FAITHFUL REPRESENTATIONS OF LIE GROUPS II

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The present paper is a continuation of Part I. In the introduction of Part I we have explained our main problems and sketched their results. In the present part we shall proceed to give complete proofs of them.

Notations and definitions in Part I shall be retained.

Semi-simple f.r. Lie groups

5. Let $\Sigma$ be a field and let $a$ and $b$ be matrices of degree $n$ and $m$ respectively with coefficients of $\Sigma$. We denote by $a \times b$ the Kronecker product of $a$ and $b$, and by $a^{(k)}$ the $k$-times Kronecker product of $a$:

- $a^{(0)} = 1_1$ (the unit matrix of degree 1),
- $a^{(k)} = \underbrace{a \times a \times \ldots \times a}_{k}$ for $k > 0$.

Let now $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the sequence of all eigen-values of $a$, repeated as many times as their multiplicities. Then $\varepsilon_1^{(i_1)} \varepsilon_2^{(i_2)} \ldots \varepsilon_n^{(i_n)}$ is an eigen-value of $a^{(k)}$ if $i(1), i(2), \ldots$ are non-negative integers such that $\sum_{j=1}^{n} i(j) = k$. Suppose now that $a^{[k]}$ is the unit matrix (of degree $n$). Since every eigen-value of $a^{[k]}$ is 1, we have $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n = \varepsilon$, where $\varepsilon$ is a $k$-th root of unity. Let $e_1$ be an eigen-vector of $a$: $ae_1 = \varepsilon e_1$, and let $e_2$ be a vector, linearly independent of $e_1$. We operate $a^{[k]}$ to $e_1 \times e_1 \times \ldots \times e_1$ and get $ae_2 = \varepsilon e_2$ easily. Hence if $a^{[k]}$ is the unit matrix, then $a = \varepsilon 1_n$ where $\varepsilon^k = 1$. Using this fact we shall prove the following algebraic

**Lemma 8.** Let $\Sigma$ be an algebraically closed field, and let $G$ be a group composed of non-singular matrices with coefficients of $\Sigma$. Let now $D$ be a finite central invariant subgroup of $G$. Then we can construct a faithful representation of $G/D$, which is an induced representation of $G$ on a certain tensor space, where we assume that the order of $D$ is indivisible by the characteristic of $\Sigma$.

**Proof.** It is clearly sufficient to prove the lemma in case when the order $q$ of $D$ is a prime number. Let $d$ ($\neq 1$) be an element of $D$. Since $D$ is comp-

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1) Gotô [22].
Let $q$ does not divide the characteristic of $\Sigma$, we may assume that $d$ is of the form

$$
d = \begin{pmatrix}
\varepsilon^{i(1)}1_{j(1)} & 0 \\
\varepsilon^{i(2)}1_{j(2)} & \ddots \\
0 & \ddots & \ddots \\
0 & \ddots & \varepsilon^{i(k)}1_{j(k)}
\end{pmatrix},
$$

where $\varepsilon$ is a $q$-th root of unity and $i(s) \equiv i(t) \mod q$ if $s \neq t$. Since $gd = dg$ for any element $g$ of $\mathfrak{G}$, every eigen-space of $d$ is allowable by $\mathfrak{G}$:

$$
\mathfrak{G} \ni g = \begin{pmatrix}
g_1 & 0 \\
g_2 & \ddots \\
0 & \ddots & \ddots \\
0 & \ddots & g_h
\end{pmatrix}.
$$

Clearly, we can determine non-negative integers $a_{ls}$ ($l = 1, 2, \ldots, h - 1; s = 1, 2, \ldots, h$) such that a solution $(x_1, \ldots, x_h)$ of the congruence equations

$$
a_{1s}x_1 + a_{2s}x_2 + \ldots + a_{hs}x_h \equiv 0 \mod q \quad l = 1, 2, \ldots, h - 1
$$

satisfies the relations

$$
x_s \equiv mi(s) \mod q \quad s = 1, 2, \ldots, h
$$

for a suitably chosen integer $m$.

Now an induced representation of $\mathfrak{G}$ is given by

$$
g \to \begin{pmatrix} g^* & 0 \\ 0 & g^{**} \end{pmatrix},
$$

where

$$
g^* = \begin{pmatrix} g_1^{[q]} & 0 \\ \ddots & \ddots \\ 0 & g_h^{[q]} \end{pmatrix},
$$

and

$$
g^{**} = \begin{pmatrix} g_1^{[a_1]} \times \ldots \times g_h^{[a_h]} & 0 \\ \ddots & \ddots \\ 0 & g_1^{[a_{h-1}]} \times \ldots \times g_h^{[a_{h-1}]} \end{pmatrix}.
$$

Let $\mathfrak{Z}$ be the kernel of the representation. Since $z^{[d]}$ is the unit matrix for $z \in \mathfrak{Z}$, we have
Now since $z^{**}$ is also the unit matrix we get that

$$a_l f(1) + a_l f(2) + \ldots + a_l f(h) \equiv 0 \mod. q, \quad l = 1, 2, \ldots, h - 1$$

so that from the definition of $a_l$ there exists an integer $m$ such that

$$f(s) \equiv m i(s) \mod. q \quad s = 1, 2, \ldots, h.$$ 

Hence we have $z \in \mathbb{D}$, i.e. $\mathbb{D} \subseteq \mathbb{D}$. That $\mathbb{D} \subseteq \mathbb{D}$ is obvious, q.e.d.

**Lemma 9.** Let $\mathfrak{G}$ be a semi-simple f.r.\(^5\) Lie group and let $\mathfrak{G}'$ be a Lie group homomorphic with $\mathfrak{G}$. Then $\mathfrak{G}'$ is also f.r.

**Proof.** We may suppose that $\mathfrak{G}/\mathfrak{R} \cong \mathfrak{G}'$, where $\mathfrak{R}$ is a closed invariant subgroup of $\mathfrak{G}$. Let $G$ be the Lie algebra of $\mathfrak{G}$, and let $\mathfrak{R}_0$ be the connected component of $\mathfrak{R}$ containing $e$. As the Lie algebra $N$ of $\mathfrak{R}$ is an ideal of semi-simple $G$, there exists an ideal $M$ of $G$ such that

$$G = M + N, \quad M \cap N = 0.$$ 

We denote by $M$ the subgroup of $\mathfrak{G}$ generated by $M$. Then since $M$ is semi-simple as an ideal of a semi-simple Lie algebra and $\mathfrak{G}$ is f.r., Corollary to Theorem 2 in Part I implies that $M$ is closed. Now from $M/M \cap \mathfrak{R}_0 \cong \mathfrak{G}/\mathfrak{R}_0$ and $G/\mathfrak{R}_0 \cong \mathfrak{G}/\mathfrak{R}$, $\mathfrak{G}'$ is homomorphic with $M: M \sim \mathfrak{G}'$. On the other hand since the Lie algebra of $\mathfrak{R}$ is isomorphic to that of $\mathfrak{G}'$, $M$ and $\mathfrak{G}'$ are locally isomorphic, so that the kernel $\mathfrak{D}$ of the homomorphism $M \sim \mathfrak{G}'$ is discrete. Then in virtue of Lemma 5 in Part I our assertion is a direct consequence of Lemma 8, q.e.d.

6. A Lie algebra $G$ is called complex if $G$ is isomorphic with a Lie algebra over the field $C$ of complex numbers, and a Lie group is called complex when its Lie algebra is complex. When the structure of a Lie algebra $G$ is given by

$$G = P x_1 + P x_2 + \ldots + P x_r, \quad [x_i, x_j] = \sum_s r_{ij}s x_s, \quad r_{ij}s \in P,$$

where $x_1, \ldots, x_r$ constitute a basis of $G$, the complex Lie algebra $G_C$ defined by

$$G_C = C x_1 + C x_2 + \ldots + C x_r, \quad [x_i, x_j] = \sum_s r_{ij}s x_s$$

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\(^2\) Cf. Malcev [13].

\(^3\) A connected Lie group is called faithfully representable (f.r.) if it admits a continuous faithful linear representation. See Part I.
is called the complex form of $G$, and conversely $G$ is called a real form of $G_c$. A complex Lie algebra may have no real form at all, or have several real forms. Note that a Lie algebra is semi-simple if and only if its complex form is semi-simple.

Let $G^*$ be a complex semi-simple Lie algebra. H. Weyl has, in his fundamental papers on semi-simple Lie groups, proved the following important results: We can select a suitable basis $x_1, x_2, \ldots, x_r$ with respect to $C$ so that

$$G^* = Cx_1 + Cx_2 + \ldots + Cx_r, \quad [x_i, x_j] = \sum_{s} \tau_{ijs} x_s, \quad \tau_{ijs} \in P$$

and moreover any Lie group which is generated by the real form $K = P x_1 + P x_2 + \ldots + P x_r$ is always compact. Such a real form, called a compact form of $G^*$, is unique up to inner automorphisms.

Now consider a general real form of $G^*$,\(^5\) Take an involutive automorphism $\tau, \tau^2 = 1_r$, of a compact form $K$, and decompose $K$ into eigen-spaces of $\tau$:

$$K = K_1 + K_{-1},$$

where $K_1, K_{-1}$ are eigen-spaces of $\tau$ belonging to eigen-values $1, -1$ respectively. Now the subalgebra of $G^*$ given by

$$G = K_1 + \sqrt{-1} K_{-1}$$

is a real form of $G^*$, and conversely all real forms of $G^*$ can be obtained in such a way. Now isomorphic real forms of $G^*$, considered as subalgebras of $G^*$, are conjugated with respect to automorphisms. Therefore since any local automorphism of a simply connected Lie group can be extended to an automorphism in the large, in the simply connected complex semi-simple Lie group generated by $G^*$, isomorphic real forms of $G^*$ generate isomorphic subgroups.

Next, we shall call a complex semi-simple Lie algebra complex simple if it has no proper complex ideal distinct from 0. It is easy to see that a complex simple Lie algebra, as well as a real form of a complex simple Lie algebra, is simple. Let $G$ be a simple Lie algebra and $G_c$ the complex form of $G$. Then only the following two cases are possible.\(^6\)

Case 1. $G_c$ is simple. Then $G$ is a real form of $G_c$, which is complex simple.

Case 2. $G_c$ is not simple. In this case $G$ itself is complex simple.

From now on we shall call a real form of a complex simple Lie algebra real
simple. A real, or complex simple Lie group is defined correspondingly.

Let now $G^*$ be a complex Lie group and $G^*$ the Lie algebra of $G^*$. We shall call a subgroup $G$ of $G^*$ generated by a real form $G$ of $G^*$ a real form of $G^*$, and call $G^*$ a complex form of $G$. While a Lie algebra always has one and only one complex form, a Lie group may have no complex form at all, or have several complex forms.

**Lemma 10.** Let $G$ be a linear Lie group. If $G$ is compact or real simple, then there exists a complex-linear Lie group $G_c$ which contains $G$ as a real form. Here we shall call a linear Lie group complex-linear if its linear Lie algebra contains any complex multiple of its matrix.

**Proof.** Let $G = P x_1 + \ldots + P x_r$ be the linear Lie algebra of $G$. It is clearly sufficient to show that $x_1$'s are linearly independent with respect to $C$.

First suppose that $G$ is compact. Then since any compact linear group is equivalent to a subgroup of the unitary group, there exists a matrix $a$ such that $a^{-1}G a$ is composed of skew hermitian matrices. Suppose now that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r = 0 \ \alpha_i \in C$. Then we have $\alpha_1 a^{-1}x_1 a + \alpha_2 a^{-1}x_2 a + \ldots + \alpha_r a^{-1}x_r a = 0$. Now since every $a^{-1}x_1 a$ is skew hermitian, we have $\overline{\alpha}_1 a^{-1}x_1 a + \ldots + \overline{\alpha}_r a^{-1}x_r a = 0$, so that $\overline{\alpha}_1 x_1 + \overline{\alpha}_2 x_2 + \ldots + \overline{\alpha}_r x_r = 0$, where $\overline{\alpha}$ denotes the conjugate complex number of $\alpha$. Then the linear independence of $x_1$'s with respect to $P$ implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$.

Next let $G$ be real simple. We know that its complex form $G_c$ is simple. Hence the complex-linear Lie algebra spanned by $x_1, C x_1 + C x_2 + \ldots + C x_r$, forms a faithful representation of $G_c$, q.e.d.

**Theorem 3.** For any simple Lie algebra $G$ there exists one and only one linear covering group $\hat{G}$. Furthermore any local isomorphism of $\hat{G}$ to any f.r. Lie group can be extended to a homomorphism in the large, in particular a local automorphism of $\hat{G}$ can be extended to an automorphism.

The linear covering group of a complex simple Lie algebra is simply connected, and that of a real simple Lie algebra can be obtained as a real form of the simply connected group generated by its complex form.

**Proof.** A compact Lie group is f.r.\(^7\) in virtue of the theory of almost pe-
periodic functions. Let \( G^* \) be a complex (semi-) simple Lie algebra and \( K \) its compact form. Since \( K \) always generate a compact group by the above mentioned theorem of Weyl, there exists a simply connected linear Lie group \( \mathfrak{X} \) generated by \( K \). Now by Lemma 10 there exists a complex-linear Lie group \( \mathfrak{X}_c \) which contains \( \mathfrak{X} \) as a real form, whence the Lie algebra of \( \mathfrak{X}_c \) is isomorphic with \( G^* \). The space of \( \mathfrak{X}_c \), the direct product of that of \( \mathfrak{X} \) and the Euclidean space of a certain dimension, is also simply connected. Therefore a simply connected complex (semi-) simple Lie group is f.r.

Next let \( G \) be a real simple Lie algebra, and let \( \mathfrak{G} \) be the simply connected Lie group corresponding to \( G \) and \( \mathcal{Z} \) the center of \( \mathfrak{G} \). Denote by \( \mathfrak{G}_c \) the simply connected Lie group corresponding to the complex form \( G_c \) of \( G \), and by \( \mathfrak{G} \) a real form of \( \mathfrak{G}_c \) whose Lie algebra is \( G \). The structure of \( \mathfrak{G} \) is then determined uniquely. (See §\( \beta \).) Let now \( \hat{\mathcal{Z}} \) be a discrete invariant subgroup of \( \mathfrak{G} \) so that \( \mathfrak{G}/\mathcal{Z} \cong \hat{\mathcal{Z}} \). Since \( \mathfrak{G} \) is f.r. and the center of \( \mathfrak{G} \) is isomorphic to \( \mathcal{Z}/\hat{\mathcal{Z}} \), the index \( [\mathcal{Z} : \hat{\mathcal{Z}}] \) must be finite by Lemma 5 in Part I.

Now let \( \mathfrak{G} \) be an f.r. Lie group corresponding to \( G \). The existence of a complex form \( \mathfrak{G}_c \) of \( \mathfrak{G} \) can be assured by Lemma 10. Since in the homomorphism \( \mathfrak{G}_c \rightarrow \mathfrak{G}_c \) there corresponds \( \mathfrak{G} \) to \( \hat{\mathcal{Z}} \), we have \( \hat{\mathcal{Z}} \sim \mathfrak{G} \sim \mathfrak{G}_c \). Hence \( \hat{\mathcal{Z}} \) is a linear covering group for \( G \). We note also the fact that the order of the center of \( \hat{\mathcal{Z}} \) is not less that that of \( \mathfrak{G} \).

Suppose now that a linear representation \( f, \mathfrak{G} \rightarrow f(\mathfrak{G}) \), gives a faithful representation of \( \hat{\mathcal{Z}} \), and let \( \sigma \) be an automorphism of \( \hat{\mathcal{Z}} \). The kernel of the representation of \( \hat{\mathcal{Z}} \) given by

\[
\begin{pmatrix}
    f(\mathfrak{g}) & 0 \\
    0 & f(\mathfrak{g}^\sigma)
\end{pmatrix}
\]

is clearly \( \hat{\mathcal{Z}} \cap \mathfrak{g}^\sigma = \mathcal{D} \). Since \( \mathfrak{G}/\mathcal{D} \) is f.r., we have that \( [\mathcal{Z} : \mathcal{D}] \leq [\mathcal{Z} : \hat{\mathcal{Z}}] < \infty \).

Since \( \mathcal{D} \) is contained in \( \mathcal{Z} \) we get \( \hat{\mathcal{Z}} = \mathfrak{g}^\sigma = \mathcal{D} \), namely \( \hat{\mathcal{Z}} \) is characteristic. The uniqueness of the linear covering group of \( G \) may be proved by a similar argument.

Next let \( \mathfrak{G} \) be an f.r. Lie group generated by \( G \), and let \( \varphi_0 \) be a local isomorphism of \( \hat{\mathcal{Z}} \) to \( \mathfrak{G} \). Since \( \mathfrak{G} \) is a characteristic subgroup of \( \mathfrak{G} \), there exists a homomorphism \( \varphi \) so that \( \varphi = \varphi_0 \) locally, q.e.d.

In virtue of Lemma 9 and Theorem 3 we can count up all simple f.r. Lie groups, since all simply connected complex simple Lie groups are known.

For example in the four grand classes \( A_n, B_n, C_n \ (n = 1, 2, \ldots) \) and \( D_n \ (n = 3, 4, \ldots) \) of complex simple Lie algebras given by Lie-Killing-Cartan,\(^3\)

\(^3\) E. g. Cartan [3].
we are familiar with the simply connected groups for the classes $A_n$ and $C_n$ i.e. the unimodular groups and the symplectic groups, and for $B_n$ and $D_n$ the classes of the proper orthogonal groups, the so-called spin representations give us simply connected groups.

The necessity of the second example of non-f.r. Lie groups given by Cartan, i.e. an arbitrary proper covering group of $\mathfrak{S}(2, P)$, is now clarified, because $\mathfrak{S}(2, P)$ is, as a real form of the complex unimodular group, which is simply connected, a linear covering group of itself.

**Theorem 4.** For any semi-simple Lie algebra $G$ there exists one and only one linear covering group $\widehat{G}$, and $\widehat{G}$ has the property stated in the first half of Theorem 3 for a simple Lie algebra.

Moreover $\widehat{G}$ is decomposed into a direct product of simple Lie groups. If $G$ is complex, $\widehat{G}$ is simply connected.

**Proof.** Decompose $G$ into simple ideals: $G = G_1 + \ldots + G_s$, and let $\widehat{G}_i$ be the linear covering group of $G_i$. Then the direct product

$$\widehat{G} = \widehat{G}_1 \times \ldots \times \widehat{G}_s$$

is clearly the uniquely determined linear covering group of $G$, and other assertions are easy to prove by an analogous argument as in Theorem 3, q.e.d.

Now we can easily prove the following

**Corollary 1.** A connected semi-simple Lie group is f.r. if and only if every (closed) simple invariant Lie subgroup is f.r.

Next let $\mathfrak{G}$ be a topological group. In the introduction of Part I we defined a notion of an $(l)$-group. We repeat the definition: If there exists a set of closed invariant subgroups $\{\mathfrak{N}_\alpha\}$ of $\mathfrak{G}$ such that

1) $\mathfrak{G}/\mathfrak{N}_\alpha$ is f.r. and 2) $\cap \mathfrak{N}_\alpha = e$,

then $\mathfrak{G}$ is called an $(l)$-group.

**Corollary 2.** A connected semi-simple Lie group is f.r. if it is an $(l)$-group.

**Proof.** Let $\mathfrak{G}$ be a connected semi-simple $(l)$-group and $\bar{\mathfrak{G}}$ the universal covering group of $\mathfrak{G}$, and let $\hat{\mathfrak{G}}$ be the linear covering group of the Lie algebra of $\mathfrak{G}$. We may suppose that $\mathfrak{G} = \bar{\mathfrak{G}}/\mathfrak{D}$ and $\hat{\mathfrak{G}} = \bar{\mathfrak{G}}/\mathfrak{B}$. Let $z$ be an element of $\mathfrak{D}$. Since $\mathfrak{G}/\mathfrak{N}_\alpha$ is f.r. there corresponds to $z$ the unit element in the homomorphism $\bar{\mathfrak{G}}(\sim \mathfrak{G}) \sim \mathfrak{G}/\mathfrak{N}_\alpha$, namely $z\mathfrak{D} \subseteq \mathfrak{N}_\alpha$, whence $z\mathfrak{D} \subseteq \mathfrak{D}$ because $\cap \mathfrak{N}_\alpha$ is the unit

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5) E. g. Weyl [24].

10) Cartan [6].

element. Therefore we have \( \mathfrak{g} \subseteq \mathfrak{d} \), namely \( \mathfrak{g} \) is f.r., q.e.d

**Solvable f.r. Lie groups**

8. We first prove the following

**Lemma 11.** A simply connected solvable Lie group is f.r.

**Proof.** Let \( G \) be a solvable Lie algebra. Then there exists a linear Lie algebra \( G_1 \) isomorphic with \( G \). Next since \( G/D(G) \) is commutative, we may easily prove that there exists a linear Lie algebra \( G_2 \), composed of nilpotent matrices, which is isomorphic with \( G/D(G) \). Then the representation of \( G \) given by

\[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\]

is obviously faithful. Furthermore in this representation there corresponds a nilpotent matrix to any element of \( D(G) \) by a theorem of Lie, and to any element which does not belong to \( D(G) \) there corresponds a non-zero nilpotent matrix in the representation \( G_2 \). Hence the linear Lie algebra generates a simply connected linear Lie group by Lemma 6 in Part I, q.e.d.

**Lemma 12.** Let \( \mathfrak{g} \) be a linear Lie group and let \( \mathfrak{h} \) be a closed invariant subgroup, \( \mathfrak{d} \) a closed vector subgroup of \( \mathfrak{g} \). Suppose that \( \mathfrak{g} = \mathfrak{g} \mathfrak{h} \), \( \mathfrak{h} \cap \mathfrak{h} = e \). If a discrete subgroup \( \mathfrak{d} \) of \( \mathfrak{g} \) is contained in the center of \( \mathfrak{g} \), then \( \mathfrak{g}/\mathfrak{d} \) is f.r.

**Proof.** Let \( m \) be the dimension of \( \mathfrak{g} \). It is clearly sufficient to prove the lemma when \( m = 1 \). We may suppose that \( \mathfrak{g} \) is a one-parameter group \( \exp \lambda x \), where \( x \) is a matrix and \( \lambda \) a real parameter, furthermore that \( \mathfrak{d} \) is generated by \( \exp x = d \). We decompose the vector space \( M \) on which \( \mathfrak{g} \) operates into eigen-spaces with respect to \( x \):

\[
M = M_\alpha + M_\beta + \ldots,
\]

where \( \alpha, \beta, \ldots \) denote eigen-values of \( x \). Next we sum up eigen-spaces whose eigen-values are the same mod. \( 2 \pi \sqrt{-1} \):

\[
\begin{align*}
M &= M_1 + M_2 + \ldots \\
M_1 &= M_{\alpha(1)} + M_{\beta(1)} + \ldots \quad \alpha(1) \equiv \beta(1) \equiv \ldots \text{ mod. } 2 \pi \sqrt{-1}, \\
M_2 &= M_{\alpha(2)} + M_{\beta(2)} + \ldots \quad \alpha(2) \equiv \beta(2) \equiv \ldots \text{ mod. } 2 \pi \sqrt{-1}, \\
&\ldots \ldots \\
M &= M_1 + M_2 + \ldots \text{ is then clearly the eigen-space decomposition with respect to } d = \exp x, \text{ whose eigen-values are } \exp \alpha(1) = \exp \beta(1) = \ldots, \exp \alpha(2), \ldots
\end{align*}
\]

\(^{12}\) Cartan [5].
Since \( d \) is contained in the center, the eigen-spaces \( M_l(l = 1, 2, \ldots) \) are allowable by \( G \), and hence \( G \) is decomposed into the representations induced on \( M_l \).

Now define matrices \( x_1 \) and \( x_2 \) by the equations
\[
x_1 e_l = \alpha(l) e_l \quad \text{for} \quad e_l \in M_l \quad l = 1, 2, \ldots,
\]
\[
x_2 e_p = \rho e_p \quad \text{for} \quad e_p \in M_p \quad \rho = \alpha, \beta, \ldots,
\]
and put
\[
x_0 = x - x_3, \quad x_2 = x_3 - x_1,
\]
then we have
\[
x = x_0 + x_1 + x_2,
\]
where \( x \)'s are mutually commutative, \( x_0 \) is nilpotent, and \( x_1 \) and \( x_2 \) are semi-simple.\(^{13}\) Since the eigen-values of \( x_2 \) are all integral multiple of \( 2 \pi \sqrt{-1} \), we have \( \exp x_2 = 1_n \) where \( n \) denotes the dimension of \( M_l \), and hence
\[
d = \exp x_0 \exp x_3.
\]
Then since \( d \) is contained in the center, we can easily prove that the matrix \( x_0 \) is commutative with any matrix of \( G \). Hence we have
\[
[\exp \lambda(x_0 + x_1), G] = 0,
\]
for any real number \( \lambda \). Therefore the correspondence defined by
\[
G \ni g = \exp \lambda x \cdot h \rightarrow \exp \lambda x_2 \cdot h \quad h \in \mathcal{R}
\]
gives a representation \( G^* \) of \( G \). Let \( \mathcal{B} \) be the kernel of the homomorphism. The relations \( \mathcal{R} \cap \mathcal{B} = e \) and \( \mathcal{D} \subseteq \mathcal{B} \) are now obvious.

Next since \( G/\mathcal{D} \mathcal{R} \) is a one-dimensional toroidal group, there exists a linear representation \( G^{**} \) of \( G \) such that the kernel coincides with \( \mathcal{D} \mathcal{R} \). Hence the representation of \( G \) given by
\[
\begin{pmatrix}
G^* & 0 \\
0 & G^{**}
\end{pmatrix}
\]
is clearly a faithful representation of \( G/\mathcal{D} \), q.e.d.

**Lemma 13.** Let \( G \) be a connected Lie group and \( \mathcal{R} \) a nilpotent Lie invariant subgroup. Then any compact subgroup of \( \mathcal{R} \) is central in \( G \).

**Proof.** Let \( G \) be the Lie algebra of \( G \), and \( N \) the subalgebra of \( G \) corresponding to \( \mathcal{R} \). Then an inner derivation of \( G \) induced by an element of \( N \) is always nilpotent. Hence the linear group \( \mathcal{R}^* \), corresponding to \( \mathcal{R} \) in the adjoint representation of \( G \), is simply connected, by Lemma 6 in Part I, and contains no compact subgroup except \( e \). Hence a compact subgroup \( \mathcal{R} \) is contained in the kernel of the adjoint representation, i.e. the center, q.e.d.

\(^{13}\) A matrix is called semi-simple if it has simple elementary divisors.
Theorem 5. Let $\mathcal{G}$ be a connected solvable Lie group. The following conditions are all necessary and sufficient for $\mathcal{G}$ to be f.r.: 1) $\mathcal{G} = \mathcal{K} \cap \mathcal{R} = e$, where $\mathcal{K}$ is a (maximal) compact subgroup and $\mathcal{R}$ is a closed simply connected invariant subgroup. 2) $C(\mathcal{G}) (= D(\mathcal{G}))$ is simply connected. 3) The center of $C(\mathcal{G})$ is connected and simply connected. 4) $\mathcal{K} \cap D(\mathcal{G}) = e$, where $\mathcal{K}$ is a maximal compact group of $\mathcal{G}$. We note here the fact that a compact connected solvable Lie group is commutative, and hence a toroidal group.

Proof. Let $G$ be the Lie algebra of $\mathcal{G}$ and $A$ a subalgebra of $G$ corresponding to a maximal compact subgroup $\mathcal{K}$. First we shall proceed to prove the following implications: ($\mathcal{G}$ is f.r.) $\Rightarrow$ 4) $\Rightarrow$ 1) $\Rightarrow$ ($\mathcal{G}$ is f.r.)

Let $\mathcal{G}$ be f.r. Since there corresponds a semi-simple matrix for an element of $A$, and a nilpotent matrix for an element of $D(G)$ in any representation of $\mathcal{G}$, we have $A \cap D(G) = 0$. On the other hand the commutator group $D(\mathcal{G})$ is closed and simply connected by Lemma 7 in Part I. Hence we get $\mathcal{K} \cap D(\mathcal{G}) = e$, i.e. 4).

Next from 4) we have $A \cap D(G) = 0$, and hence there exists a subspace $N$ of $G$ such that $N \subseteq D(G)$, $G = A + N$ $A \cap N = 0$. $N$ is then obviously an ideal. Let $\tilde{\mathcal{G}}$ be the universal covering group of $\mathcal{G}$ and let $\tilde{\mathcal{K}}, \tilde{\mathcal{R}}$ be (closed) subgroups of $\tilde{\mathcal{G}}$ generated by $A, N$ respectively. Then the simple connectedness of the factor group $\tilde{\mathcal{G}}/\tilde{\mathcal{R}}$ by Lemma 3 in Part I implies that $\tilde{\mathcal{K}} \cap \tilde{\mathcal{R}} = e$. Let $\mathcal{D}$ be the kernel of the homomorphism $\mathcal{G} \sim \mathcal{G}$. Then $\mathcal{D}$ is obviously contained in $\tilde{\mathcal{K}}$ because $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$. Hence $N$ generates a closed simply connected group $\mathcal{R}$ in $\mathcal{G}$. That $\mathcal{K} \cap \mathcal{R} = e$ follows immediately. Thus we get 1).

Now suppose that the condition 1) is satisfied. Then by Lemma 11 the universal covering group of $\mathcal{G}$ is f.r., and using Lemma 12 we get a faithful representation of $\mathcal{G}$. Thus we have the equivalence of the faithful representability, 1) and 4).

Now from the simple connectedness of the adjoint group of a connected nilpotent Lie group and by Lemma 3 in Part I we can easily prove that 2) is equivalent to 3). Next from 1) we get 2) because $C(\mathcal{G})$ is contained in $\mathcal{R}$. Suppose now 2) is satisfied for $\mathcal{G}$. Then $D(\mathcal{G})$ is, as a Lie subgroup of the simply connected $C(\mathcal{G})$, closed, and hence coincides with $C(\mathcal{G})$, q.e.d.

From Theorem 5 and Lemma 13 we get the following

Corollary 1. A connected nilpotent Lie group is f.r. if and only if it is

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15) Cf. Birkhoff [19].
a direct product of a compact group and a simply connected Lie group.

**Corollary 2.** A connected solvable Lie (I)-group is f.r.

**Proof.** Let $\mathcal{G}$ be a connected solvable Lie group. Suppose that $\mathcal{G}$ is not f.r. Then the center of $C(\mathcal{G})$ contains a compact group $\mathcal{K}(\equiv e)$ by Theorem 5. Since $\mathcal{K}$ is mapped into the unit matrix in any linear representation of $\mathcal{G}$, $\mathcal{G}$ is not an (I)-group. q.e.d.

**Remark.** A connected non-f.r. solvable Lie group has no locally isomorphic (univalent) representation.

**F.R. Lie groups**

9. Let $G$ be a Lie algebra and $R$ the radical of $G$, and let $R_i$ be the intersection of $R$ and $D(G)$. Then the factor algebra $G/R_i$ is decomposed into a direct sum of a semi-simple ideal $S$ and the center $Z$: 

$$G/R_i = S + Z \quad S \cap Z = 0 \quad [S, Z] = 0$$

Let $\hat{S}$ be the linear covering group of $S$, and $Z$ the $m$-dimensional vector group, where $m$ denotes the dimension of $Z$. The direct product $\hat{S} \times Z$ is then clearly f.r., and furthermore there exists a faithful representation of $\hat{S} \times Z$ such that the center of the corresponding linear Lie algebra consists of nilpotent matrices. Let $G_1$ be the linear Lie algebra of such a representation. Let now $G_2$ be an arbitrary faithful representation of $G$. Then the representation of $G$ defined by

$$G^* = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

is of course faithful.

Let now $\bar{\mathcal{G}}$ be the linear Lie group generated by $G^*$. Then we may easily prove that the radical $\mathcal{K}$ of $\bar{\mathcal{G}}$ is simply connected in such a way as in the proof of Lemma 11. Let $\mathcal{S}$ be a maximal semi-simple Lie subgroup of $\bar{\mathcal{G}}$. Since $\mathcal{S}$ is closed and the center of $\mathcal{S}$ is finite in virtue of § 4 in Part I, $\mathcal{S} \cap \mathcal{K}$ is also a finite group, whence the simple connectedness of $\mathcal{K}$ implies that $\mathcal{S} \cap \mathcal{K} = e$. On the other hand since $\mathcal{S}$ has a representation which is isomorphic to $\hat{S}$, we get that $\mathcal{S} \cong \hat{S}/\mathcal{K} \sim \hat{S}$, whence we have $\mathcal{S} \cong \hat{S}$ comparing the orders of the centers of $\mathcal{S}$ and $\hat{S}$.

Now we shall prove that our $\bar{\mathcal{G}}$ is a linear covering group of $G$. First let $\tilde{\mathcal{G}}$ be the universal covering group of $\bar{\mathcal{G}}$ and let $\mathcal{G}$ be any f.r. Lie group corresponding to $G$. We may suppose that $\mathcal{G}/\mathcal{D} = \bar{\mathcal{G}}$ and $\mathcal{G}/\mathcal{Z} = \bar{\mathcal{G}}$. Now let $\mathcal{S}$ be a maximal semi-simple Lie subgroup of $\bar{\mathcal{G}}$. Since the space of $\bar{\mathcal{G}}$ is a direct product of the space of $\mathcal{S}$ and the Euclidean space, $\mathcal{Z}$ is contained in $\mathcal{S}$. Since
\(\mathfrak{E}/\mathfrak{D}(\cong \mathfrak{E}/\mathfrak{E} \cap \mathfrak{D})\), a maximal semi-simple Lie subgroup of \(\mathfrak{G}\), is f.r., we get that \(\mathfrak{E} \cap \mathfrak{D} \cong \mathfrak{B}\), whence \(\mathfrak{B}\) is contained in \(\mathfrak{D}\): \(\mathfrak{B} \subseteq \mathfrak{D}\). Hence \(\mathfrak{G}\) is homomorphic with \(\mathfrak{E}\), namely \(\mathfrak{E}\) is a linear covering group. Here we note the fact that the discrete invariant subgroup \(\mathfrak{B}\) is characteristic, which will be seen easily. Now the uniqueness of the linear covering group for any Lie algebra may be proved as in the proof of Theorem 3.

Next let \(\mathfrak{E}\) be a linear covering group and \(\mathfrak{N}\) a Lie invariant subgroup of \(\mathfrak{E}\). An analogous argument as in Lemma 3 in Part I shows that \(\mathfrak{R}\) is closed and \(\mathfrak{N}\) and \(\mathfrak{E}/\mathfrak{N}\) are both linear covering groups.

From the above considerations we may easily prove the following

**Theorem 6.** For any Lie algebra \(G\) there exists one and only one linear covering group \(\hat{\mathfrak{E}}\).

The radical \(\mathfrak{R}\) of \(\mathfrak{E}\) is simply connected, and a maximal semi-simple Lie subgroup \(\mathfrak{E}\) is a linear covering group such that

\[\mathfrak{E} = \hat{\mathfrak{E}}/\mathfrak{R}, \ \hat{\mathfrak{E}} \cap \mathfrak{R} = e.\]

In particular, a linear covering group of a complex Lie algebra is simply connected.\(^{16}\)

Any local homomorphism of a linear covering group onto any f.r. Lie group can be extended to a homomorphism in the large.

**Remark.** A connected Lie group is homomorphic with the linear covering group of its Lie algebra if its maximal semi-simple Lie subgroup is f.r.

**Theorem 7.**\(^{17}\) Let \(\mathfrak{E}\) be a connected Lie group and \(\mathfrak{R}\) the radical of \(\mathfrak{E}\). If a maximal semi-simple Lie subgroup \(\mathfrak{E}\) and \(\mathfrak{R}\) are f.r., then \(\mathfrak{E}\) is also f.r., and vice versa.

**Proof.** Let \(G\) be the Lie algebra of \(\mathfrak{E}\) and \(R\) the radical of \(G\). Since the radical \(\mathfrak{R}\) is f.r. and solvable there exist closed subgroups \(\mathfrak{A}\) and \(\mathfrak{A}'\) of \(\mathfrak{R}\) such that

\[\mathfrak{R} = \mathfrak{A}/\mathfrak{A}', \ \mathfrak{A} \cap \mathfrak{A}' = e,\]

where \(\mathfrak{A}\) is a toroidal group and \(\mathfrak{A}'\) is simply connected and invariant in \(\mathfrak{R}\). Let \(h(\lambda)\) be an everywhere dense one-parameter subgroup of \(\mathfrak{A}\): \(h(\lambda) = \mathfrak{A}\), and let \(x\) be an element of \(G\) which generates \(h(\lambda)\). Now we decompose \(G\) and \(R\) into eigenspaces of the inner derivation \(\delta_{x}\) induced by \(x\). Since \(\delta_{x}\) is semi-simple because of the compactness of \(\mathfrak{A}\), we may put

\[16\] Matsushima [15].

\[17\] Malcev [13].
\[ G = G_0 + G_1, \quad \delta_x G_0 = 0, \quad \delta_x G_1 = G_1, \]
\[ R = R_0 + R_1, \quad \delta_x R_0 = 0, \quad \delta_x R_1 = R_1, \]
where \( G_0 \) and \( R_0 \) are subalgebras. Since it is clear that \( G_0 \cap R = R_0 \) and \( G_0 + R = G \), we have
\[ G_0/R_0 \cong G/R \quad \text{(semi-simple)}. \]

Hence \( R_0 \) is the radical of \( G_0 \). Let \( S \) be a maximal semi-simple subalgebra of \( G \). Then \( S \) is evidently maximal semi-simple in \( G \). Let \( A \) be the Lie algebra of \( S \). Since \( \delta_x G_0 = 0 \), we have \([A, G_0] = 0\), and in particular \([A, S] = 0\). Now let us consider \( R \) as an \( S \)-module. Then \( A \) and \( D(R) \) are both \( S \)-modules such that \( A \cap D(R) = 0 \). Hence from the complete reducibility of \( R \) there exists a submodule \( N \) containing \( D(R) \) such that
\[ R = A + N, \quad A \cap N = 0 \]
\( N \) is, as an ideal of \( R \) and an \( S \)-module, an ideal of \( G \). Let now \( \mathfrak{H} \) be the subgroup of \( G \) generated by \( N \). \( \mathfrak{H} \) is easily seen to be closed and simply connected. Thus we get the decomposition
\[ R = \mathfrak{H} \quad \mathfrak{H} \cap N = e, \]
where \( \mathfrak{H} \) is a toroidal group and \( \mathfrak{H} \) is a closed simply connected invariant subgroup of \( G \).

Next let \( \mathfrak{S} \) be the subgroup generated by \( S \). Since the center of \( \mathfrak{S} \) is finite in virtue of Lemma 5 in Part I because \( \mathfrak{S} \) is f.r., \( \mathfrak{S} \) is closed by Theorem 2 in Part I. Then the compactness of \( \mathfrak{H} \) implies that \( \mathfrak{S} \mathfrak{H} \) is closed. Now since the center of \( \mathfrak{S} \mathfrak{H} \) is compact and \( \mathfrak{H} \) contains no compact subgroup except \( e \), we have \( \mathfrak{S} \mathfrak{H} \cap \mathfrak{H} = e \), whence \( \mathfrak{S} \) is topologically a direct product of \( \mathfrak{S} \mathfrak{H} \) and \( \mathfrak{H} \): \( \mathfrak{S} = (\mathfrak{S} \mathfrak{H}) \mathfrak{H} \).

Let now \( \hat{\mathfrak{S}} \) be the linear covering group of \( G \). Then by the remark to Theorem 6, \( \hat{\mathfrak{S}} \) is a covering group of \( \mathfrak{S} \). Let \( \hat{\mathfrak{S}}, \mathfrak{H} \) and \( \hat{\mathfrak{H}} \) be subgroups of \( \hat{\mathfrak{S}} \) generated by \( S, A \) and \( N \) respectively. Then \( \hat{\mathfrak{S}} = \hat{\mathfrak{S}} \mathfrak{H} \mathfrak{H} \) and the space of \( \hat{\mathfrak{S}} \) is the direct product of the spaces of \( \hat{\mathfrak{S}}, \mathfrak{H} \) and \( \hat{\mathfrak{H}} \). Since \( \mathfrak{H} \) is simply connected, we have \( \mathfrak{D} \cong \hat{\mathfrak{S}} \mathfrak{H} \), where \( \mathfrak{D} \) denotes the kernel of the homomorphism \( \hat{\mathfrak{S}} \sim \mathfrak{S} \).

\[ ^{15} \text{Matsushima in [15] proved the following theorem: } \text{Let } \mathfrak{G} \text{ be a connected Lie group and } \mathfrak{H} \text{ and } \mathfrak{N} \text{ closed subgroups. Let } H, N \text{ be their Lie algebras. Then } \mathfrak{G} \text{ is f.r. if the following conditions are satisfied. 1) } \mathfrak{N} \text{ is a simply connected solvable invariant subgroup and } \mathfrak{H} \text{ is f.r. 2) } \mathfrak{G} = \mathfrak{H} \mathfrak{N} \mathfrak{N} \cap \mathfrak{N} = e. \text{ 3) } N \text{ is completely reducible as an } H\text{-module. Using this theorem we get another proof of Theorem 7, because } \mathfrak{S} \mathfrak{H}, \text{ which is a direct product of f.r. } \mathfrak{S} \text{ and a toroidal group } \mathfrak{H} \text{ mod. a finite group, is f.r. by Lemma 8, and our } N \text{ is clearly a completely reducible } (S + A)\text{-module. We note also the fact that our Theorem 7 and our decomposition } \mathfrak{G} = (\mathfrak{S} \mathfrak{H}) \mathfrak{H} \text{ show the validity of the converse of the above Theorem of Matsushima.} \]
Now let $\mathfrak{B}$ be a maximal compact subgroup of $\mathfrak{G}$, and $B$ its Lie algebra. Then $\mathfrak{K} = \mathfrak{B}A$ is obviously a maximal compact subgroup of $\mathfrak{G}A$, and the Lie algebra $K$ of $\mathfrak{K}$ coincides with $A + B$. Let $\mathfrak{B}$ be the subgroup of $\mathfrak{G}$ generated by $B$. Since $\mathfrak{G}$ is a covering group of $\mathfrak{G}$ of finite order, $\mathfrak{B}$ is compact. Now since $\mathfrak{K}$ is a maximal compact subgroup of $\mathfrak{G}A$, we may easily conclude that $\mathfrak{K} = \mathfrak{B}A \supseteq \mathfrak{D}$. Let $A$ be of $m$ dimensions. Then $\mathfrak{A}$ is the $m$-dimensional vector group. Hence $\mathfrak{D}$ is, as a discrete invariant subgroup of a direct product of a compact group and an $m$-dimensional vector group, of rank at most $m$. On the other hand, $\mathfrak{D}_1 = \mathfrak{D} \cap \mathfrak{A}$ is obviously a free commutative group of rank $m$. Hence the factor group $\mathfrak{D}/\mathfrak{D}_1$ is finite. Now since $\mathfrak{B}A$ is a closed invariant subgroup and $\mathfrak{A}$ is a vector group, Lemma 12 shows that $\mathfrak{G}/\mathfrak{D}_1$ is f.r. Then $\mathfrak{D}/\mathfrak{D}_1$ being a finite group, the relation

$$\mathfrak{G} \cong \mathfrak{G}/\mathfrak{G}_1 \cong \mathfrak{G}/\mathfrak{D}_1/\mathfrak{D}/\mathfrak{D}_1,$$

implies the faithful representability of $\mathfrak{G}$ by Lemma 8, q.e.d.

**Corollary 1.** Let $\mathfrak{G}$ be an f.r. Lie group, and $\mathfrak{K}$ a maximal compact subgroup of $\mathfrak{G}$. Then there exists a closed simply connected solvable subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that

$$\mathfrak{G} = \mathfrak{KH} = \mathfrak{K}H, \quad \mathfrak{H} \cap \mathfrak{K} = e.$$

**Proof.** The assertion was proved by K. Iwasawa\(^{10}\) for adjoint groups of semi-simple Lie groups. By our Lemma 5 in Part I it is also valid for semi-simple f.r. Lie groups. Then the decomposition of an f.r. Lie group $\mathfrak{G}$:

$$\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{K}, \quad \mathfrak{S}\mathfrak{A} \cap \mathfrak{K} = e,$$

given in the proof of Theorem 7, easily implies our corollary, q.e.d.

Next let us call an f.r. group $\mathfrak{G}$ *completely reducible* if any linear representation of $\mathfrak{G}$ is completely reducible. Semi-simple, or compact f.r. groups are completely reducible, but a vector group is not.

Let $\mathfrak{G}$ be a completely reducible f.r. group and $\mathfrak{G}$ its Lie algebra. By a theorem of Cartan, an irreducible linear Lie algebra is a direct sum of a semi-simple ideal and its center which is composed of scalar matrices. Using this we can readily prove that $\mathfrak{G}$ is decomposed into a direct sum of its center $Z$ and a semi-simple ideal $S$: $\mathfrak{G} = S + Z$, $[S, Z] = 0$. Now let $\mathfrak{S}$ and $\mathfrak{Z}$ be subgroups of $\mathfrak{G}$ generated by $S$ and $Z$ respectively. Assume that $\mathfrak{Z}$ is not compact. Then the finiteness of the center of $\mathfrak{S}$ by Lemma 5 in Part I obviously implies that there exists a closed vector invariant subgroup of $\mathfrak{G}$ which is a direct factor. Hence $\mathfrak{G}$ is not completely reducible, contrary to the assumption.

\(^{10}\) Iwasawa [10].
Thus $\mathfrak{Z}$ must be compact. Since the converse proposition is clear, we have the following.

**Lemma 14.** An f.r. group $\mathfrak{G}$ is completely reducible if and only if the radical of $\mathfrak{G}$ is compact.

Now the following corollary is an immediate consequence of Theorem 7. (Cf. also Note (11) of p. 103).

**Corollary 2.** Let $\mathfrak{G}$ be a connected Lie group. If there exist closed subgroups $\mathfrak{I}$ and $\mathfrak{I}_1$ of $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{I} \mathfrak{I}_1$, where $\mathfrak{I}$ is a completely reducible f.r. group and $\mathfrak{I}_1$ is a simply connected solvable invariant subgroup, then $\mathfrak{G}$ is f.r., and vice versa.

10. In this § three theorems shall be proved concerning f.r. Lie groups.

**Theorem 8.** A connected Lie group is f.r. if it is an (I)-group.

**Proof.** Let $\mathfrak{G}$ be a connected Lie group and let $\mathfrak{Z}$ and $\mathfrak{R}$ be a maximal semi-simple Lie subgroup and the radical of $\mathfrak{G}$ respectively. Suppose that $\mathfrak{G}$ is an (I)-group. Then $\mathfrak{Z}$ and $\mathfrak{R}$ are also (I)-groups. Now Corollary 2 to Theorem 4 and Corollary 2 to Theorem 5 imply that $\mathfrak{Z}$ and $\mathfrak{R}$ are both f.r., whence $\mathfrak{G}$ is f.r. by Theorem 7, q.e.d.

**Theorem 9.** Any f.r. Lie group is isomorphic with a closed subgroup of the general linear group of a certain degree. In other words, for any f.r. Lie group there exists a topologically isomorphic linear representation with respect to the induced topology.

**Proof.** Let $\mathfrak{G}$ be an f.r. Lie group and $\mathfrak{R}$ the radical of $\mathfrak{G}$. The commutator subgroup $D(\mathfrak{G})$ is closed and the radical $\mathfrak{R}_1$ of $D(\mathfrak{G})$ is simply connected by Lemma 7 in Part I. Denote by $\mathfrak{A}$ a maximal compact subgroup of $\mathfrak{R}$. Then there exists a closed simply connected invariant subgroup $\mathfrak{A}_1$ such that $\mathfrak{A} = \mathfrak{A}_1 \mathfrak{R}_1$, $\mathfrak{A} \cap \mathfrak{R} = e$, and $\mathfrak{R} \supseteq \mathfrak{R}_1$. As $\mathfrak{A}$ is compact, $\mathfrak{A} D(\mathfrak{G})$ is a closed invariant subgroup. Let

\[ \mathfrak{G} = (\mathfrak{A} \mathfrak{R}) \mathfrak{A}_1, \quad \mathfrak{A} \cap \mathfrak{R} = e \]

be a decomposition as in the proof of Theorem 7. Since $\mathfrak{A} \mathfrak{R}_1 \mathfrak{A}_1$ coincides with $\mathfrak{A} D(\mathfrak{G})$ locally and they are both connected, we have

\[ \mathfrak{A} \mathfrak{R}_1 = \mathfrak{A} D(\mathfrak{G}) \],

whence $\mathfrak{A} D(\mathfrak{G}) \cap \mathfrak{R} = \mathfrak{R}_1$. Now from

\[ \mathfrak{G} / \mathfrak{A} D(\mathfrak{G}) = \mathfrak{A} D(\mathfrak{G}) \mathfrak{R} / \mathfrak{A} D(\mathfrak{G}) \cong \mathfrak{R} / \mathfrak{R} \cap \mathfrak{A} D(\mathfrak{G}) = \mathfrak{R} / \mathfrak{R}_1, \]

we see that $\mathfrak{G} / \mathfrak{A} D(\mathfrak{G})$ is a vector group. Hence there exists surely a representation of $\mathfrak{G}$ such that the kernel coincides with $\mathfrak{A} D(\mathfrak{G})$ and whose linear Lie
algebra consists of nilpotent matrices. Let $\mathfrak{g}^*$ be such a representation.

Next let $\mathfrak{g}^{**}$ be a faithful representation of $\mathfrak{g}$. The representation defined by

$$
\begin{pmatrix}
\mathfrak{g}^* & 0 \\
0 & \mathfrak{g}^{**}
\end{pmatrix}
$$

is of course faithful. On the other hand in this representation any non-zero element of the Lie algebra of $\mathfrak{m}$ is represented by a matrix with non-zero 0-eigenspace, whence $\mathfrak{m}$ is closed in the general linear group by Lemma 6 in Part I. Since $\mathfrak{m}$ is compact, the radical $\mathfrak{r} = \mathfrak{m}\mathfrak{m}$ is also represented by a closed group. Thus Lemma 5 and Theorem 2 in Part I imply our assertion, q.e.d.

**Remark.** The above two theorems give us apparently weaker but equivalent definitions of f.r. groups.

Examples of non-f.r. Lie groups have been given by Cartan and Birkhoff. (See § 4 in Part I and § 7.) Here we shall prove a theorem which shows the necessity of the above examples, namely the following

**Theorem 10.** A connected Lie group $\mathfrak{g}$ is f.r. if and only if the following two conditions are satisfied.

1) All simple Lie subgroups are f.r.

2) The radical of $C(\mathfrak{g})$ is simply connected.

**Proof.** If $\mathfrak{g}$ is f.r., the above two conditions are obvious. Conversely let us suppose that 1) and 2) are satisfied. Let $\mathfrak{s}$ be a maximal semi-simple Lie subgroup of $\mathfrak{g}$ and $\mathfrak{r}$ the radical of $\mathfrak{g}$. Since any simple invariant Lie subgroup of $\mathfrak{s}$ is f.r., $\mathfrak{s}$ is also f.r. by Corollary 1 to Theorem 4. Now since $C(\mathfrak{r})$ is contained in the radical of $C(\mathfrak{g})$, it is simply connected, whence $\mathfrak{r}$ is f.r. by Theorem 5. Therefore the theorem follows from Theorem 7, q.e.d.

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