

CERTAIN RESULTS OF REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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(Received 4 August 2010; revised 17 December 2010; accepted 11 April 2011;
first published online 2 August 2011)

Abstract. First, we classify a real hypersurface of a non-flat complex space form with (i) semi-parallel $T(= \mathcal{L}_\xi g)$, and (ii) recurrent T . Next, we characterise a real hypersurface admitting the generalised η -Ricci soliton in a non-flat complex space form.

2010 *Mathematics Subject Classification.* 53B20, 53C15, 53C25.

1. Introduction. A complex n -dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form and is denoted by $\overline{M}^n(c)$. A complete and simply connected complex space form is a complex Euclidean space C^n , if $c = 0$, a complex projective space $P_n(C)$, if $c > 0$ or a complex hyperbolic space $H_n(C)$, if $c < 0$. Takagi [17, 18] first characterised all homogeneous real hypersurfaces in $P_n(C)$ into six model spaces A_1, A_2, B, C, D and E . Thereafter, Cecil and Ryan [2] (see also [9]) studied extensively that when the structure vector field ξ is principal and showed that they are realised as the tubes over certain submanifolds in $P_n(C)$ by using its focal map. On the other hand, Berndt [1] classified all homogeneous real hypersurfaces in $H_n(C)$ with ξ as principal vector and divided into four model space A_0, A_1, A_2 and B . Let M be a real hypersurface of a non-flat complex space form. Then M has an almost contact metric structure (φ, ξ, η, g) induced from the complex structure J . Many differential geometers studied real hypersurfaces of a complex space form under various conditions on the Ricci tensor, the shape operator A (in the direction of the unit normal of M), curvature tensor etc. For a real hypersurface of a complex space form, we now define the tensor T by

$$g(TX, Y) = (\mathcal{L}_\xi g)(X, Y) = g((\varphi A - A\varphi)X, Y), \quad (1)$$

for all vector fields X, Y tangent to M . A typical characterisation for a real hypersurface M of type A in a complex space form $\overline{M}^n(c)$ was given under the condition $g(TX, Y) = 0$, for any tangent vector fields X and Y on M . Under this condition Okumura [15], for $c > 0$, and Montiel-Romero [13], for $c < 0$ proved the following:

THEOREM A. *Let M^{2n-1} be a real hypersurface in a non-flat complex space form. If M satisfies $A\varphi = \varphi A$, then M is locally congruent to real hypersurface of type A .*

Let M be a real hypersurface of type A in $\overline{M}^n(c)$. Then it follows from Theorem A that M naturally satisfies $\nabla_X T = 0$. Thus, as a generalisation of Okumura's condition $g(TX, Y) = 0$, for any tangent vector fields X and Y on M , here we consider the real

hypersurfaces M of a non-flat complex space form $\overline{M}^n(c)$ with semi-parallel tensor T (i.e. $R.T = 0$, where R is the curvature tensor of M) and prove that such hypersurface is the Hopf hypersurface and also locally congruent to one of type A in $P_n(C)$ or $H_n(C)$. We also consider a real hypersurface of a non-flat complex space form with recurrent T and prove that such hypersurface is locally congruent to one of type A in $P_n(C)$ or $H_n(C)$. We discuss these issues in Section 3.

It is well known [5] that there are no real hypersurfaces with parallel Ricci tensor in a non-flat complex space form $\overline{M}^n(c)$ when $n \geq 3$. This is also true for $n = 2$ as was pointed out by Kim [7]. Since the Einstein manifold has parallel Ricci tensor, it is easy to observe that there do not exist the Einstein real hypersurfaces in a non-flat complex space form. For this Kon [10], studied and classified the *pseudo-Einstein* (that is there exist constants λ, μ such that the Ricci tensor S satisfies $S = \lambda I + \mu \eta \otimes \eta$) real hypersurfaces of a complex space form $\overline{M}^n(c)$ when $n \geq 3$. Recently, Kim–Ryan [8] proved that every pseudo-Einstein hypersurface in $P_2(C)$ or $H_2(C)$ is the Hopf hypersurface. Now we recall some classification theorems of the pseudo-Einstein type real hypersurfaces in $P_n(C)$ (see [2, 10]) or $H_n(C)$ (see [12]).

THEOREM B. *Let M^{2n-1} ($n \geq 3$) be a real hypersurface of $P_n(C)$ with Fubini-study metric of constant holomorphic sectional curvature 4. Then M is pseudo-Einstein if and only if M is locally congruent to one of the following:*

(A₁) *A geodesic hypersphere of radius r , where $0 < r < \frac{\pi}{2}$.*

(A₂) *A tube of radius r over a totally geodesic $P_k(C)$ ($1 \leq k \leq n - 2$), where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{k}{n-k-1}$.*

(B) *A tube of radius r over a complex quadric Q^{n-1} and $P_n\mathbb{R}$, where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n - 2$.*

THEOREM C. *Let M^{2n-1} ($n \geq 3$) be a real hypersurface of $H_n(C)$ with Bergman metric of constant holomorphic sectional curvature -4 . Then M is pseudo-Einstein if and only if M is locally congruent to one of the following:*

(A₀) *A horosphere.*

(A₁) *A geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}(C)$.*

Moreover, we remark that a tube over a totally geodesic $H_l(C)$ ($1 \leq l \leq n - 2$) is known as a A_2 -type hypersurface of $H_n(C)$, $n \geq 3$. Note that real hypersurfaces of types A_1 and A_2 (without extra restriction $\cot^2 r = \frac{k}{n-k-1}$) in $P_n(C)$ and of types A_0, A_1 and A_2 in $H_n(C)$ are simply known as a real hypersurfaces of type A .

A Ricci soliton is a generalisation of Einstein metric and is defined on a Riemannian manifold (M, g) by a vector field V and a constant λ

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (2)$$

where \mathcal{L}_V denotes the Lie-derivative operator along V , S is the Ricci tensor of g and X, Y are arbitrary vector fields on M . It can be viewed as a fixed point of the Hamilton's Ricci flow: $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. For details we refer to Chow–Knopf [4]. Recently, Cho–Kimura [3] considered real hypersurfaces of a non-flat complex space form that admits the Ricci soliton with $V = \xi$ and proved that such hypersurface does not exist. For this reason, Cho–Kimura [3] defined the so-called η -Ricci soliton by taking $V = \xi$ and adding an extra term $\mu \eta \otimes \eta$ in the left-hand side

of (2), i.e.

$$\frac{1}{2} \mathcal{L}_\xi g + S + \lambda g + \mu \eta \otimes \eta = 0,$$

for constants λ and μ . Under this assumption they proved that M is *pseudo-Einstein* (or η -umbilical). Moreover, as a generalisation of η -Ricci soliton, one may consider real hypersurfaces M of a complex space form $\overline{M}^n(c)$ satisfying

$$\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) + S(X, Y) + \lambda g(X, Y) = 0, \tag{3}$$

for all tangent vectors X, Y orthogonal to ξ and λ is constant. We call this a generalised η -Ricci soliton. Note that there exist real hypersurfaces that admit a η -Ricci soliton and hence generalised η -Ricci soliton. In fact, it is straight forward to see that any η -umbilical real hypersurface of a complex space form admits such a structure. Thus, as a generalisation of Cho–Kimura’s result we classify real hypersurfaces M of complex space form $\overline{M}^n(c)$ satisfying equation (3). We discuss this matter in Section 4.

2. Real hypersurfaces in a complex space form. In this section we recall some basic equations and formulas that we shall use later on. For details about the real hypersurfaces of a complex space form we refer to Niebergall–Ryan [14]. Let M be a real hypersurface of a Kaehler manifold $(\overline{M}, J, \overline{g})$. For any vector field X tangent to M , we put

$$JX = \varphi X + \eta(X)\xi, \tag{4}$$

$$JN = -\xi, \tag{5}$$

where φ is a tensor field of type $(1, 1)$, η is a 1-form and ξ is a unit vector field on M . We denote the induced metric of M by g . From equation (4) it is easy to see that (φ, ξ, η, g) gives an almost contact metric structure on M , that is

$$\varphi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{6}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

for all vector fields X, Y on M . From these equations it is easy to see that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. The Gauss and Weingarten formulas for M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y), \overline{\nabla}_X N = -AX,$$

where $\overline{\nabla}$ and ∇ are the Levi-Civita connection of \overline{M} and M , respectively. Making use of these formulas, equations (4) and (5) and $\overline{\nabla}J = 0$ (as \overline{M} is Kaehler) it follows that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y), \tag{8}$$

$$\nabla_X \xi = \varphi AX, \tag{9}$$

where A is the second fundamental tensor of M . Now we suppose that the Kaehler manifold $\overline{M} = \overline{M}(c)$ is a complex space form. Then we have the following Gauss and

Codazzi equations:

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} + g(AY, Z)AX - g(AX, Z)AY, \quad (10)$$

for any tangent vector fields X, Y, Z on M . From equation (10), we get

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \quad (11)$$

where h is the trace of A . If the vector field ξ is a principal curvature vector in a non-flat complex space form, i.e. $A\xi = \alpha\xi$, then M is called the Hopf hypersurface of $\overline{M}(c)$. Such hypersurfaces have some remarkable properties. Note that for $c \neq 0$, α is constant (see [6, 10, 11, 14]).

3. Real hypersurfaces with semi parallel T .

THEOREM 1. *Let M be real hypersurface of a non-flat complex space form. If the tensor T is semi-parallel, then M is locally congruent to a type A hypersurface.*

Proof. By hypothesis, we have

$$R(X, Y)T - TR(X, Y) = 0,$$

from which we get

$$g(R(X, Y)TZ, W) - g(R(X, Y)Z, TW) = 0. \quad (12)$$

Setting $Z = W = \xi$ the foregoing equation yields

$$g(R(X, Y)T\xi, \xi) = 0. \quad (13)$$

Now, from equation (1), $T\xi = \varphi A\xi$ and hence equation (13) reduces to $g(R(X, Y)\varphi A\xi, \xi) = 0$. Thus, in view of this we obtain from equation (10)

$$\begin{aligned} & \frac{c}{4}\{g(Y, \varphi A\xi)\eta(X) - g(X, \varphi A\xi)\eta(Y)\} + g(AY, \varphi A\xi)g(AX, \xi) \\ & - g(AX, \varphi A\xi)g(AY, \xi) = 0. \end{aligned}$$

Next, putting $Y = \varphi A\xi$ and since $g(A\varphi A\xi, \xi) = 0$, the foregoing equation implies that

$$\frac{c}{4}g(\varphi A\xi, \varphi A\xi)\eta(X) + g(A\varphi A\xi, \varphi A\xi)g(AX, \xi) = 0. \quad (14)$$

Finally, taking $X = \varphi A\varphi A\xi$ in equation (14) provides $g(A\varphi A\xi, \varphi A\xi) = 0$. Making use of this in equation (14) and since M is non-flat, we see that ξ is principal, i.e. $A\xi = \alpha\xi$. Utilising this and taking $Y = Z = \xi$ in equation (12), we get $TR(X, \xi)\xi = 0$. Let X be any principal vector orthogonal to ξ corresponding to the principal curvature λ , i.e. $AX = \lambda X$. Then $R(X, \xi)T\xi = 0$ since $T\xi = 0$. Also from the Gauss equation (10) it follows that $R(X, \xi)\xi = (\alpha\lambda + \frac{c}{4})X$. Thus,

$$0 = TR(X, \xi)\xi = \left(\alpha\lambda + \frac{c}{4}\right)(A\varphi - \varphi A)X,$$

so that unless there is a principal curvature satisfying $\alpha\lambda + \frac{\epsilon}{4} = 0$, we are finished by Theorem A. Suppose λ is such a principal curvature so that $AX = \lambda X$ and $(A\varphi - \varphi A)X \neq 0$. The well-known properties of principal curvatures of Hopf hypersurfaces (see [14, pp 245–246]) give a principal curvature μ such that $A\varphi X = \mu\varphi X$. Since $(A\varphi - \varphi A)X = (\mu - \lambda)X$ we have $\mu \neq \lambda$. This is a contradiction as the same argument applied to μ and φX gives $\alpha\mu + \frac{\epsilon}{4} = 0$. This completes the proof. \square

REMARK 1. In [16], Pyo–Suh proved that a real hypersurface M of a non-flat complex space form $\overline{M}^n(c)$, $n \geq 2$, satisfying $\mathcal{L}_\xi R = 0$ is of type A . We can prove this result by applying Theorem 1. In fact, Lie differentiating the identity

$$g(R(X, Y)Z, W) + g(R(X, Y)W, Z) = 0,$$

using $\mathcal{L}_\xi R = 0$ and (1), it follows that $(R(X, Y)T)Z = 0$.

Next we prove the following.

THEOREM 2. *Let M be real hypersurface of a non-flat complex space form with recurrent T . Then M is locally congruent to one of type A in $P_n(C)$ or $H_n(C)$.*

Proof. By hypothesis T is recurrent, i.e. there exists a 1-form π such that

$$(\nabla_X T)Y = \pi(X)TY, \tag{15}$$

for all vector fields Y, Z on M . Clearly T is symmetric. Suppose T has a non-zero eigenvalue σ , for otherwise $T = 0$ and by Theorem A, M will be congruent to one of type A in $P_n(C)$ or $H_n(C)$. Let Y be a unit vector and $TY = \sigma Y$. Then by (15), we have

$$\pi(X)g(TY, Y) = g((\nabla_X T)Y, Y) = g(\nabla_X(TY), Y) - g(\nabla_X Y, TY).$$

Using $TY = \sigma Y$ the foregoing equation shows that

$$(X\sigma)g(Y, Y) + \sigma g(\nabla_X Y, Y) - \sigma g(\nabla_X Y, Y) = \sigma\pi(X)g(Y, Y),$$

which, in turn, gives $X\sigma = \sigma\pi(X)$. Writing this consequence as $d\sigma = \sigma\pi$ and operating this by d (operator of exterior differentiation) and using the Poincaré lemma, $d^2 = 0$, we obtain

$$0 = d^2\sigma = d\sigma \wedge \pi + \sigma d\pi = \sigma(\pi \wedge \pi) + \sigma d\pi,$$

i.e. $\sigma d\pi = 0$. At this point we take an open set N of all points p of M such that $\sigma(p) \neq 0$. Then on N , $d\pi = 0$, i.e.

$$(\nabla_X \pi)Z = (\nabla_Z \pi)X. \tag{16}$$

Now, for any X, Y and $Z \in T_p M$ and $p \in N$, by differentiating (15) covariantly with respect to Z , we obtain

$$(\nabla_Z \nabla_X T)Y = \{(\nabla_Z \pi)X\}TY + \pi(Z)\pi(X)TY.$$

Interchanging Z and X we have

$$(\nabla_X \nabla_Z T)Y = \{(\nabla_X \pi)Z\}TY + \pi(X)\pi(Z)TY.$$

Making use of these equations, together with the Ricci identity and (16) we find that

$$R(X, Z)TY - TR(X, Z)Y = 0.$$

Therefore, following the proof of Theorem 1 it is easy to see that $T = 0$ and so $\sigma = 0$ on N . Thus, we arrive at a contradiction and hence $\varphi A = A\varphi$. Using Theorem A, we complete the proof. \square

4. Generalised η -Ricci soliton.

THEOREM 3. *Let M be real hypersurface of a non-flat complex space form admitting a generalised η -Ricci soliton. If the tensor $g(TX, Y)$ of M vanishes for all X, Y orthogonal to ξ , then M is pseudo-Einstein.*

Proof. In view of equation (1), the hypothesis $g(TX, Y) = 0$, for all X, Y orthogonal to ξ implies $g((A\varphi - \varphi A)X, Y) = 0$, for all X, Y orthogonal to ξ , which is equivalent to

$$\varphi A\varphi^2 X - \varphi^2 A\varphi X = 0,$$

for all X tangent to M . Operating this by φ and replacing X by φX , the foregoing equation provides

$$(A\varphi - \varphi A)X - g(A\varphi X, \xi)\xi + \eta(X)\varphi A\xi = 0. \quad (17)$$

Since M admits a generalised η -Ricci soliton, equation (3) is equivalent to

$$g(\nabla_{\varphi X}\xi, \varphi Y) + g(\nabla_{\varphi Y}\xi, \varphi X) + 2S(\varphi X, \varphi Y) + 2\lambda g(\varphi X, \varphi Y) = 0 \quad (18)$$

for all vectors X, Y tangent to M . Making use of equations (9) and (11), the foregoing equation yields

$$\varphi A\varphi^2 X - \varphi^2 A\varphi X + \varphi A^2\varphi X - h\varphi A\varphi X - \left\{ 2\lambda + \frac{(2n+1)c}{2} \right\} \varphi^2 X = 0$$

for all vectors X tangent to M . Therefore, use of (6) the last equation entails that

$$\begin{aligned} & (A\varphi - \varphi A)X - g(A\varphi X, \xi)\xi + \eta(X)\varphi A\xi + \varphi A^2\varphi X \\ & - h\varphi A\varphi X - \left\{ 2\lambda + \frac{(2n+1)c}{2} \right\} \varphi^2 X = 0. \end{aligned} \quad (19)$$

Feeding equation (17) into (19) provides

$$\varphi A^2\varphi X - h\varphi A\varphi X - \left\{ 2\lambda + \frac{(2n+1)c}{2} \right\} \varphi^2 X = 0. \quad (20)$$

Operating equation (20) by φ we get an equation and replacing X by φX in equation (20) gives another equation. Differentiating them yields

$$\begin{aligned} & (\varphi A^2 - A^2\varphi)X + g(A^2\varphi X, \xi)\xi - \eta(X)\varphi A^2\xi \\ & + h\{(A\varphi - \varphi A)X - g(A\varphi X, \xi)\xi + \eta(X)\varphi A\xi\} = 0. \end{aligned}$$

Thus, in view of equation (17), the preceding equation shows that

$$(\varphi A^2 - A^2\varphi)X + g(A^2\varphi X, \xi)\xi - \eta(X)\varphi A^2\xi = 0. \tag{21}$$

In other words

$$g((\varphi A^2 - A^2\varphi)X, Y) = 0, \tag{22}$$

for all tangent vectors X, Y orthogonal to ξ . Now, operating equation (17) by A gives

$$(A^2\varphi - A\varphi A)X - g(A\varphi X, \xi)A\xi + \eta(X)A\varphi A\xi = 0. \tag{23}$$

Further, replacing X by AX , equation (17) transforms into

$$(A\varphi A - \varphi A^2)X - g(A\varphi AX, \xi)\xi + g(AX, \xi)\varphi A\xi = 0. \tag{24}$$

Adding equation (23) with (24) and taking into account equation (22) it follows that

$$g(AX, \xi)g(\varphi A\xi, Y) + g(\varphi A\xi, X)g(AY, \xi) = 0, \tag{25}$$

for all tangent vectors X, Y orthogonal to ξ . Since $\varphi\xi = 0$, the vector fields $\varphi^2A\xi$ and $\varphi A\xi$ are orthogonal to ξ . Therefore, if we replace X by $\varphi^2A\xi$ and Y by $\varphi A\xi$, then equation (25) shows $|g(\varphi A\xi, \varphi A\xi)|^4 = 0$, which implies $\varphi A\xi = 0$, that is $A\xi = \alpha\xi$. This, together with the hypothesis $(g(A\varphi - \varphi A)X, Y) = 0$, for all X, Y orthogonal to ξ implies that $A\varphi = \varphi A$. Moreover, using $A\xi = \alpha\xi$ in equation (11), we see that $S\xi = \beta\xi$, where $\beta = \frac{c(n-1)}{2} + h\alpha - \alpha^2$. Making use of equation (9), $(g(A\varphi - \varphi A)X, Y) = 0$ in equation (18), we find that

$$S(\varphi X, \varphi Y) + \lambda g(\varphi X, \varphi Y) = 0.$$

Finally, replacing X by φX and Y by φY in the foregoing equation and since $S\xi = \beta\xi$ we see that M is pseudo-Einstein. □

ACKNOWLEDGEMENTS. The author is very much grateful to the referee for reading the manuscript carefully and offering many valuable comments on it. In particular, the referee kindly suggested a shorter proof of Theorem 1 than its earlier version.

The author has been supported by the UGC (India) under the scheme Minor Research Project in Science, sanction no. F.PSW-106/09-10, dated 8 October 2009.

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