A, \mathbb{R}, \sigma)\) be a C*-dynamical system and let \(\delta\) denote the generator of \(\sigma\). Then, for each \(A \in A\), the following conditions are equivalent:

1. \[\sup_{|t| > 0} \|(\sigma_t - 1)^2 A / t^2\| < +\infty,\]
2. \[\sup_{|t| > 0} \sup_{|s| > 0} \|(\sigma_t - 1) (\sigma_s - 1) A / ts\| < +\infty,\]
3. \(A \in D(\delta)\) and \[\sup_{|t| > 0} \|(\sigma_t - 1) \delta(A) / t\| < +\infty.\]

Moreover, if these conditions are satisfied, then the three suprema are equal.

PROOF. 3 \(\Rightarrow\) 2. This follows from the triangle inequality once one observes that

\[
\frac{(\sigma_t - 1) A}{s} = -\frac{1}{2} \int_0^s \sigma_r \delta(A) \, dr.
\]

2 \(\Rightarrow\) 1. This is obvious.

1 \(\Rightarrow\) 3. First we prove that \(A \in D(\delta)\). Set

\[a = \sup_{|t| > 0} \|(\sigma_t - 1)^2 A / t^2\|.
\]

Then note that

\[
\left( \frac{\sigma_t - 1}{t} - \frac{\sigma_{t/2} - 1}{t/2} \right) A = \left( \frac{\sigma_{t/2} - 1}{t/2} \right) \left( \frac{\sigma_{t/2} + 1}{2} - 1 \right) A
\]

\[= \frac{(\sigma_{t/2} - 1)^2}{t/2} A / 2.
\]

Replacing \(t\) by \(t_m = t/2^m\) and summing from \(m = 0\) to \(m = n - 1\), one finds that

\[
\left( \frac{\sigma_t - 1}{t} - \frac{\sigma_{t_n} - 1}{t_n} \right) A = \sum_{m=0}^{n-1} \frac{(\sigma_{t_{m+1}} - 1)^2}{t_{m+1}} A/2.
\]

Therefore

\[
(*) \quad \left\| \left( \frac{\sigma_t - 1}{t} - \frac{\sigma_{t_n} - 1}{t_n} \right) A \right\| \leq (a/2) \sum_{m=0}^{\infty} |t_{m+1}|
\]

\[\leq (a/2) |t| \sum_{m=0}^{\infty} \frac{1}{2^{m+1}}
\]

\[= (a/2) |t|.
\]

Consequently, for all \(m, n \geq 0\), one has

\[
\left\| \left( \frac{\sigma_{t_n} - 1}{t_n} - \frac{\sigma_{t_{m+1}} - 1}{t_{m+1}} \right) A \right\| \leq a |t|,
\]

and, replacing \(t\) by \(t_p\), one concludes that

\[
\left\| \left( \frac{\sigma_{t_n} - 1}{t_n} - \frac{\sigma_{t_{m+1}} - 1}{t_{m+1}} \right) A \right\| \leq a |t| / 2^p
\]
for all \( m, n \geq p \). This proves that
\[
\lim_{n \to \infty} \frac{(\sigma_n - 1)}{t_n} A = A_t
\]
exists, but in principle it could depend upon \( t \). Nevertheless,
\[
\frac{(\sigma_n + t - 1)}{s_n + t} A = \frac{t_n}{s_n + t} \frac{(\sigma_n - 1)}{t_n} A + \frac{s_n}{s_n + t} \frac{(\sigma_n - 1)}{s_n} A,
\]
and hence
\[
(s + t) A_{s+t} = sA_s + tA_t,
\]
i.e. the function \( t \in \mathbb{R} \mapsto tA_t \in A \) is additive. But \((*)\) implies that
\[
\|tA_t\| \leq 2\|A\| + (a/2)t^2,
\]
and consequently, by classical reasoning, the function \( t \in \mathbb{R} \mapsto tA_t \in A \) must be linear, i.e. \( A_t = A_1 \) is independent of \( t \). Now referring to \((*)\) once again, one sees that
\[
\lim_{t \to 0} \|(a, - 1)^t A/t - A_1\| = 0,
\]
i.e. \( A \in D(\delta) \) and \( \delta(A) = A_1 \).

Finally, using the foregoing identification and the estimate \((*)\), one has
\[
\|(a, - 1)^2 A/t^2 - (a, - 1)\delta(A)/t\| \leq a.
\]
Hence the supremum in Condition 3 is finite. This establishes that 1 \( \Rightarrow \) 3.

Now consider the last statement of the proposition. Let \( a_1, a_2, a_3 \) denote the values of the suprema occurring in Conditions 1, 2, and 3 respectively. Then by elementary reasoning \( a_1 \leq a_2 \leq a_3 \). But using \((\sigma, - 1) = (\sigma/2 - 1)(\sigma/2 + 1)\), one deduces that
\[
\|(\sigma, - 1)^2 A/t^2\| \leq \|(\sigma, - 1)(\sigma, - 1) A/t_n\| \leq \|(\sigma, - 1)^2 A/t^2\| \leq a_1,
\]
where we have once again used the notation \( t_n = t/2^n \). Therefore, taking the limit as \( n \to \infty \), one finds that
\[
\|(\sigma, - 1)^2 A/t^2\| \leq \|(\sigma, - 1)\delta(A)/t\| \leq a_1.
\]
Finally, taking the supremum over \( t \) gives \( a_1 \leq a_3 \leq a_1 \), and hence \( a_1 = a_2 = a_3 \).

**Appendix-semigroup convergence**

In Section 3 we make several applications of a version of the Trotter-Kato theorem on semigroup convergence. The resolvent formulation of this result is given in [5, Chapter IV], but it can also be stated in terms of the semigroup. The complete result, for contraction semigroups, is summarized in the following proposition.
PROPOSITION. Let $S^a$ be a net of $C_0$-contraction semigroups on a Banach space $B$ and denote the generator of $S^a$ by $H_a$. The following conditions are equivalent.

1. The strong limit of $S^a_t$ exists for all small $t > 0$, and for each $a \in B$, 
   $$\lim_{t \to 0^+} \| (S^a_t - I) a \| = 0$$
   uniformly in $a$.

2. The strong limit of $(I + \varepsilon H_a)^{-1}$ exists for all small $\varepsilon > 0$, and for each $a \in B$, 
   $$\lim_{\varepsilon \to 0^+} \| (I + \varepsilon H_a)^{-1} - I \| a \| = 0$$
   uniformly in $a$.

Moreover, these conditions imply that there exists a $C_0$-contraction semigroup $S$, with generator $H$, such that $S^a_t \to S$ uniformly for $t$ in any finite interval of $\mathbb{R}_+$, and such that $(I + \varepsilon H_a)^{-1} \to (I + \varepsilon H)^{-1}$ uniformly for $\varepsilon > 0$.

PROOF. Assume that Condition 1 holds and let $S_t$ denote the strong limit of $S^a_t$. Since $\| S^a_t \| \leq 1$ for all $t \geq 0$, it readily follows that the strong limit exists for all $t \geq 0$ and that $S_t S_{t'} = S_{t+t'}$ for all $s, t \geq 0$. But $S_0 = I$, and one automatically has $\| S_t \| \leq 1$. Therefore, to conclude that $S$ is a $C_0$-contraction semigroup, it remains to prove continuity at the origin. But given $a \in B$ and $\varepsilon > 0$, one can choose $t_0$ such that 
   $$\|(S^a_t - I) a\| < \varepsilon/2$$
   for all $0 \leq t \leq t_0$, uniformly in $a$. Then for $0 < t \leq t_0$ fixed, one can choose $a_0$ such that 
   $$\|(S^a_t - S_t) a\| < \varepsilon/2$$
   for $a > a_0$. Therefore, by the triangle inequality, 
   $$\|(S_t - I) a\| < \varepsilon,$$
   and this is valid for any $0 \leq t \leq t_0$.

Now since $S^a$ converges to the $C_0$-contraction semigroup $S$, it follows from the usual Trotter-Kato theorem (see, for example, [7], Theorem 3.1.26) that $(I + \varepsilon H_a)^{-1}$ converges strongly to $(I + \varepsilon H)^{-1}$, where $H$ is the generator of $S$, and the convergence is uniform in $\varepsilon$. But 
   $$\left[ (I + \varepsilon H_a)^{-1} - I \right] a = \int_0^\infty dt \varepsilon^{-1} \left( S^a_t - I \right) a,$$
   and hence, for any $M > 0$, one estimates that 
   $$\| \left[ (I + \varepsilon H_a)^{-1} - I \right] a \| \leq 2\varepsilon^{-M} \| a \| + \sup_{|t| < M} \| (S^a_t - I) a \|.$$

It follows immediately that $(I + \varepsilon H_a)^{-1} \to I$ as $\varepsilon \to 0$ uniformly in $a$. 

This establishes that Condition 2, and also the last statement of the proposition, follow from Condition 1.

Next assume that Condition 2 holds. Then the existence of $S$ and $H$ and the identification

$$(I + \epsilon H)^{-1} = \lim_{\alpha} (I + \epsilon H_{\alpha})^{-1}$$

follow from [5, Chapter IX, Theorem 2.17]. The proof can be summarized as follows.

Let $R_{\epsilon}$ denote the strong limit of $(I + \epsilon H_{\alpha})^{-1}$. Since $\|(I + \epsilon H_{\alpha})^{-1}\| \leq 1$, and since the resolvent relation

$$\epsilon_{1}(I + \epsilon_{1}H_{\alpha})^{-1} - \epsilon_{2}(I + \epsilon_{2}H_{\alpha})^{-1} = (\epsilon_{1} - \epsilon_{2})(I + \epsilon_{1}H_{\alpha})^{-1}(I + \epsilon_{2}H_{\alpha})^{-1}$$

is valid, it follows that

$$\|R_{\epsilon}\| \leq 1$$

and that

$$\epsilon_{1}R_{\epsilon_{1}} - \epsilon_{2}R_{\epsilon_{2}} = (\epsilon_{1} - \epsilon_{2})R_{\epsilon_{1}}R_{\epsilon_{2}}.$$

But given $a \in \mathcal{B}$ and $\kappa > 0$, one can choose $\epsilon_{0} \geq 0$ such that

$$\|(I + \epsilon H_{\alpha})^{-1} - I)a\| < \kappa/2$$

for $0 < \epsilon \leq \epsilon_{0}$ uniformly in $\alpha$. Then, for $0 < \epsilon \leq \epsilon_{0}$ fixed, one can choose $\alpha_{0}$ such that

$$\|(I + \epsilon H_{\alpha})^{-1} - R_{\epsilon})a\| < \kappa/2$$

for $\alpha > \alpha_{0}$. Hence, by the triangle inequality,

$$\|(R_{\epsilon} - I)a\| < \kappa,$$

and this is valid for any $0 \leq \epsilon \leq \epsilon_{0}$. Consequently,

$$\lim_{\epsilon \to 0} R_{\epsilon} = I.$$

It immediately follows that $R_{\epsilon} = (I + \epsilon H)^{-1}$, where $H$ is the generator of a $C_{0}$-contraction semigroup $S$. Then $S^{\alpha} \to S$ by the usual Trotter-Kato convergence theorem, and this implies the last statement of the proposition.

It remains to prove that $S_{\epsilon}^{\alpha} \to I$ as $t \to 0$ uniformly in $\alpha$. Now given $a \in \mathcal{B}$, set $a_{\alpha} = (I + \kappa H_{\alpha})a^{-2}$. Then $a_{\alpha} \in D(H_{\alpha}^{2})$,

$$a_{\alpha} - a = ((I + \kappa H_{\alpha})^{-1} + I)((I + \kappa H_{\alpha})^{-1} - I)a,$$

and

$$H_{\alpha}^{2}a_{\alpha} = \frac{1}{\kappa^{2}}((I + \kappa H_{\alpha})^{-1} - I)^{2}a.$$
In particular, \( \|a_{\alpha} - a\| \leq 2\|((I + \kappa H_a)^{-1} - I) a\| \), and \( \|H_a^2 a_\alpha\| \leq 4\|a\|/\kappa^2 \). But by a standard estimate (see, for example, [5, Chapter IX, Section 1.2]), we have
\[
\left\| \left( S_t^\alpha - \left( I + \frac{t}{n} H_a \right)^{-n} \right) a_\alpha \right\| \leq \frac{t^2}{2n} \| H_a^2 a_\alpha \| \leq 2t^2 \|a\|/n\kappa^2.
\]
Therefore
\[
\| (S_t^\alpha - I) a \| \leq \left\| \left( I + \frac{t}{n} H_a \right)^{-n} - I \right\| a_\alpha \| + \| S_t^\alpha - \left( I + \frac{t}{n} H_a \right)^{-n} \right\| a_\alpha \| + 2\|a - a_\alpha\|
\leq n\left\| \left( I + \frac{t}{n} H_a \right)^{-1} - I \right\| a \| + 2t^2 \|a\|/n\kappa^2
+ 4\| (I + \kappa H_a)^{-1} - I \right\| a \|.
\]
Hence, by first choosing \( \kappa \) and then \( t \), one deduces that \( S_t^\alpha \to I \) uniformly in \( \alpha \) as \( t \to 0 \). This completes the proof.

References


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