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MOVING BOUNDARY PROBLEMS IN THE FLOW OF LIQUID THROUGH POROUS MEDIA

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Abstract

The movement of the interface between two immiscible fluids flowing through a porous medium is discussed. It is assumed that one of the fluids, which is a liquid, is much more viscous than the other. The problem is transformed by replacing the pressure with an integral of pressure with respect to time. Singularities of pressure and the transformed variable are seen to be related.

Some two-dimensional problems may be solved by comparing the singularities of certain analytic functions, one of which is derived from the new variable. The implications of the approach of a singularity to the moving boundary are examined.

1. Introduction

A viscous liquid flows through a porous medium. It will be assumed that Darcy's law holds so that the velocity of the liquid is proportional to the gradient of pressure; the constant of proportionality is negative. By fluid velocity we mean the local velocity averaged throughout some representative region, which is much larger than the pores but much smaller than the total system, and which includes both the pores and the matrix (the local velocity is taken to be zero in the matrix). The porous medium is isotropic and homogeneous [2].

The region occupied by the viscous liquid bounds a region in which there is a much less viscous fluid. This is immiscible with the former liquid and it is assumed that if one fluid displaces the other it does so totally so that there is a well defined interface between the two regions.

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If the viscosity of the first fluid, μ_1, is considerably larger than the viscosity of the second, μ_2, then the pressure gradient in the second is much less than that in the first; so that to leading order the pressure along the boundary of the first fluid may be considered to be independent of position. Taking asymptotic expansions of pressure and velocity for $\varepsilon = \mu_2/\mu_1 \to 0$, *e.g.*, pressure $\sim p_0 + \varepsilon p_1 + \cdots$, then the model considered in this paper describes the first order terms (*e.g.*, p_0) in the first fluid. In the other fluid p_0 is a function only of time and may be considered to be zero. The interface between the fluids is a material boundary so that its normal velocity is that of the liquid.

This problem is motivated by the system of oil and water (or gas) flowing through a porous rock.

A two-dimensional problem which is modelled by the same equations concerns the injection of molten plastic into moulds. In this case a viscous fluid flows between two parallel plates, this is Hele-Shaw flow [4].

Again the average fluid velocity (averaged across the gap) is proportional to the pressure gradient. The liquid is in contact with air. At the interface the air pressure is taken to be constant and the pressure difference between the liquid and air, due to surface tension, is also assumed to be constant.

The flow of the liquid is driven by a singularity within the region, for example a source or sink (corresponding to an oil well) or by having a specified flow or pressure at some fixed boundary. The moving boundary problem for pressure is investigated by replacing pressure with a transformed variable. The resulting free boundary problem has time appearing only as a parameter in known boundary conditions.

The same mathematical system models electro-chemical machining [1]. In this problem electric potential takes the place of pressure and current density replaces fluid velocity. The rate of dissolution of the anode, on which the potential is constant, is proportional to the normal current density.

2. Formulation

We scale the pressure p so that the velocity $\mathbf{v}$ satisfies

$$\mathbf{v} = -\nabla p \tag{1}$$

in the fluid-occupied region $D(t)$.

We are able to do this since the medium is homogeneous and isotropic. This equation is assumed to hold up to the interface. Edge effects, such as the fountain effect [5], are neglected. For an incompressible fluid in a homogeneous medium conservation of mass gives

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } D. \tag{2}$$

The interface is a material boundary so that it moves with normal velocity v_n equal to the normal velocity of the fluid. Also on the free boundary $\partial D(t)$,

$$p = 0. \tag{3}$$

Eliminating $\mathbf{v}$, the moving boundary problem that is to be solved becomes

$$\nabla^2 p = 0 \quad \text{in } D(t) \tag{4}$$

the normal velocity of ∂D, v_n, satisfies

$$v_n = -\frac{\partial p}{\partial n}, \tag{5}$$

also p has specified singularities in D and (3) holds on $\partial D(t)$.

3. The transformation of the dependent variable

The problem (3)–(5) has the disadvantage of time appearing implicitly in the moving boundary condition (5) through v_n. We seek to transform the problem, replacing p by a variable u which will be seen to be related to the volume occupied by the fluid, so that the time only appears as a parameter in a free boundary problem. In this way the position of the boundary may be located at any time t_1 without having to compute its location for all intervening times t, $0 < t < t_1$.

As long as the solution to (3), (4) and (5) exists, which is to say that our model is valid, the equation describing the location of the boundary $\partial D(t)$ can be written in the form

$$\omega(\mathbf{r}) = t. \tag{6}$$

We expect that the system is well behaved so that ω is a smooth function. In this case we take the total time derivative of equation (6) and use (5) to find

$$\nabla p \cdot \nabla \omega + 1 = 0 \quad \text{on } \partial D(t). \tag{7}$$

In writing ∂D in the form (6) we assume that the boundary crosses no point more than once. The quantity ω is not defined for points $\mathbf{r}$ which at no time lie on the boundary.

For any point that has been crossed during the interval $[0, t]$ we can define a new dependent variable u by

$$u(\mathbf{r}, t) = \int_{\omega(\mathbf{r})}^{t} p(\mathbf{r}, t_1)\, dt_1. \tag{8}$$

If the fluid-occupied region is expanding, u is defined for $\mathbf{r}$ in the region $D(t) - D(0)$ ($\mathbf{r}$ is in D). If the region is contracting, (8) applies in $D(0) - D(t)$, which is to say, since $\mathbf{r}$ is outside D, that p is actually the analytic continuation of the pressure.

If p and ω are sufficiently smooth (8) can be differentiated twice within the region in which u is defined:

$$\nabla^2 u = 1. \tag{9}$$

Equation (9) can be used to continue u analytically elsewhere, up to any singularities, branch cuts, or natural boundaries of u.

The definition (8) also reveals that on the boundary $\partial D(t)$

$$u = 0, \tag{10}$$

$$\frac{\partial u}{\partial n} = 0, \tag{11}$$

or equivalently (disregarding a constant),

$$\nabla u = \mathbf{0}. \tag{11'}$$

We can now see that if we know what the region D is at any time t we have a Cauchy problem, (9), (10) and (11), for u. This is necessarily ill-posed. In particular, since we know the initial boundary, $u(\mathbf{r}, 0)$ is, in principle at least, determined.

If we differentiate u with respect to time we see from (9) that

$$\nabla^2 \frac{\partial u}{\partial t} = 0 \tag{12}$$

away from singularities of u. We also have, from (8),

$$\frac{\partial u}{\partial t} = p \tag{13}$$

in some region (p is the continuation of pressure for a contracting region). Equation (4) can be used to analytically continue p. Hence, apart from the singularities and branch cuts of u and p, (13) applies everywhere. Outside $D(t)$, $\partial u/\partial t$ is simply the continuation of p; its exterior singularities may be regarded as applied singularities of pressure, which, in conjunction with the real (and known) internal singularities, give rise to a moving boundary which has normal velocity equal to $-\partial p/\partial n$ and on which p vanishes.

Integrating (13) between 0 and t gives

$$u(\mathbf{r}, t) = u(\mathbf{r}, 0) + \int_0^t p(\mathbf{r}, t_1)\, dt_1 \tag{14}$$

except at the singular points of p and u. Since *a priori* p is unknown, (14) cannot be used to determine u directly. However, the singularities of p within D, for example applied sources and sinks, are known. Thus (14) defines the corresponding interior singularities of u. The exterior singularities are of course as yet undetermined since there is no direct information on the behaviour of p outside D. Other internal singularities of $u(\mathbf{r}, t)$ are those of $u(\mathbf{r}, 0)$ which are given by the solution to (9), (10) and (11) at $t = 0$.

The variable u is related to the amount of fluid distributed in the region D. Taking some region Ω lying within D such that there is no singularity of u between $\partial\Omega$ and ∂D we see that the volume of fluid in $D - \Omega$ is

$$\int_{D-\Omega} 1\, d^3x = \int_{D-\Omega} \nabla^2 u\, d^3x = -\int_{\partial\Omega} \frac{\partial u}{\partial n} d^2x.$$

Hence the amount of fluid lying outside Ω is given by the integral of $-\partial u/\partial n$ over its boundary. (If there are no singularities of u in Ω, then the volume in Ω is $\int_{\partial\Omega} \partial u/\partial n\, d^3x$.) Any change in the volume outside ω is seen to be given by the integral of the change of $-\partial u/\partial n$, but this is also the time integral of $-\int_{\partial\Omega} \partial p/\partial n\, d^2x$ (directly from (1)). Note from (13) the identity of the two expressions.

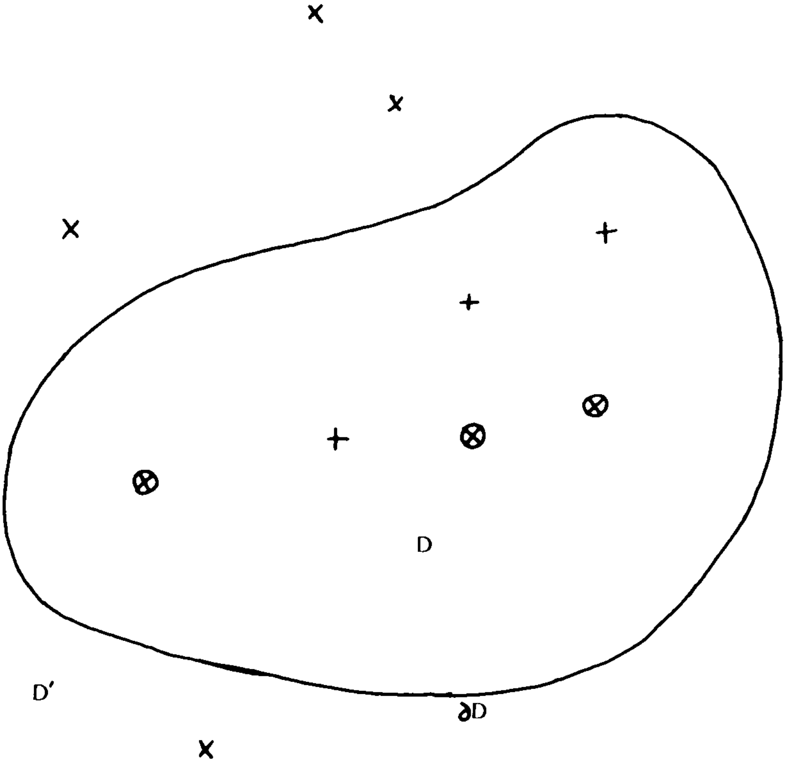

Figure 1. Fluid-occupied region D and the singularities of u. + constant singularities of $u(\mathbf{r}, t) =$ singularities of $u(\mathbf{r}, 0)$. ⊗ singularities of pressure, *e.g.* sources: the singularities of u are known but not constant. × unknown exterior singularities of u.

Applying (13) to the integral $-\int_{\partial\Omega} \partial p/\partial n \, d^2x$ for some small region around an isolated source (or sink) we again see that the increase in $-\int_{\partial\Omega} \partial u/\partial n \, d^2x$ is just the volume added at the source.

We are now in a position to restate the moving boundary problem for p as a free boundary problem for u (see Figure 1):

(i) $\nabla^2 u = 1$ in D except at the singularities and branch cuts,

(ii) u has specified internal singularities which are the singularities of $u(\mathbf{r}, 0)$ that lie within $D(0)$ together with the time integrals of the known singularities of pressure,

(iii) $u = \partial u/\partial n = 0$ on the unknown boundary ∂D.

The last condition may be replaced by $\nabla u = 0$ on ∂D. If the problem (i), (ii) and (iii) has solutions the solution which admits a continuous deformation of ∂D from $\partial D(0)$ is selected.

Having solved the problem for u and ∂D the pressure p, if required, can be determined in two ways. Firstly u can be differentiated to give $p = \partial u/\partial t$. Secondly the equation $\nabla^2 p = 0$ in D can be solved subject to $p = 0$ on ∂D with p having the required singularities.

We now give an example of the type of problem to be posed in two dimensions. We write the similar three-dimensional problem in brackets, { }, underneath.

3.1 Problem of a circle or sphere with an off-centre source.

A source, strength $Q(t)$, is located at $\mathbf{r} = \mathbf{a}$. The point $\mathbf{a}$ lies within the initial region which is a circle {sphere} of radius R centered at $\mathbf{r} = \mathbf{b}$.

It is easily found that

$$u(\mathbf{r}, 0) = \tfrac{1}{4}|\mathbf{r} - \mathbf{b}|^2 - \tfrac{1}{2}R^2 \ln|\mathbf{r} - \mathbf{b}| - \tfrac{1}{4}R^2 + \tfrac{1}{2}R^2 \ln R, \tag{15}$$

$$\left\{u(\mathbf{r}, 0) = \tfrac{1}{6}|\mathbf{r} - \mathbf{b}|^2 + \tfrac{1}{3}R^3/|\mathbf{r} - \mathbf{b}| - \tfrac{1}{2}R^2\right\}, \tag{15$'$}$$

so that $u(\mathbf{r}, 0)$ has a singularity

$$u \sim -\tfrac{1}{2}R^2 \ln|\mathbf{r} - \mathbf{b}|, \tag{16}$$

$$\left\{u \sim R^2/(3|\mathbf{r} - \mathbf{b}|)\right\}, \tag{16$'$}$$

at $\mathbf{r} = \mathbf{b}$, lying within $D(0)$.

The pressure has a singularity

$$p \sim -[Q(t)/2\pi] \ln|\mathbf{r} - \mathbf{a}|, \tag{17}$$

$$\left\{p \sim Q(t)/(4\pi|\mathbf{r} - \mathbf{a}|)\right\}, \tag{17$'$}$$

at $\mathbf{r} = \mathbf{a}$.

Thus, at time t, u has the known singularities in D:

$$-\tfrac{1}{2}R^2 \ln|\mathbf{r} - \mathbf{b}| - [A(t)/2\pi] \ln|\mathbf{r} - \mathbf{a}|, \tag{18}$$

$$\left\{R^3/(3|\mathbf{r} - \mathbf{b}|) + A(t)/(4\pi|\mathbf{r} - \mathbf{a}|)\right\}, \tag{18$'$}$$

where

$$A(t) = \int_0^t Q(t_1)\, dt_1 \tag{19}$$

is the amount of fluid added at $\mathbf{r} = \mathbf{a}$ between 0 and t. The two-dimensional problem of off-centre blowing has been solved in [4] using a complex variable method (see also below).

We are not limited to considering cases where p has isolated singularities. If fluid is added or removed through a line or surface (line or surface sources or sinks) then p or $\partial p/\partial n$, or indeed some linear combination of the two, may be known on a fixed line or surface. In this case (14) can be applied to give the appropriate condition for u at this fixed boundary (see Figure 2). If this boundary is closed it must be remembered that the contained region is not part of D.

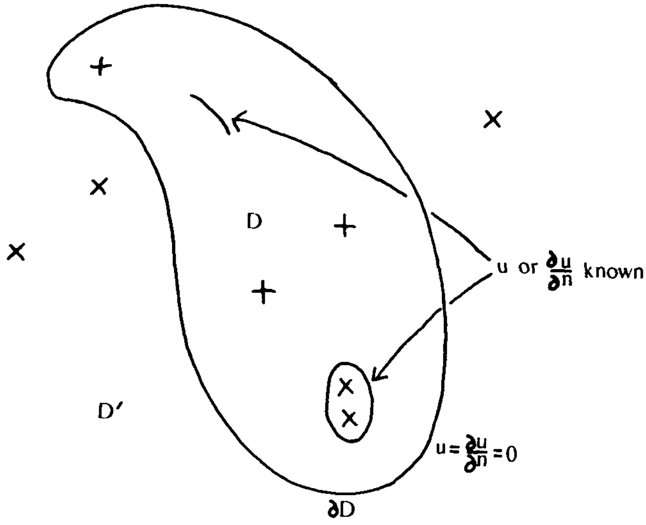

Figure 2. Distributed pressure singularities. + known singularity of u. × unknown singularity of u.

4. Application of complex variable theory

For two-dimensional problems we write $\mathbf{r} = (x, y)$ and $z = x + iy$. Since $\nabla^2 u = 1$, the difference $u - \frac{1}{4}(x^2 + y^2)$ is harmonic. So $\partial/\partial x\{u - \frac{1}{4}(x^2 + y^2)\} - i\,\partial/\partial y\{u - \frac{1}{4}(x^2 + y^2)\} = \partial u/\partial x - i\,\partial u/\partial y - \frac{1}{2}(x - iy)$ is an analytic function of z away from singularities of u.

If ∂D is analytic its equation may be written in the form

$$\bar{z} = g(z) \tag{20}$$

where g is analytic in some neighbourhood of ∂D, [4].

On this boundary $\partial u/\partial x = \partial u/\partial y = 0$ so

$$\partial u/\partial x - i\,\partial u/\partial y - \tfrac{1}{2}(x - iy) = -\tfrac{1}{2}\bar{z} = -\tfrac{1}{2}g(z). \tag{21}$$

Both sides of (21) are analytic so this equation holds everywhere, apart from singularities and branch cuts. Thus u and g are related by

$$\partial u/\partial x - i\,\partial u/\partial y = \tfrac{1}{2}[\bar{z} - g(z)]. \tag{22}$$

Equations (22) and (14) yield a relationship between $g(z, t)$ and $g(z, 0)$:

$$g(z,t) = g(z,0) - 2\int_0^t \{\partial p/\partial x(z, t_1) - i\,\partial p/\partial y(z, t_1)\}\,dt_1. \tag{23}$$

Equation (23) can be used directly to find the internal singularities of g.

The relationship between the internal singularities of g and those of p obtained from (23) was found for the special case of a single point source (or sink) in a bounded region in [4].

Knowing the internal singularities of g is equivalent to knowing g_e where $g = g_e + g_i$ and g_i is analytic in D whereas g_e is analytic outside D. This decomposition for g can always be carried out for bounded D. Defining g_e and g_i by

$$g_e(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{g(z_1)}{z - z_1}\,dz_1, \qquad z \text{ outside } D, \tag{24}$$

$$g_i(z) = -\frac{1}{2\pi i}\int_{\partial D} \frac{g(z_1)}{z - z_1}\,dz_1, \qquad z \text{ inside } D, \tag{25}$$

results in the required functions.

As a simple example of the use of (23) we consider the problem of a source or sink at $x = 1$, $y = 0$ in a region D which is initially the right hand half plane $x > 0$.

At $t = 0$ the boundary ∂D is $x = 0$ which may be written

$$\bar{z} = -z, \tag{26}$$

and $g(z, 0) = -z$ has the singularity $-z$ located at infinity in $D(0)$. The source at $z = 1$ has strength $Q(t)$ so

$$p \sim -[Q(t)/2\pi]\ln|z - 1| \quad \text{as } z \to 1. \tag{27}$$

($Q < 0$ for a sink).

We deduce from (23) that the internal singularities of g at time t are at $z = 1$ and at $z = \infty$.

$$g(z,t) \sim -z + o(1) \quad \text{as } z \to \infty, \tag{28}$$

$$g(z,t) \sim A(t)/\left(\pi(z - 1)\right) + O(1) \quad \text{as } z \to 1, \tag{29}$$

where $A(t) = \int_0^t Q(t_1)\,dt_1$.

We now conformally map the unit circle in the ζ plane onto D with a function f: $z = f(\zeta)$. We specify that the point $\zeta = 0$ is mapped onto $z = 1$ and that the boundary point $\zeta = 1$ is mapped to $z = \infty$. The latter condition requires that f has a singularity at $\zeta = 1$. Comparing with the transformation at $t = 0$ (where $A = 0$), $f = (1 + \zeta)/(1 - \zeta)$, we expect f to have a simple pole at $\zeta = 1$.

The boundary condition $\bar{z} = g(z)$ is now used. For $|\zeta| = 1$, $\bar{\zeta} = 1/\zeta$ so

$$\overline{f(\zeta)} = g(f(\zeta)) = \bar{f}(1/\zeta). \tag{30}$$

Hence $\bar{f}(1/\zeta) = g(f(\zeta))$ away from the boundary since both sides are analytic functions of ζ.

For $|\zeta| < 1$, $f(\zeta)$ is analytic and $z = f(\zeta)$ lies in D, where $g(z)$ has the known singularities (28) and (29). Also $|1/\zeta| > 1$ so $\bar{f}(1/\zeta)$ can have singularities. Letting ζ tend to zero we find that $\bar{f}(1/\zeta)$ must have a simple pole at $\zeta = 0$.

This suggests that $f(\zeta)$ is of the form $a/(\zeta - 1) + b\zeta + c$. From the choice of $f(0)$ and $f(1)$ the constants a, b, c will be real and $\bar{f} = f$.

Taking the limits $\zeta \to 0$ and $\zeta \to 1$ in (30) gives the values of a, b and c:

$$a = -\tfrac{2}{3}(\rho + 2), \tag{31}$$

$$b = \tfrac{1}{3}(\rho - 1), \tag{32}$$

$$c = -\tfrac{1}{3}(2\rho + 1), \tag{33}$$

where

$$\rho = [1 + 3A(t)/\pi]^{1/2}. \tag{34}$$

The continuity of the boundary, which requires that $b \to 0$ as $A \to 0$, is used in determining the sign of ρ.

The above solution applies for $A \geqslant -\pi/3$. At the critical value $A = -\pi/3$ a cusp develops in the boundary (located at $z = \frac{2}{3}$) and the solution cannot be extended for more negative A. This indicates that the model we have been using can no longer be valid for the physical problem. As A approaches $-\pi/3$ terms that we have been neglecting, *e.g.*, inertial terms or the variation of pressure in the less viscous fluid, become significant. The method of determining the mapping function f by comparison of the internal singularities comes from [4]. In [4] the method is limited to bounded regions unlike the above example (however this may be thought of as a limiting case of bounded regions D, say circles).

It may be noted that for more singularities the algebraic equations derived from consideration of the singularities become too complicated to solve explicitly. Moreover, for the problems for which this method of determining f works and which have a single source or sink, the function f can actually be found directly from the motion of the boundary.

If the internal singularities g at $t = 0$ are of types other than poles we are no longer able to compare the singularities to obtain a finite system of algebraic equations for the parameters in f. This happens for the initial region being an ellipse. In this case there are square-root singularities at the foci. To get the correct corresponding singularities at later time the terms which are locally $z^{3/2}, z^{5/2}, \ldots$ near the foci must be considered in addition to the basic $z^{1/2}$.

5. Three-dimensional problems

The equation $\bar{z} = g(z)$ for the two-dimensionl analytic boundary ∂D may be written in the form

$$x - iy = \Phi_x - i\Phi_y \tag{35}$$

for some harmonic function Φ. For the three-dimensional problem with an analytic boundary we write ∂D as

$$\mathbf{r} = \nabla\Phi \tag{36}$$

where Φ is harmonic in some neighbourhood of the surface. With such a Φ we can easily verify that up to a constant

$$u = \tfrac{1}{6}\sum_j r_j^2 - \tfrac{1}{3}\Phi. \tag{37}$$

Taking the gradient of (37),

$$\nabla u = \tfrac{1}{3}(\mathbf{r} - \nabla\Phi) \tag{38}$$

which vanishes on the boundary. Thus $\partial u/\partial n$ vanishes and u is a constant on ∂D. This constant is zero if the arbitrary constant in Φ is chosen suitably. The divergence of (38) shows that $\nabla^2 u = 1$.

The free boundary problem for u in Section 3, (i)–(iii) can now be re-posed in terms of the function Φ. From (14) and (37)

$$\Phi(\mathbf{r}, t) = \Phi(\mathbf{r}, 0) - 3\int_0^t p(\mathbf{r}, t_1)\, dt_1. \tag{39}$$

Hence the internal singularities of $\Phi(\mathbf{r}, 0)$ and the specified singularities of pressure are sufficient to give the internal singularities of $\Phi(\mathbf{r}, t)$. If Φ is decomposed so that $\Phi = \Phi_i + \Phi_e$ with Φ_i analytic in D and Φ_e analytic outside D, then the singularities of Φ_e are known.

5.1 Relationship between the variable u and a gravitational potential

We note in passing that for bounded regions D there is a relationship between u and the gravitational potential U due to matter of unit density occupying D. They are related by

$$u = U_i - U_e, \tag{40}$$

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where U_i is the internal potential or its analytic continuation into the exterior of D ($\nabla^2 U_i = 1$), and U_e is the external potential or its continuation into D ($\nabla^2 U_e = 0$). The singularities of U_e and U_i lie within and outside D respectively. The boundary conditions for u, (10) and (11) follow from the continuity of U and ∇U at ∂D. Using (40) and the results of [6] we can derive (24) and (25) in two dimensions. In three dimensions (40) and [7] give (37). Papers [6] and [7] were concerned with the location of concentrations of heavy minerals from the measurement of their resulting gravitational potential U_H outside D. The quantity U_H is given by $U_M - U_e$ where U_M is the total measured potential while U_e is the potential from an assumed unit mass density in D. The problem of determining ∂D that we are discussing here is an inverse problem: we wish to find D from knowledge of the external potential U_e assuming that the mass in D has unit density.

It may be possible to use the equivalence to some effect. Any known solutions to the gravitional problem with external potential U_e provide solutions to the free boundary problem for u with internal singularities equal and opposite to those of U_e.

6. Examples

The one-dimensional problem of sucking from a point, at which there is a jump discontinuity in the slope of pressure, may be solved very simply either directly or by the u method. Two-dimensional problems may be solved by determining a mapping function f from the unit circle to D by examination of the singularities of g (see [4] or above).

6.1 Distributed singularity in an oblate spheroid

For three dimensions there is a difficulty in that there is no result analogous to the composition of analytic complex functions being analytic as in two dimensions. This property was crucial to the success of the solution to two-dimensional problems.

We can use (10) and (11′) to derive the initial singularities of an oblate spheroid. This result could be used to determine the position of the boundary after sucking or blowing through a disc has occured in certain special ways.

Taking an oblate spheroid to have its axis of symmetry along the z-axis with its centre at the origin, the oblate spherical co-ordinates ξ, η, φ are used:

$$x = \kappa \cosh \xi \cos \eta \cos \varphi, \tag{41}$$

$$y = \kappa \cosh \xi \cos \eta \sin \varphi, \tag{42}$$

$$z = \kappa \sinh \xi \sin \eta. \tag{43}$$

The oblate spheroid

$$(x^2 + y^2)/\cosh^2\xi_0 + z^2/\sinh^2\xi_0 = \kappa^2 \tag{44}$$

is given by

$$\xi = \xi_0. \tag{45}$$

The axisymmetric function u is independent of φ so

$$\begin{aligned}\nabla^2 u = &\left\{\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\cosh\xi\right)/\cosh\xi + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\cos\eta\right)/\cos\eta\right\} \\ &\times\left\{\kappa^2(\sinh^2\xi + \sin^2\eta)\right\}^{-1} = 1.\end{aligned} \tag{46}$$

The boundary conditions (11′) may be written

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta} = 0. \tag{47}$$

The solution is

$$\begin{aligned}u = &(\kappa^2/6)(\cos^2\eta + \sinh^2\xi) + A_1\tan^{-1}\sinh\xi + A_2 + (3\sin^2\eta - 1) \\ &\times\left\{A_3(3\sinh^2\xi + 1) + A_4(3\sinh^2\xi + 1)\tan^{-1}\sinh\xi + 3A_4\sinh\xi\right\},\end{aligned} \tag{48}$$

where A_1, A_3 and A_4 are chosen to satisfy (47) so

$$A_1 = -(\kappa^2/3)\cosh^2\xi_0\sinh\xi_0 \tag{49}$$

and

$$A_4 = -(\kappa^2/12)\cosh^2\xi_0\sinh\xi_0. \tag{50}$$

From (48), (49) and (50) this u is continuous but has a jump in its z derivative across the disc $x^2 + y^2 < \kappa^2$, $z = 0$ (see Figure 3):

$$\frac{\partial u}{\partial z} \sim \pm\cosh^2\xi_0\sinh\xi_0(\kappa^2 - x^2 - y^2)^{1/2} \quad \text{as } z \to 0\pm. \tag{51}$$

This initial singularity can be used to predict the deformation of the spheroid given a distributed source or sink on the disc $x^2 + y^2 < \kappa^2$, $z = 0$.

Taking note of the above solution u we may solve a class of problems of blowing through a disc when there is initially no fluid present. We expect to be able to find a solution of the form of (48) if we can utilise oblate spherical co-ordinates. Consider the case of addition of fluid through the disc $x^2 + y^2 < \kappa^2$, $z = 0$ such that:

(i) The pressure gradient normal to the disc (the normal fluid velocity) is a smooth function of distance from the centre of the disc $(x^2 + y^2)^{1/2}$, and is independent of φ (where $y/x = \tan\phi$).

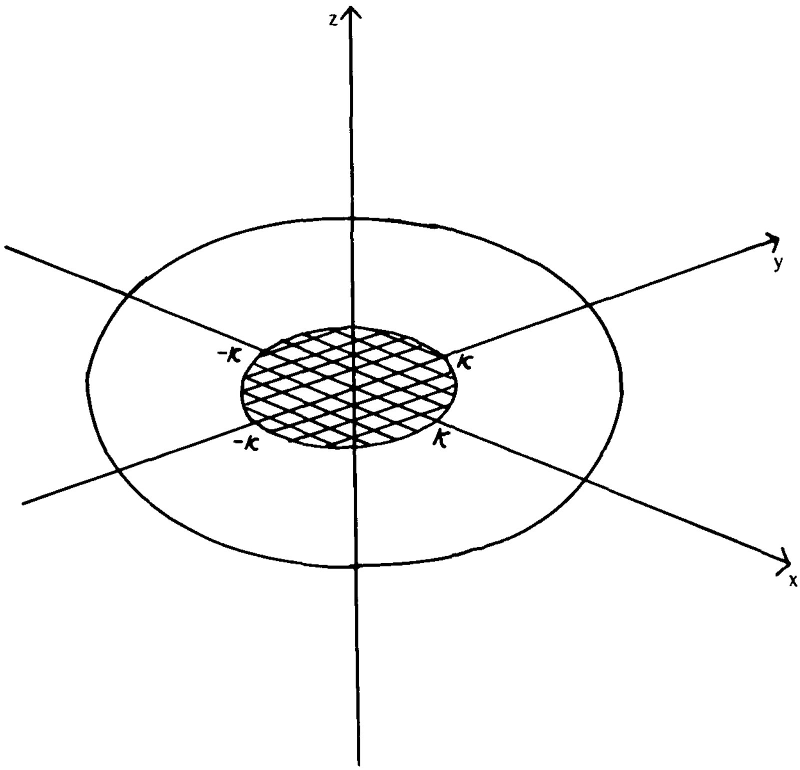

Figure 3. Singularities of u for an oblate spheroid. The shaded region denotes the surface of the jump in $\partial u/\partial z$.

(ii) The normal pressure gradient is the same above and below the disc so that there is symmetry about the x, y plane.

$$\frac{\partial p}{\partial n}(x, y, 0, t) = P_1(x^2 + y^2, t). \tag{52}$$

(iii) No fluid is added at the edge of the disc, in other words the normal derivative of pressure vanishes at the edge.

$$P_1 \to 0 \quad \text{as } x^2 + y^2 \to \kappa^2. \tag{53}$$

(iv) P_1 is non-negative for $x^2 + y^2 < \kappa^2$.

A simple case satisfying (i)–(iv) is

$$P_1(x^2 + y^2, t) = (\kappa^2 - x^2 - y^2)^{1/2} P_2(t). \tag{54}$$

For this example the function u satisfies

$$-\frac{\partial u}{\partial z}(x, y, 0+, t) = \frac{\partial u}{\partial z}(x, y, 0-, t) = (\kappa^2 - x^2 - y^2)^{1/2} B(t) \tag{55}$$

where

$$B(t) = \int_0^t P_2(t_1)\,dt_1. \tag{56}$$

Equation (56) follows from the z derivative of (14) with $u(\mathbf{r}, 0) \equiv 0$.

By comparison with (51) we deduce that the region occupied by the fluid produced entirely by the distributed source

$$\frac{\partial p}{\partial z}(x, y, 0\pm, t) = \mp P_2(t)\left(\kappa^2 - x^2 - y^2\right)^{1/2}, \qquad x^2 + y^2 < \kappa^2,$$

is the oblate spheroid

$$\left(x^2 + y^2\right)/\cosh^2\xi_0 + z^2/\sinh^2\xi_0 < \kappa^2$$

where ξ_0 and t are related by

$$\cosh^2\xi_0 \sinh\xi_0 = B(t). \tag{57}$$

Equally well, we can see that if we initially have the oblate spheroid with boundary $\partial D(0)$ given by (44) then if we suck through the disc,

$$\frac{\partial p}{\partial z}(x, y, 0\pm, t) = \pm P_2(t)\left(\kappa^2 - x^2 - y^2\right)^{1/2}, \qquad x^2 + y^2 < \kappa^2,$$

then the region shrinks to the disc when $B(t) = \cosh^2\xi_0 \sinh\xi_0$.

Similar results may be obtained for a prolate spheroid. This has a distributed logarithmic singularity on the straight line joining its foci.

7. The external singularities of u and D

As can be seen from the example of off-centre suction from a circle, [4], the solution to the problem ceases to exist if an external singularity of g approaches the boundary. This, together with the relationships between g, u and $p(= \partial u/\partial t)$, suggests in general the importance of locating all the singularities of u. Clearly the solution "blows-up" if a singularity of p, and hence of u, reaches ∂D.

It has been shown that the internal singularities of u are determined by the initial boundary shape through $u(\mathbf{r}, 0)$ and by the singularities of pressure within D. By the very definition of internal singularity, the boundary ∂D never crosses any of these points. For a system with a contracting region, if an internal singularity were approached at some time the boundary would become non-analytic at that point so the solution would fail to exist for later times. However, for some simple two-dimensional problems with suction from a point (such as the example in Section 4 or that of [4]) it is easily seen that the model breaks down with non-analyticity in ∂D due to an external singularity overtaking the boundary. The form of this external singularity of g, being associated with a stationary value of the mapping function f, is in general that of a square root.

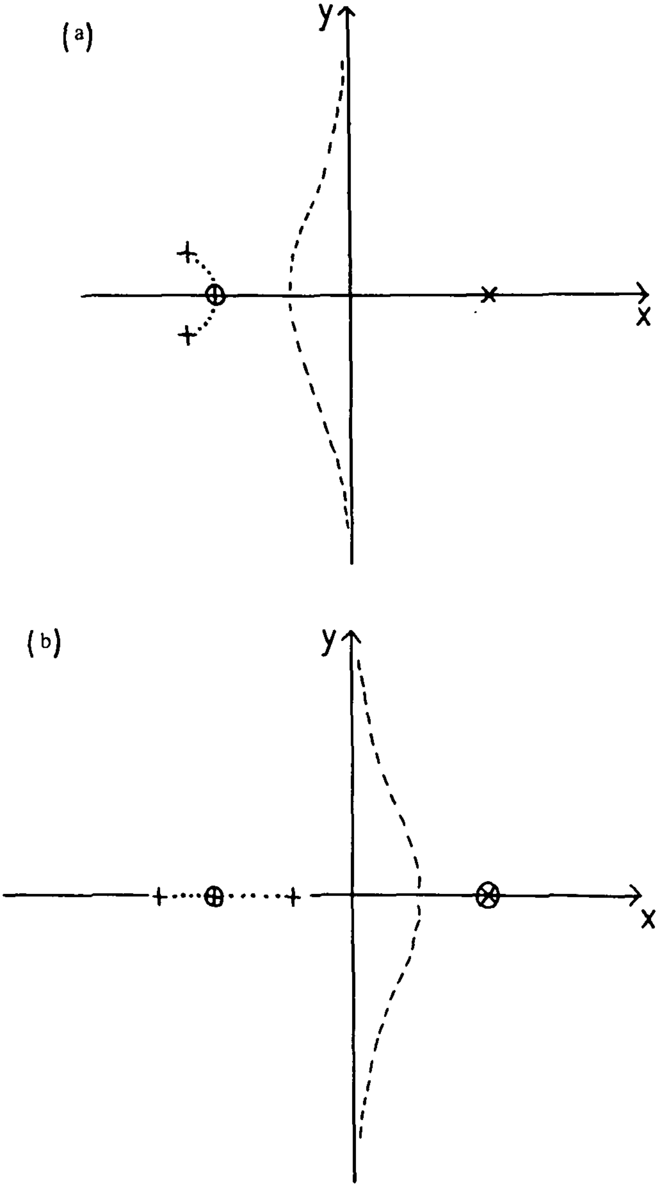

Figure 4. The movement of external singularities for an initial half-plane $x > 0$, (a) blowing, (b) suction. $\times$ point source, $\otimes$ point sink. $\oplus$ position of external singularity at $t = 0$, $+$ external square-root singularity. --- deformed boundary, $\cdots$ trajectory of external square-root singularity.

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The typical behaviour upon the approach of such a singularity at time t_c is for both the singularity and boundary to move at a speed of $O([t_c - t]^{-1/2})$. The singularity in g becomes of type $z^{3/2}$ and the boundary forms a cusp at $t = t_c$. This behaviour is most easily seen in the example of a point sink located at $z = 0$ within the cardioid whose boundary is described by

$$z = a(t)\zeta + b(t)\zeta^2 \tag{58}$$

for $|\zeta| = 1$. Here $0 < b < \frac{1}{2}a$ for $t < t_c$ (see [3]). The cusp forms at $z = b(t_c) - a(t_c)$, t_c being given by $a(t_c) = 2b(t_c)$.

Previously considered examples ([4] or above) show the following features:

(i) Initially the pressure has an induced logarithmic singularity at a point, z_s, outside D. This point is related to the position of the applied singularity by reflection for the half plane and by conjugacy for off-centre suction or blowing in a circle.

(ii) The function g develops two branch points of square-root type with a branch cut linking them. These form in pairs at the point z_s in (i). For blowing they initially move parallel to the boundary (Figure 4a), whereas for suction one moves directly towards the boundary while the other moves directly away (Figure 4b).

(iii) There is no induced singularity at infinity.

(iv) The function g_t may be written as a Cauchy integral along the branch cut joining the square-root singularities.

(v) For small values of time, so that the external singularities are close to the point z_s at which they form, and for distances from the singularities much greater than the length of the branch cut, to leading order p behaves as a logarithm about z_s. The leading term for g has a simple pole at z_s.

In connection with property (v), solutions for small time may be sought as asymptotic expansions for $t \to 0$. In particular the half-plane problem can be discussed in this way. The expansions for p and s, where the boundary ∂D is written in the form $x = s(y, t)$, are power series in t:

$$p \sim p_0 + tp_1 + \cdots, \qquad s \sim ts_1 + t^2 s_2 + \cdots.$$

The leading term p_0 has the logarithmic behaviour of (i) and (v) at $z = z_s$. Using p_0 to determine s_1 the position of the boundary is correctly found to an error of $O(t^2)$. Then p_1 is found to have $r^{-2}\cos 2\theta$ type behaviour near the external singularity.

In general in the expansion of p the $O(t^m)$ term p_m has a singularity of type $r^{-2m}\cos 2m\theta$ near z_s. Thus (23) predicts that in the expansion $g \sim g_0 + tg_1 + \cdots$,

$$g_m \sim a_m / (z - z_s)^{2m+1} \quad \text{as } z \to z_s, \qquad m = 0, 1, \ldots, \tag{59}$$

where the a_m are constants.

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The expansions of p and g are non-uniform for $|z - z_s| \sim O(t^{1/2})$. To determine the precise nature of the external singularities, or, equivalently, to evaluate the leading term in the expansion for g in the small region $|z - z_s| \sim O(t^{1/2})$, the g_m must be determined for all m since there are no boundary conditions imposed near z_s. Matching between the regions $|s - z_s| \sim O(1)$ and $|z - z_s| \sim O(t^{1/2})$ could then be used to find each term in the expansion of g in the "inner region" as a descending power series in $(z - z_s)^2/t$. Effectively the full problem must be solved to obtain p to leading order everywhere although t is very small.

For the half-plane problem we can deduce some information about the boundary by writing g_i as an integral as in (iv) and examining the consequences of $\bar{z} = g(z)$ on the boundary.

We have seen above that, apart from the role played by the external singularities in describing the boundary through $\mathbf{r} = \nabla(\Phi_i + \Phi_e)$, they can have the additional direct importance in the breakdown of the model as they approach ∂D. We believe that in general the boundary fails to exist due to an exterior singularity catching up with the boundary both for two and three dimensions. If the boundary approached an internal singularity at $r = 0$ where $u \sim O(r^\beta)$ as $r \to 0$ for some $\beta < 1$ then, unless some exterior singularity was also in the neighbourhood of $r = 0$, $\partial u/\partial n$ would become large, contradicting (11).

It is desirable to have some means of calculating these external singularities for three dimensional problems.

For the problem of suction from $(0, 0, 1)$ in the half-space $z > 0$ a small time perturbation analysis can be used to find there is an induced singularity of pressure,

$$p \sim Q(0)/\left(4\pi\left[x^2 + y^2 + (z+1)^2\right]^{1/2}\right), \qquad (Q > 0) \quad \text{at} \quad (0, 0, -1).$$

The position of the perturbed boundary is

$$z \sim Q(0)t/\left(2\pi\left(x^2 + y^2\right)^{1/2}\right) + O(t^2) \quad \text{for } t \to 0. \tag{60}$$

Following the ideas suggested by (iii) and (iv) we can try to write Φ_i as an integral along some interval of the negative z-axis. [We conjecture that for three dimensions the singularities of Φ are, in general, of three types: (a) Point $1/r$ singularities as for a sphere, (b) logarithmic singularities along an arc as for a prolate spheroid, (c) jumps in derivative across a surface as for an oblate spheroid.] However, after consideration of large values of $x^2 + y^2$ we are led to the conclusion that the boundary moves a distance of less than $O([x^2 + y^2]^{-m/2})$

for any m. This contradicts the small time analysis. We find a similar contradiction if we assume that

$$\Phi_t = \sum_{m=1}^{\infty} a_m r_1^{-m} P_{m-1}((z+1)/r_1)$$

where $r_1 = [x^2 + y^2 + (z+1)^2]^{1/2}$ and the P_m are the Legendre polynomials.

This suggests that in this case at least some singularities of Φ (and p) are induced at infinity. This possibility is given some strength by the two-dimensional example of a sink within an ellipse. In this case the small time solution indicates an infinity of induced external singularities of p; the set of these singularities is unbounded.

8. Some limits on the existence of a solution

We shall now use the function u to estimate some upper bounds on the times for which solutions to some of our problems can exist. The fact that the boundary cannot cross internal singularities of u (as noted in the previous section) will be applied.

If there is suction from a fixed sink at $\mathbf{r} = \mathbf{a}$ in D, $p \sim -Q(t)/(4\pi|\mathbf{r}-\mathbf{a}|)$ as $\mathbf{r} \to \mathbf{a}$ ($Q > 0$). For such a system for $\mathbf{r}$ in D with $\mathbf{r} \neq \mathbf{a}$, $p < 0$. We define a potential ϕ in D by

$$\phi = p + Q(t)/(4\pi|\mathbf{r}-\mathbf{a}|). \tag{61}$$

If we suppose that the point $E(t)$ is the point of the boundary nearest to $\mathbf{a}$, at a distance $\rho(t)$, then the sphere $|\mathbf{r}-\mathbf{a}| = \rho$ is tangent to ∂D at E and lies within D elsewhere. Thus $\phi = Q(t)/4\pi\rho$ at E and $\phi \leqslant Q(t)/4\pi\rho$ on $|\mathbf{r}-\mathbf{a}| = \rho$. Since $\nabla^2\phi = 0$ in $|\mathbf{r}-\mathbf{a}| < \rho$ the maximum principle leads to $\phi \leqslant Q(t)/4\pi\rho$ in D. Hence at E

$$\frac{\partial\phi}{\partial n} \geqslant 0 \quad \text{and} \quad \frac{\partial p}{\partial n} \geqslant Q(t)/4\pi\rho^2. \tag{62}$$

The inward normal speed of the boundary at E is greater than or equal to $Q(t)/4\pi\rho^2$, so $d\rho/dt \leqslant -Q(t)/4\pi\rho^2$ and

$$\rho(t) \leqslant [\rho(0) - 3A(t)/4\pi]^{1/3}, \tag{63}$$

where $A(t) = \int_0^t Q(t_1)\,dt_1$.

Equation (63) predicts that ρ must vanish no later than the time t_0 at which $\rho^3(0) = 3A(t_0)/4\pi$. We deduce that the solution to the problem exists up to some time t_c where $t_c \leqslant t_0$.

Using the estimate of (63) to obtain an upper bound on t_c we find that, for off-centre suction from a sphere of radius R and centre $\mathbf{b}$,

$$A(t_c) \leqslant 4\pi(R - |\mathbf{a} - \mathbf{b}|)^3/3.$$

For the half-space problem of suction from $(0, 0, 1)$ the estimate is

$$A(t_c) \leqslant 4\pi/3.$$

Similar analysis in two dimensions (for the purposes of gaining information about some of the unsolvable problems, for example, the ellipse) gives $A(t_c) \leqslant \pi\rho^2(0)$. For the half plane problem this estimate gives $A(t_c) \leqslant \pi$ compared to the true value $A(t_c) = \pi/3$ (see Section 4).

The estimate $A(t_c) < 4\pi\rho^3(0)/3$ comes from the boundary failing to travel past the sink. We now wish to use the fact that ∂D cannot cross the internal singularities of $u(0)$.

Clearly the problem of off-centre suction from a sphere does not admit an improved estimate using this property. We shall consider instead a dipole in a sphere.

8.1 A dipole singularity of pressure within a sphere

At $\mathbf{r} = (0, 0, a)$, which lies in $D(0)$, the unit sphere $|\mathbf{r}| < 1$, there is a dipole so that

$$p \sim (z - a)\left[x^2 + y^2 + (z - a)^2\right]^{-3/2} \quad \text{as } x, y \to 0 \text{ and } z \to a.$$

At $t = 0$ we can find p exactly:

$$p = (z - a)\left[x^2 + y^2 + (z - a)^2\right]^{-3/2}$$
$$+ a^{-2}(x^2 + y^2 + z^2 - z/a)\left[x^2 + y^2 + (z - 1/a)^2\right]^{-3/2}.$$

From this we find that initially the point of the boundary which moves most quickly inwards is at $(0, 0, -1)$: this is where $\partial p/\partial n$ is largest. We expect at any later time the point which is travelling inwards most rapidly to be that which lies on the negative z-axis. (The speed is expected to increase as the distance to $(0, 0, a)$ decreases, moreover the decrease in curvature of the boundary will, being most marked where the inward velocity is greatest, tend to reinforce this effect.)

Examining the potential function $\phi = p - (z - a)[x^2 + y^2 + (z - a)^2]^{-3/2}$ in a manner similar to that for the sink, we find that the inward speed of the boundary at the point E, where E is the point on ∂D at which $(z - a)[x^2 + y^2 + (z - a)^2]^{-3/2}$ is least, is greater than or equal to $\rho^{-4}[\rho^2 + 3(z - a)^2]^{1/2}$. Here, ρ is the distance between $(0, 0, a)$ and E.

From the above E will be lying on the negative z-axis at $(0,0,-[\rho - a])$. From consideration of the speed of E

$$\frac{d\rho}{dt} \leqslant -2/\rho^3.$$

Initially $\rho = 1 + a$ so at later times

$$\rho(t) < \left[(1+a)^4 - 8t\right]^{1/4}. \tag{64}$$

From (64) the boundary must reach $(0,0,0)$ by

$$t = \tfrac{1}{8}\left[(1+a)^4 - a^4\right].$$

Since there is a singularity at $(0,0,0)$, $t_c \leqslant \frac{1}{8}[(1+a)^4 - a^4]$.

8.2 Two-dimensional parabola with suction from a point on the axis

Finally as a two-dimensional example we consider a point sink $p \sim \ln|z+a|$, where $z = x + iy$, on the axis of symmetry of the parabola $x = \frac{1}{2}(1-y^2)$ (see Figure 5). The function g at $t = 0$ is $2 + z - 2(2z)^{1/2}$ where $z^{1/2}$ is real positive for z real positive and the branch cut of $z^{1/2}$ lies along the negative real axis. We now seek conditions on the number a so that initially the boundary point moving most quickly is at $z = \frac{1}{2}$ so we may expect that the boundary point closest to $z = 0$ (and the fastest travelling) lies on the positive real axis.

Writing $z = \frac{1}{2}\zeta^2$ we conformally map the region $0 < \xi < 1$, where $\zeta = \xi + i\eta$, onto $D(0)$ less the branch cut. The line $\xi = 1$ is mapped onto $\partial D(0)$ and $\xi = 0$ onto the branch cut. We must solve

$$\nabla_\zeta^2 p = 0$$

in the strip, subject to $p = 0$ on $\xi = 1$, $\partial p/\partial\xi = 0$ on $\xi = 0$ for $\eta \neq 0$, and $p \sim \ln|\zeta \pm ib|$ as $\zeta \to \mp ib$ where $a = \frac{1}{2}b^2$. The solution gives

$$\frac{\partial p}{\partial \xi} = -\frac{\pi}{2}\operatorname{Re}\{1/\sin[\pi(\zeta+ib)/2] + 1/\sin[\pi(\zeta - ib)/2]\}.$$

Since $\partial p/\partial n$ on $\partial D(0)$ is given by $\partial p/\partial n = (1+\eta^2)^{-1/2}\partial p/\partial\xi$, we can now determine those values of a which give rise to a local maximum of $\partial p/\partial n$ at $\eta = 0$ (which corresponds to $z = \frac{1}{2}$).

Since

$$\frac{\partial p}{\partial n} = \frac{\pi}{2}(1+\eta^2)^{-1/2}\left\{1/\cosh\frac{\pi}{2}(\eta+b) + 1/\cosh\frac{\pi}{2}(\eta+b)\right\},$$

$$\frac{\partial p}{\partial n} \sim \pi\{1 + (2T^2 - 3/2)\eta_1^2 + (2T^4 - 10T^2/3 + 31/24)\eta_1^4 + \cdots\}/2\cosh\frac{\pi b}{2}$$

for small $\eta_1 = \pi\eta/2$. Here $T = \tanh(\pi b/2)$.

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Hence we need to have $b < 2\tanh^{-1}(\sqrt{3}/2)/\pi$. Taking a with $0 \leqslant a \leqslant 2\{\pi^{-1}\tanh^{-1}(\sqrt{3}/2)\}^2$ the condition that the boundary does not cross $z = 0$ together with the two-dimensional form of (63) yields $t_c \leqslant \frac{1}{2}[(a + \frac{1}{2})^2 - a^2] = \frac{1}{2}a + \frac{1}{8}$.

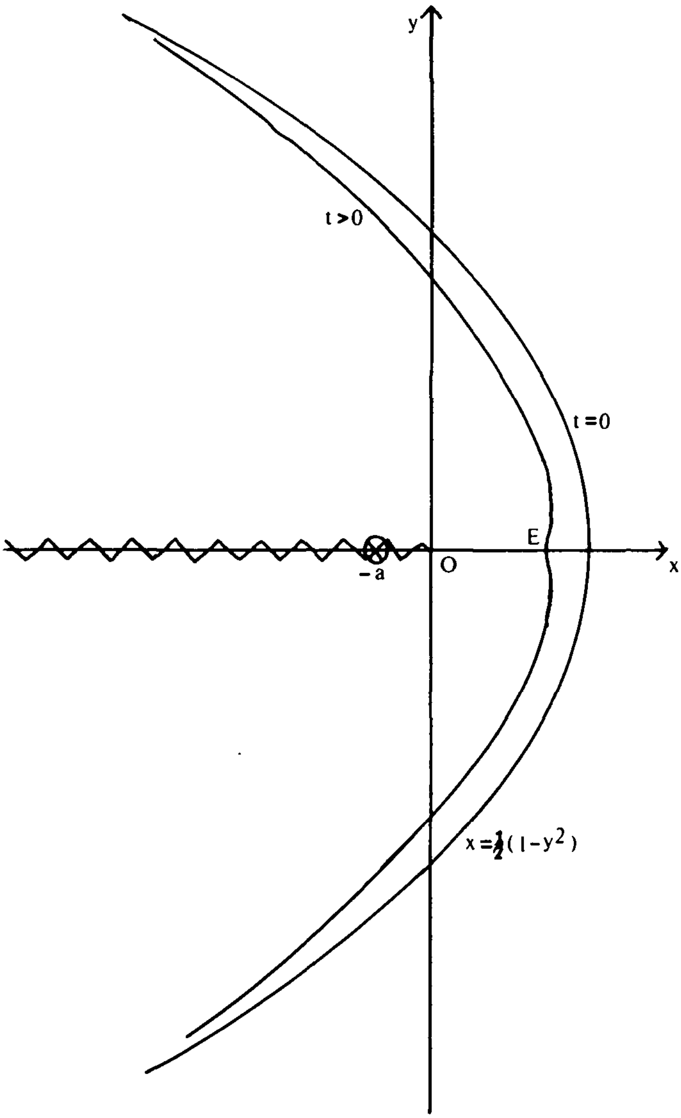

Figure 5. Suction from a parabola.

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9. Discussion of results

We have found that the moving boundary problem (1)–(3), (5) describing the flow of a liquid in a porous medium with the liquid in contact with an immiscible fluid of much lower viscosity can be related to a free boundary problem for the variable u, which is a transformation of the pressure. The new problem takes the form of Poisson's equation for u inside the unknown region except at certain known singularities, with u and $\partial u/\partial n$ both zero on the free boundary.

For two-dimensional problems (Section 4) where the initial boundary is described in the form $\bar{z} = g_0(z)$ (where $z = x + iy$) and g_0 is an analytic function, the problem can be solved exactly if the driving singularities of pressure take the form of stationary point sources, sinks and dipoles (or other reasonably simple isolated types) and g_0 is a rational function. To do this the function describing the boundary, g, is related to the variable u so that the internal singularities of g are known. For other initial boundaries the problem can still be solved approximately by taking some rational approximation g_r in place of the true g_0.

For three-dimensional problems we are limited in our use of the method. We are able to determine the internal singularities of u_0 (the initial function u) and hence those of u at any later time (see *e.g.*, Section 3.1); so in principle we can determine u, ∂D and p at any time $t > 0$ without having to consider intermediate times. However, except for certain special cases, such as a spheroid with suction or blowing in an appropriate manner from a distributed singularity (Section 6.1), we are unable to solve general problems exactly.

But, as we have seen in Section 8, it is still possible to obtain information about the behaviour of some problems, even if we cannot solve them exactly, by use of the variable u. We have been able to obtain bounds on the time for which the model is valid for some problems involving suction (considering a dipole to be a combination of a source and a sink), which exhibit the appearance of a singularity in the boundary at some finite time, this being less than our bound.

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