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HYPERCENTRAL INJECTORS IN INFINITE SOLUBLE GROUPS

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The Carter subgroups of a finite soluble group may be characterised either as the self-normalising nilpotent subgroups or as the nilpotent projectors. Subgroups with properties analogous to both of these have been considered by Newell (2, 3) in the class of $\mathfrak{S}_1$-groups. The results obtained are necessarily less satisfactory than in the finite case, the subgroups either being almost self-normalising (i.e. having finite index in their normaliser) or having an almost-covering property. Also the subgroups are not necessarily conjugate but lie in finitely many conjugacy classes.

Here we consider the dual concept of nilpotent injectors. A nilpotent subgroup V of a finite soluble group G is a nilpotent injector if, for each subnormal subgroup S of G, $V \cap S$ is a maximal nilpotent subgroup of S. These subgroups may also be characterised as the maximal nilpotent subgroups containing the Fitting subgroup. In $\mathfrak{S}_1$-groups, maximal hypercentral subgroups containing the hypercentral radical necessarily exist and one might therefore expect the nilpotent injectors to generalise to $\mathfrak{S}_1$-groups in a more satisfactory way than the nilpotent projectors.

We shall see that this is in fact the case, proving that in an $\mathfrak{S}_1$-group the hypercentral injectors coincide with the maximal hypercentral subgroups containing the hypercentral radical and form a unique conjugacy class. The results depend heavily on the fact that $\mathfrak{S}_1$-groups are nilpotent-by-abelian-by-finite and so the hypercentral subgroups containing the radical are all finite extensions of the radical. The definition and basic results concerning $\mathfrak{S}_1$-groups may be found in (4).

The radical R of the $\mathfrak{S}_1$-group G has a finite series

$$1 = R_0 \leqslant \cdots \leqslant R_n = R$$

of normal subgroups of G such that each factor R_i/R_{i-1} is either

(1) torsion-free abelian and rationally irreducible (as a G-module)

or (2) an abelian Černikov p-group.

The torsion-free factors in this series are central in R. If we define

$$N = \bigcap \{C_G(R_i/R_{i-1});\ R_i/R_{i-1} \text{ is torsion-free}\}$$

then $N \geqslant R$.

We write $\varpi(R)$ for the set of primes $\{p;\ R_i/R_{i-1} \text{ is a } p\text{-group for some } i\}$. For each $p \in \varpi(R)$, define

$$N_p = \bigcap \{C_N(\Omega_1(R_i/R_{i-1}));\ R_i/R_{i-1} \text{ is a } p\text{-group}\}.$$

Then N/N_p is finite and N_p centralises $\Omega_{n+1}(R_i/R_{i-1})/\Omega_n(R_i/R_{i-1})$. Thus each p-factor R_i/R_{i-1} is hypercentral in N_p.

For each $p \in \varpi(R)$, let $C_p = \bigcap_{q \neq p} N_q$ and define

$$M = \bigcap_{p \in \varpi(R)} C_p = \bigcap_{p \in \varpi(R)} N_p;$$

then N/M is finite and $M \leq R$.

We relate the maximal hypercentral subgroups of G containing R to certain subgroups of N/M. The characterisation given in this result may be compared with a characterisation given by Fischer in the finite case (see (1) VI.7.18).

Theorem 1. *V is a maximal hypercentral subgroup of G containing R if and only if $V \leq N$ and each Sylow p-subgroup of V/M is a Sylow p-subgroup of C_p/M.*

Proof. Let W_p/M be a p-subgroup of C_p/M and $W = \langle W_p; p \in \varpi(R)\rangle$. If p and q are distinct primes in $\varpi(R)$, then $[W_p, W_q] \leq C_p \cap C_q = M$ and so $W_p \lhd W$. Since each p'-factor and each torsion-free factor of M is hypercentral in W_p it follows that W_p is a hypercentral group. Thus W is a product of normal hypercentral subgroups and so is itself hypercentral.

Let U be a hypercentral subgroup of G containing R. Then U/R is finite and so U centralises each torsion-free factor R_i/R_{i-1}; hence $U \leq N$. Let U_p/M be the Sylow p-subgroup of U/M; then U_p centralises each p'-factor of R. Thus if $p \notin \varpi(R)$ then $U_p \leq M$ i.e. the Sylow p-subgroup of U/M is trivial, and if $p \in \varpi(R)$ then $U_p \leq C_p$.

A straightforward argument using the results of these two paragraphs yields the required result.

Theorem 2. *The maximal hypercentral subgroups of G containing R are conjugate in G.*

Proof. Let V_1 and V_2 be maximal hypercentral subgroups of G containing R. For $i = 1, 2$, there is a Sylow basis $S_i = \{S_p^{(i)}/M\}$ of N/M reducing into V_i/M. Thus the Sylow p-subgroup of V_i/M is $(V_i \cap S_p^{(i)})/M = (C_p \cap S_p^{(i)})/M$. Since N/M is a finite soluble group there is an element $x \in N$ such that $x^{-1}S_p^{(1)}x = S_p^{(2)}$ for each prime p. Hence $x^{-1}(V_1 \cap S_p^{(1)})x = x^{-1}(C_p \cap S_p^{(1)})x = C_p \cap S_p^{(2)} = V_2 \cap S_p^{(2)}$ and so $x^{-1}V_1x = V_2$.

We define injectors in $\mathfrak{S}_1$-groups to have a rather stronger condition than in the finite definition. A hypercentral subgroup V of the $\mathfrak{S}_1$-group G is a *hypercentral injector* of G if, for each ascendant subgroup A of G, $V \cap A$ is a maximal hypercentral subgroup of A.

Theorem 3. *V is a hypercentral injector of G if and only if V is a maximal hypercentral subgroup of G containing R.*

Proof. In one direction the result is obvious and so we have to prove that if V is a maximal hypercentral subgroup of G containing R and if A is an ascendant subgroup of G, then V is a maximal hypercentral subgroup of A.

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Consideration of the subgroups $A \leqslant AR \leqslant G$ shows that we may consider the separate cases $A \geqslant R$ and $AR = G$.

(1) $A \geqslant R$.

The Sylow p-subgroup of V/M is a Sylow p-subgroup of C_p/M and hence the Sylow p-subgroup of $(V \cap A)/M$ is a Sylow p-subgroup of $(C_p \cap A)/M$, since $C_p \cap A$ is a subnormal subgroup of C_p. Also $C_p \cap A = C_p(A)$ and so Theorem 1 shows that $V \cap A$ is a maximal hypercentral subgroup of A.

(2) $AR = G$.

There is an ascending series

$$A = A_0 \lhd \cdots \lhd A_\alpha \lhd \cdots \lhd A_\gamma = G.$$

If W is a hypercentral subgroup of A containing $V \cap A$, then $[W, R \cap A_{\alpha+1}] \leqslant R \cap A_\alpha$ and so $WR \cap A_\alpha \lhd WR \cap A_{\alpha+1}$. Thus $W(= WR \cap A)$ is an ascendant subgroup of WR which, being a product of two ascendant hypercentral subgroups, is itself hypercentral. But $WR \geqslant (V \cap A)R = V$ and so $WR = V$. Hence $W = V \cap A$, as required.

In the characterisation given in Theorem 1, a hypercentral injector V of G corresponds to a subgroup V/M of a finite soluble section of G. In the case of polycyclic groups we can determine a different finite soluble section K/L such that the subgroups V/L are the nilpotent injectors of K/L. The simple example of C_{3^∞} extended by its involution automorphism shows that no similar result can be obtained for $\mathfrak{S}_1$-groups.

Letting R_i denote the ith term of the upper central series of R and using the residual finiteness of polycyclic groups there is an integer n such that

$$(R^n \cap R_i)/(R^n \cap R_{i-1})$$

is torsion-free. We put $L = R^n$ so that R/L is finite and L has a central series

$$1 = L \cap R_0 \leqslant \cdots \leqslant L \cap R_k = L$$

with torsion-free factors. We define

$$K = \bigcap_{i=1} C_G((L \cap R_i)/(L \cap R_{i-1})).$$

Theorem 4. *Let G be a polycyclic group. With the above notation, K/L is a finite soluble group and R/L is its radical.*

V is a maximal nilpotent subgroup of G containing R if and only if V/L is a maximal nilpotent subgroup of K/L containing R/L.

Proof. G/L is finite-by-abelian-by-finite and is also residually finite; hence G/L is abelian-by-finite.

Let F/L be the radical of K/L. Then F is nilpotent and normal in G and so $F \leqslant R$. Hence the radical of K/L is the finite subgroup R/L. But K/L is abelian-by-finite and so if K/L is infinite, then its radical is infinite. Hence K/L is a finite soluble group.

If V is a nilpotent subgroup of G containing R, then V/R is finite and hence V/L is finite. Thus V centralises each of the torsion-free factors $(L \cap R_i)/(L \cap R_{i-1})$ and so $V \leqslant N$.

If W/L is a nilpotent subgroup of K/L, then W is nilpotent. The result now follows easily.

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