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REALIZABILITY PROBLEM FOR COMMUTING GRAPHS

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Abstract

We investigate properties which ensure that a given finite graph is the commuting graph of a group or semigroup. We show that all graphs on at least two vertices such that no vertex is adjacent to all other vertices is the commuting graph of some semigroup. Moreover, we obtain complete classifications of the graphs with an isolated vertex or edge that are the commuting graph of a group and the cycles that are the commuting graph of a centrefree semigroup.

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1. Introduction and preliminaries

Let S be a semigroup with centre $Z(S) := \{x \in S \mid xs = sx \text{ for all } s \in S\}$. The *commuting graph* $\Gamma(S)$ is the simple graph with vertex set $S \setminus Z(S)$, and two distinct vertices x and y are adjacent if and only $xy = yx$. This notion can be traced back at least as far as the paper by Brauer and Fowler [6] who used commuting graphs to study the distances between involutions in finite groups (it should be mentioned, however, that the vertices of their graph consisted of all nonidentity elements).

Solomon and Woldar [16] showed that a commuting graph distinguishes finite simple nonabelian groups. More precisely, if the commuting graph of a group is isomorphic to the commuting graph of some finite nonabelian simple group, then the two groups are isomorphic. In general, commuting graphs do not distinguish groups as the commuting graph of Q_8 and D_8 both consist of three disjoint edges.

In the present paper, we will be concerned with the following inverse problem for commuting graphs: given a simple graph Γ, can we find a group or semigroup whose commuting graph is isomorphic to Γ? We say that such a graph Γ is *realizable* over groups or over semigroups, respectively. Part of the motivation behind the present study are the results of Pisanski [15] who showed that any graph on n vertices is

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isomorphic to the induced subgraph of commuting graph of S_3^n, a direct product of n copies of the symmetric group S_3. However, the vertices of this subgraph, in general, do not form a semigroup as they may not be closed under multiplication. Unaware of [15], the second author with collaborators [2] proved that every finite simple graph is isomorphic to the induced subgraph of the commuting graph of complex matrices with sufficiently large size. The vertices of this subgraph can be taken to be projections. Further, they showed that, for any positive integer n, there exists a finite graph of order $n^2 + 2$ which is isomorphic to no induced subgraph of the commuting graph of n–by–n matrices.

Araújo *et al.* [4, Problems (3) and (4)] asked to classify the commuting graphs of semigroups and to prove or disprove that there is a semigroup whose commuting graph has clique number, girth or chromatic number n for any integer n. We give a complete answer to these two questions in Theorem 2.2 by showing that any finite graph on at least two vertices that does not have a vertex adjacent to all other vertices is the commuting graph of a semigroup. This also gives an alternative proof for the result of Araújo *et al.* [4] that, for every integer n, there is a semigroup whose commuting graph has diameter n. The semigroup in their proof has no centre while the semigroup in our construction to prove Theorem 2.2 has a centre of order two. Given Theorem 2.2, it is natural to restrict the realization question for semigroups to semigroups from particular classes, for example centrefree semigroups. We undertake some investigations along these lines with our main result being the following (the proof will be given after Theorem 2.12).

THEOREM 1.1. *A cycle is the commuting graph of a centrefree semigroup if and only if its length is divisible by four.*

Commuting graphs of groups are much more restrictive. For example, a GAP [9] computation shows that no graph of order less than five is a commuting graph of some group. Moreover, 30 is the smallest order of a connected graph that is the commuting graph of some group. In fact if G were a group with connected commuting graph on less then 30 vertices then, since $|Z(G)|$ divides $|G|$, we would have $|G| < 60$ but a GAP calculation shows that no such group exists. There are seven groups with a connected commuting graph with 30 vertices; each has order 32 and a centre of order two.

There are also many other restrictions on the possible graphs realizable as the commuting graph of a group. Indeed, even though there is no bound for the diameter of the commuting graph of a group [10], Morgan and Parker [14] have shown that every connected component of the commuting graph of a group with trivial centre has diameter at most 10. Moreover, they showed that if Δ is a connected component of such a graph for a nonsoluble group with Δ containing no involutions, then Δ must be a clique. Furthermore, Afkhami *et al.* [1] have shown that only 17 groups have a planar commuting graph. This result was also obtained by Das and Nongsiang [7], who further proved that, for a given genus g, there are only a finite number of groups whose commuting graph has genus g. In addition, [7] also shows that only three groups have triangle-free commuting graphs.

We collect some simple observations about the structure of commuting graphs of groups in Section 3. In particular, Lemma 3.4 shows that any nonisolated edge of the commuting graph of a group must lie in a triangle. We then determine the structure of commuting graphs of groups that contain an isolated vertex or edge. This work is summarised in the following two theorems.

THEOREM 1.2. *Let G be a group and suppose that $\Gamma(G)$ has an isolated vertex. Then $\Gamma(G)$ has exactly $|G|/2$ isolated vertices and the remaining vertices form a clique.*

THEOREM 1.3. *Let G be a group and suppose that $\Gamma(G)$ has an isolated edge. Then $\Gamma(G)$ consists of isolated edges, cliques and at most one noncomplete connected component Δ. Moreover, Δ has diameter at most five.*

More specific information about the groups involved in Theorem 1.2 is given in Theorem 3.9, while more details about the graphs and groups involved in Theorem 1.3 are given in Lemma 3.12, Theorem 3.13, Lemma 3.15 and Theorem 3.16. Combined, they show how much the commutativity relation determines finite groups that contain nontrivial self-centralizing subgroups of order at most three. For example, it follows, from Theorems 3.9, 3.13 and 3.16, that if $\Gamma(G) = 2^{k-2}K_2 + K_{2^{k-1}-2}$ for $k \geq 4$, then G is dihedral, semidihedral or a generalized quaternion group. The proof of Theorem 1.3 uses some deep group theoretical results by Feit and Thompson [8], Mazurov [13] and Wong [19] about groups with a self-centralizing subgroup of order three or four.

The groups that occur in Theorems 1.2 and 1.3 have centre of order at most two and the diameter of each connected component is at most five. Given the result of Morgan and Parker [14] that the diameter of each connected component of the commuting graph of a group with trivial centre is at most 10, the following question seems natural.

QUESTION 1.4. Is there some function $f(d)$ such that if G is a group with $|Z(G)| \leqslant d$, then each connected component of $\Gamma(G)$ has diameter at most $f(d)$?

Note that there is no bound on the order of the centre for the family of examples in [10] with unbounded diameter.

1.0.1. Notation. Given a graph, Γ we denote the vertex set of Γ by $V(\Gamma)$ and the edge set by $E(\Gamma)$. Also, we denote by $|V(\Gamma)|$ the cardinality of the vertex set of Γ. Given vertices $x, y \in \Gamma$ we denote by $x \sim y$ the fact that they form an edge in Γ. The distance, $d(x, y)$ between connected vertices x and y is the length of a minimal path between x and y. We set $d(x, y) = \infty$ if there is no path from x to y. We let K_n denote the complete graph on n vertices.

Unless otherwise stated, all semigroups and groups are written multiplicatively and the identity in a group is one. Let $|G|$ denote the order of the (semi)group G and, given $g \in G$, let $|g|$ denote the order of g. Let $Z(G)$ be the centre of a (semi)group G, let $C_G(A)$ be the centralizer of the subset A of a (semi)group G and, in the case where G is a group, let $\mathcal{N}_G(A) := \{x \in G \mid x^{-1}Ax = A\}$ be its normalizer. Given elements $x, y \in G$,

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we denote by $\langle x, y\rangle$ the subgroup generated by x and y, so $\langle x\rangle$ is the subgroup generated by x. By $H \leqslant G$ we denote that H is a subgroup of a group G, and $\mathrm{Aut}(G)$ denotes the automorphism group of G.

Let S_n and A_n be the symmetric group and alternating group on n elements, respectively, let $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$ be a cyclic group of order p, let $\mathrm{SL}(n, p^k)$ be the special linear group of $n \times n$ matrices with determinant one over the Galois field $\mathrm{GF}(p^k)$ and let $\mathrm{PSL}(n, p^k) = \mathrm{SL}(n, p^k)/Z(\mathrm{SL}(n, p^k))$ be the projective special linear group. As usual, given a group homomorphism $\phi\colon G \to G$, its action on an element $g \in G$ is denoted by $g\phi \in G$.

2. Realizability over semigroups

We start with two basic obstructions that prevent realizability over semigroups.

Lemma 2.1. *Let Γ be a graph. If either:*

(i) $|V(\Gamma)| = 1$*; or*

(ii) Γ *contains a vertex adjacent to all vertices in* $V(\Gamma)\backslash\{v\}$,

then Γ is not the commuting graph of a semigroup.

Proof. Suppose that $\Gamma = \Gamma(S)$ for some semigroup S and that $v \in V(\Gamma)$ such that either $V(\Gamma) = \{v\}$ or v is adjacent to all vertices in $V(\Gamma)\backslash\{v\}$. Then v commutes with itself and all elements of $Z(S) \cup \Gamma = S$ so $v \in Z(S)$, which is a contradiction. □

In particular, Lemma 2.1 shows that complete graphs are not commuting graphs of semigroups. However, the obstructions in Lemma 2.1 are the only ones that prevent realizability over semigroups.

Theorem 2.2. *Every finite graph Γ with at least two vertices and such that no vertex is adjacent to all other vertices is the commuting graph of some semigroup S with $|S| = |\Gamma| + 2$.*

Proof. Let $V(\Gamma) = \{v_1, \dots, v_n\}$ be the ordered vertex set of Γ. Pick a set $Z = \{0, z\}$, with two distinct elements disjoint from $V(\Gamma)$. On the set $S = V(\Gamma) \cup Z$ define a multiplication by $SZ = ZS = \{0\}$, and

$$v_i v_j = \begin{cases} 0 & \text{if } i < j \text{ or } (v_i, v_j) \text{ is an edge in } \Gamma, \\ z & \text{if } i > j \text{ and } (v_i, v_j) \text{ is not an edge in } \Gamma. \end{cases}$$

Clearly, $S^2 \subseteq Z$ so $S(S^2) \subseteq SZ = \{0\}$ and $(S^2)S \subseteq ZS = \{0\}$ and hence the product of any three elements gives zero. Therefore, the multiplication is associative and so S is a semigroup with zero. Moreover, since no vertex in Γ is adjacent to all other vertices, $Z(S) = Z$. Also, $v_i, v_j \in S \setminus Z(S)$ commute if and only if they form an edge in Γ. □

Remark 2.3. Since each set can be well ordered, only cosmetic modification is required to show that every infinite graph such that no vertex connects to all other vertices is a commuting graph of a semigroup.

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Remark 2.4. The upper bound on the order of S is exact. Namely, it can be shown that the six-cycle graph $\Gamma = C_6$ is not the commuting graph of a semigroup of order seven.

Things are more complicated if we restrict ourselves to centrefree semigroups, that is, to semigroups whose centre is the empty set. We demonstrate this with the next lemma and its corollary.

Lemma 2.5. *Let Γ be the commuting graph of a centrefree semigroup S and let $a, b, c \in V(\Gamma)$ such that:*

(i) *$a \sim b \sim c$ and $a \nsim c$; and*

(ii) *no triangle in Γ contains $\{a, b\}$ or $\{b, c\}$.*

Then

$$\text{either } ab = a \text{ and } bc = c \quad \text{or} \quad ab = b = bc. \tag{2.1}$$

Proof. Note first that $ab = ba \in S$ commutes with both a and b and hence, in $\Gamma = \Gamma(S)$, ab is adjacent to a and b. By the assumption (ii), it follows that $ab \in \{a, b\}$ and, similarly, $bc = cb \in \{b, c\}$. Now, assume that $ab = a$ and $bc = b$. Then

$$ac = (ab)c = a(bc) = ab = a = ba = (cb)a = c(ba) = ca,$$

which contradicts the fact that $a \nsim c$. Similarly, $ab = b$ and $bc = c$ is not possible since, otherwise,

$$ac = a(bc) = (ab)c = bc = c = cb = c(ba) = (cb)a = ca.$$ □

Corollary 2.6. *Let Γ be the commuting graph of a centrefree semigroup S. Assume that Γ contains a cycle C of length $n \geq 4$ as an induced subgraph such that no two adjacent vertices of C are contained in a triangle from Γ. Then n is even.*

Proof. Suppose that Γ contains a cycle C. By Lemma 2.5, the vertices of C can be labelled by $\{\pm 1\}$ such that, if $ab = a$ and $bc = c$, we label vertex b with -1; otherwise, if $ab = b = bc$, we label vertex b with $+1$. Hence, walking around the cycle, the labelling alternates and so $|C|$ is even. □

Example 2.7. The assumption in Corollary 2.6 that no edge from an induced cycle forms a triangle in Γ is essential. For example, the multiplication table

$$\begin{array}{c|c} & \begin{matrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{matrix} \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{matrix} & \begin{pmatrix} s_1 & s_1 & s_1 & s_1 & s_1 & s_1 \\ s_1 & s_1 & s_1 & s_1 & s_2 & s_2 \\ s_1 & s_1 & s_1 & s_1 & s_3 & s_3 \\ s_4 & s_4 & s_4 & s_4 & s_4 & s_4 \\ s_1 & s_2 & s_2 & s_4 & s_5 & s_5 \\ s_1 & s_3 & s_3 & s_4 & s_6 & s_6 \end{pmatrix} \end{array}$$

makes the set $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ into a centrefree semigroup with $\Gamma(S)$ containing the induced 5-cycle $s_2 \sim s_3 \sim s_6 \sim s_4 \sim s_5 \sim s_2$. This is possible because s_1 forms a triangle with edges $\{s_5, s_2\}$, $\{s_2, s_3\}$ and $\{s_3, s_6\}$.

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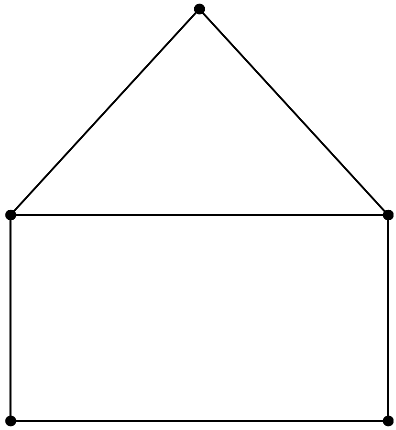

FIGURE 1. A house graph.

COROLLARY 2.8. *If $n \geq 5$ then there exists a graph on n vertices that is the commuting graph of a semigroup but not the commuting graph of a centrefree semigroup.*

PROOF. If n is odd, consider an n-cycle. If n is even, consider the disjoint union of an $(n-1)$-cycle and an isolated vertex. □

EXAMPLE 2.9. The edgeless graph Γ on n vertices is the commuting graph of the centrefree semigroup $S = \{v_1, \ldots, v_n\}$ with multiplication defined by $v_i v_j = v_i$.

EXAMPLE 2.10. GAP calculations show that each graph on at most four vertices that does not satisfy the obstructions in Lemma 2.1 is the commuting graph of some centrefree semigroup.

EXAMPLE 2.11. GAP calculations show that there exist exactly two simple graphs on five vertices, each vertex with valency at most three, which are not the commuting graph of a centrefree semigroup. One such graph is the 5-cycle, while the other is a *house* (see Figure 1).

Note that, by Theorem 2.2, both graphs are commuting graphs of a semigroup with nontrivial centre.

Next, we give a complete picture of when a cycle is the commuting graph of some centrefree semigroup and so prove Theorem 1.1. It turns out that if such a semigroup exists, then it is essentially unique. Clearly, this is no longer the case if the restriction about being centrefree is removed because S and its unitization $S^1 = S \cup \{1\}$ have the same commuting graph. Note the following consequence of the Theorem below: there is no upper bound on the diameter of a connected commuting graph of centrefree semigroups (see, also, [4, Theorem 4.1]).

Semigroups S_1 and S_2 are (anti)isomorphic if there exists a bijection $\phi\colon S_1 \to S_2$ such that $(ab)\phi = (a\phi)(b\phi)$ for each $a, b \in S_1$ (that is, ϕ is an isomorphism) or $(ab)\phi = (b\phi)(a\phi)$ for each $a, b \in S_1$ (that is, ϕ is an antiisomorphism).

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THEOREM 2.12. *A cycle is the commuting graph of some centrefree semigroup if and only if its length is divisible by four. If a cycle has at least 5 vertices then up to (anti)isomorphism there exists at most one centrefree semigroup whose commuting graph is a given cycle.*

PROOF. By Lemma 2.1, a triangle is not the commuting graph of a semigroup. So, by Corollary 2.6, it only remains to consider even cycles.

Suppose, therefore, that an even cycle C_{2k} is the commuting graph of a centrefree semigroup S. Each vertex $x \in \Gamma(S)$ has exactly two neighbours, say, y, z. It follows that $x^2 \in C_S(\{y, x, z\}) = \{x\}$, so S is a band.

We label the vertices of $\Gamma(S)$ by $x_0, x_1, \ldots, x_{2k-1}$ such that $x_i \sim x_{i\pm1}$ where addition of subscripts is done modulo $2k$. Now, by Lemma 2.5, the identity (2.1) holds for $\Gamma(S)$ and so, as in the proof of Corollary 2.6, the vertices of $\Gamma(S)$ can be labelled by $\{\pm1\}$. Without loss of generality, we label the vertices with even subscripts by $+1$ and those with odd subscripts by -1. Recall that this means that

$$x_{2i}x_{2i\pm1} = x_{2i\pm1}x_{2i} = x_{2i} \quad i \in \{0, 1, \ldots, k-1\}. \tag{2.2}$$

Step 1. If $x, y \in C_{2k}$ are at distance two then

$$xy \in \{x, y\}. \tag{2.3}$$

To see this, let $x \sim b \sim y$ be a path of length two from x to y. Note that xy commutes with b and so $xy \in \{x, b, y\}$. Suppose that $xy = b$. Then xy commutes with x and so

$$bx = (xy)x = x(xy) = xy = b.$$

Thus b is labelled $+1$ and we also have $by = b$. Now yx also commutes with b and, since $x \nsim y$, we must have $yx \in \{x, y\}$. If $yx = x$, then, using the fact that $x^2 = x$, we have that $xyx = x$ and so $bx = x$, which contradicts b being labelled $+1$. Thus $yx = y$, but then $yxy = y$, and so $by = y$, which is another contradiction. Thus (2.2) holds.

Step 2. By (2.3), $x_0x_2 \in \{x_0, x_2\}$ and, by considering, if necessary, the opposite semigroup (with multiplication given by $a \cdot b := ba$), we may assume that

$$x_0x_2 = x_0. \tag{2.4}$$

Step 3. We prove by induction that

$$x_{2i}x_{2i+2} = x_{2i}, \quad i \in \{0, 1, \ldots, k-1\}. \tag{2.5}$$

The base step is given by (2.4). To prove the induction step, assume we have already shown that vertices a, c with $d(a, c) = 2$ and with labels $+1$ satisfy

$$ac = a.$$

Choose $e \neq a$ with $d(c, e) = 2$ and assume, to the contrary, that

$$ce = e. \tag{2.6}$$

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By (2.3), $ca \in \{c, a\}$ and, as $ca \neq ac = a$,

$$ca = c.$$

Likewise, $ec = c$. Moreover, if $d \in C_S(\{c, e\})$ is the unique vertex commuting with both c and e, then, since both c, e have label $+1$,

$$dc = c = c\,d \quad \text{and} \quad e\,d = e = de,$$

and, from the inductive hypothesis (2.6), we deduce that

$$ad = (ac)\,d = a(c\,d) = ac = a. \tag{2.7}$$

Furthermore, $d(da) = d^2a = da = d(ad) = (da)d$, so $da \in C_S(d) = \{c, d, e\}$. Note that $da = d$ is impossible because then da would commute with c and we would get that $c = cd = c(da) = (da)c = d(ac) = da = d$, which is a contradiction. Hence

$$da \in \{c, e\}. \tag{2.8}$$

Now, by our assumption, $cea = ea = eac$, and, by (2.7), also $dea = ea = ead$, so $ea \in C_S(\{c, d\}) = \{c, d\}$. If $ea = d$ then, by (2.8), $d = ea = ea^2 = (ea)a = da \in \{c, e\}$, which is a contradiction. Hence

$$ea = c.$$

Next, in (2.8), $da = e$ is impossible, since then $e = ce = c(da) = (cd)a = ca = c$. So,

$$da = c.$$

For the unique $b \in C(\{a, c\})$, which clearly satisfies

$$ba = a = ab \quad \text{and} \quad bc = c = cb$$

this further gives $(bd)a = b(da) = bc = c$. Thus $bd \neq b$ (because $ba = a = a^2$) and, as (2.3) implies $bd \in \{b, d\}$, we see that

$$bd = d.$$

We also have $db \in \{b, d\}$ and, since $d \nsim b$, it follows that $db = b$. However, we then have $a = ba = (db)a = d(ba) = d(ab) = (da)b = cb = c$, which is again a contradiction. Hence (2.6) is contradictory and we must have $ec = e$, which proves the induction step and hence the equation (2.5).

Step 4. We claim that

$$x_{2i}y = x_{2i} \quad \text{for all } x_{2i}, y \in S. \tag{2.9}$$

In fact, (2.5) implies $x_0x_4 = (x_0x_2)x_4 = x_0(x_2x_4) = x_0x_2 = x_0$ and, by induction, $x_0x_{2i} = x_0$. Likewise, we see that

$$x_{2i}x_{2j} = x_{2i} \quad \text{for all } 0 \leq i, j \leq k-1.$$

Furthermore, by (2.2), $x_{2j}x_{2j+1} = x_{2j+1}x_{2j} = x_{2j}$, which implies that

$$x_{2i}x_{2j+1} = (x_{2i}x_{2j})x_{2j+1} = x_{2i}(x_{2j}x_{2j+1}) = x_{2i}x_{2j} = x_{2i},$$

which proves (2.9).

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Step 5. By (2.9),

$$x_{2i+1}x_{2j} \in C_S(x_{2i+1}) = \{x_{2i}, x_{2i+1}, x_{2i+2}\}$$

for all i, j. Actually, $x_{2i+1}x_{2j} = x_{2i+1}$ is impossible, since then $x_{2i+1}x_{2i} = x_{2i}$ would give

$$x_{2i} = x_{2i+1}x_{2i} = (x_{2i+1}x_{2j})x_{2i} = x_{2i+1}(x_{2j}x_{2i}) = x_{2i+1}x_{2j} = x_{2i+1},$$

which is a contradiction. Hence

$$x_{2i+1}x_{2j} \in \{x_{2i}, x_{2i+2}\}. \tag{2.10}$$

In particular, $x_{2i+1}x_{2j}$ has label $+1$ for all i and j and, for a fixed i, can take only two values as j varies.

Step 6. Assume that we have $x_{2i+1}x_{2j} \neq x_{2i+1}x_{2j+2}$ for some i, j. We claim that then

$$x_{2i+1}x_{2j+1} = x_{2i+1}. \tag{2.11}$$

To see this, we first show that the product $t := x_{2i+1}x_{2j+1}$ cannot have label $+1$. Otherwise, by (2.9), $ty = t$ for every $y \in S$. Combined with $x_{2j+1}x_{2j} = x_{2j}$ and $x_{2j+1}x_{2j+2} = x_{2j+2}$ we would have

$$t = tx_{2j} = (x_{2i+1}x_{2j+1})x_{2j} = x_{2i+1}(x_{2j+1}x_{2j}) = x_{2i+1}x_{2j},$$

and, likewise,

$$t = tx_{2j+2} = (x_{2i+1}x_{2j+1})x_{2j+2} = x_{2i+1}x_{2j+2},$$

which contradicts the fact that $x_{2i+1}x_{2j} \neq x_{2i+1}x_{2j+2}$. Thus $x_{2i+1}x_{2j+1}$ has label -1 and, as such, commutes with exactly two vertices with label $+1$, namely, t_{prec} and t_{succ}. By (2.9) and (2.10), given $s \in \{t_{\text{prec}}, t_{\text{succ}}\}$,

$$s = s(x_{2i+1}x_{2j+1}) = (x_{2i+1}x_{2j+1})s \in x_{2i+1}\{x_{2j}, x_{2j+2}\} \subseteq \{x_{2i}, x_{2i+2}\}.$$

Therefore the only option is $\{t_{\text{prec}}, t_{\text{succ}}\} = \{x_{2i}, x_{2i+2}\}$, and hence $t = x_{2i+1}$, as this is the only vertex with label -1 that commutes with $\{t_{\text{prec}}, t_{\text{succ}}\} = \{x_{2i}, x_{2i+2}\}$.

Step 7. We claim that $d(x_{2j}, x_{2t}) = 2$ implies $x_{2i+1}x_{2j} \neq x_{2i+1}x_{2t}$ for each $i \in \mathbb{Z}_k$. Without loss of generality, we assume that $2t = 2j + 2$.

In fact, this claim is trivial whenever x_{2i+1} is adjacent to both x_{2j} and x_{2t} because then, by (2.2), $x_{2i+1}x_{2j} = x_{2j} \neq x_{2t} = x_{2i+1}x_{2t}$.

Next, suppose that x_{2i+1} is adjacent to only one of x_{2j} and x_{2t}. By symmetry, we may assume that $x_{2i+1} \sim x_{2j} \sim x_{2j+1} \sim x_{2j+2} = x_{2t}$. Let us denote $b := x_{2i+1}$, $c := x_{2i+2}$, $d := x_{2i+3}$ and $e := x_{2i+4} = x_{2t}$. Suppose, contrary to the claim, that $bc = c = be$. By (2.3), $bd, db \in \{d, b\}$. Now, $bd = d$ implies $e = de = (bd)e = b(de) = be = c$, which is a contradiction. Hence $bd = b$. Since $b \nsim d$, we have $db = d$. However, then $c = dc = d(be) = (db)e = de = e$, which is a contradiction. Thus $bc = c$ and $be \neq c$.

Now assume the claim does not hold and let x_{2i_0+1} and x_{2j_0} be the vertices with least distance for which

$$x_{2i+1}x_{2j} = x_{2i+1}x_{2t}, \quad d(x_{2j}, x_{2t}) = 2.$$

Clearly, $2t_0 = 2j_0 + 2$ or $2t_0 = 2j_0 - 2$. By the symmetry, we may assume the former, so $d(x_{2i_0+1}, x_{2t_0}) = d(x_{2i_0+1}, x_{2j_0+2}) \geq d(x_{2i_0+1}, x_{2j_0}) =: \delta_0$. By the first part of the proof of Step 7, we must have $\delta_0 \geq 5$. Then, there exists a vertex $\hat{d} = x_{2j_0\pm1}$ with label -1 (on the short arc from x_{2i_0+1} to x_{2j_0}) such that, simultaneously, $d(x_{2i_0+1}, \hat{d}) \leq \delta_0 - 1$ and $d(\hat{d}, x_{2j_0+2}) = 3$. By the minimality of distance δ_0 we have, $\hat{d}x_{2i_0} \neq \hat{d}x_{2i_0+2}$, and hence, by Step 6, $\hat{d}x_{2i_0+1} = \hat{d}$. Therefore

$$\begin{aligned}\hat{d}x_{2j_0} &= (\hat{d}x_{2i_0+1})x_{2j_0} = \hat{d}(x_{2i_0+1}x_{2j_0}) = \hat{d}(x_{2i_0+1}x_{2j_0+2}) = (\hat{d}x_{2i_0+1})x_{2j_0+2}\\ &= \hat{d}x_{2j_0+2},\end{aligned}$$

which contradicts the fact that δ_0 was the minimal distance with such equality possible, while $d(\hat{d}, x_{2j_0+2}) = 3 < \delta_0$.

Step 8. Combining the previous step with (2.10) shows that x_1x_{2j} alternates between x_0 and x_2 as j varies over $\mathbb{Z}_k$. This is only possible if j is even or, equivalently, if the number of vertices in the cycle is divisible by four.

Conversely, given a cycle with $4k$ vertices $C_{4k} = \{x_0, x_1, \ldots, x_{4k-1}\}$, define the multiplication in the only possible way (up to considering the opposite semigroup, that is, up to antiisomorphism), determined by (2.2), (2.5), (2.9)–(2.11): that is,

$$\begin{aligned}x_{2i}y &:= x_{2i},\\ x_{2i+1}x_{2i+4j} &:= x_{2i}, \quad x_{2i+1}x_{2i+4j+2} := x_{2i+2},\\ x_{2i+1}x_{2j+1} &:= x_{2i+1}\end{aligned}$$

for all $i, j \in \mathbb{Z}_{2k}$ and $y \in C_{4k}$, where addition of subscripts in modulo $4k$. It is straightforward to verify that this multiplication is associative, and hence makes C_{4k} into a semigroup, and that $xy = yx$ if and only if x and y are adjacent vertices in the cycle C_{4k}. Moreover, as shown in previous steps, up to considering the opposite semigroup this is the only option for a centrefree semigroup whose commuting graph is the $4k$-cycle. □

3. Commuting graph of a group

We now turn our attention to realizability over groups. Our main results classify graphs with an isolated vertex or edge that are commuting graphs of a group. Conversely, we also classify all groups whose commuting graphs have an isolated vertex or edge.

Let us start with several results of a general nature. Our first lemma is well known, but we give a proof for the sake of completeness. Recall that $d(v)$ denotes the valency of a vertex v, that is, the cardinality of the set of all vertices adjacent to v.

LEMMA 3.1. *Let G be a finite group with centre Z and Γ its commuting graph. Then $|Z|$ is a common divisor of the integers $\{d(v) + 1 \mid v \in \Gamma\}$.*

In particular, if $d_{\min}$ is the minimal valency of vertices in Γ, then Z has at most $d_{\min} + 1$ elements.

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Proof. Take any $v \in \Gamma$. Its neighbourhood equals $C_G(v) \setminus (Z \cup \{v\})$. Observe that $C_G(v)$ is a group which contains Z as a subgroup and, therefore, $|C_G(v)|$ is divisible by $|Z|$. Hence $|Z|$ also divides $|C_G(v)\backslash Z| = d(v) + 1$. □

Lemma 3.2. *Let Γ_1 be a connected component with diameter at most two of the commuting graph of a group G. Then $\Gamma_1 \cup Z(G)$ is a subgroup of G.*

Proof. Choose any $v \in \Gamma_1$. Then $v^{-1} \in C_G(v)$ and is clearly not in $Z(G)$. So either v is an involution or $v^{-1} \sim v$, and hence, in both cases, v belongs to the connected component containing v, that is, to Γ_1. It remains to show that the product of any $v, w \in Z(G) \cup \Gamma_1$ is also inside $Z(G) \cup \Gamma_1$. This is trivial if v, w commute, since their product is either in $Z(G)$ or else commutes with v and w, and hence is adjacent to both v and w. Assume that v, w do not commute. Since the diameter of Γ_1 is at most two, there exists $g \in C_G(v) \cap C_G(w) \cap \Gamma_1$. Thus $v, w \in C_G(g)$, so, also, $vw \in C_G(g) \subseteq Z(G) \cup \Gamma_1$. □

Our next example shows that the previous lemma does not hold for components with larger diameters.

Example 3.3. Let $\Gamma = \Gamma(S_4)$. It is an elementary calculation that the set of elements of order two or four form a connected component of Γ on 15 vertices with diameter three. However, S_4 contains no subgroups of order $15 + |Z(S_4)|$.

A proper subgroup M of a group G is called a *CC-subgroup* if $C_G(m) \leqslant M$ for all $m \in M\backslash\{1\}$. By Lemma 3.2, if $Z(G) = 1$ and Γ_1 is a connected component with diameter at most two of the commuting graph of G, then $\Gamma_1 \cup \{1\}$ is a CC-subgroup of G. The structure of finite groups with a CC-subgroup was determined by Arad and Herfort [3]. Note, however, that there are groups with nontrivial centre whose commuting graph contains a connected component of diameter at most two (see, for example, Theorem 3.16 below). Such a group cannot have a proper CC-subgroup.

We now state a simple obstruction for realizability among groups. It reflects sharply against the realizability with centrefree semigroups (cf. Theorem 2.12). We remark that Theorems 3.9 and 3.13 give additional obstructions.

Lemma 3.4. *Let Γ be the commuting graph of a group G. Then each edge $\{a, b\}$ that is not an isolated edge lies on a triangle.*

Proof. Let $\{a, b\}$ be a nonisolated edge and let Γ_1 be the connected component containing $\{a, b\}$. Then either every vertex in $\Gamma_1\backslash\{a, b\}$ is at distance one from both a and b and so forms a triangle with $\{a, b\}$, or, without loss of generality, there is some vertex c with $a \sim b \sim c$ and $a \nsim c$. Since a and b commute, their product, ab, commutes both with a and with b. Since a and b are nontrivial and distinct, $\{a, ab, b\}$ are distinct, pairwise commuting elements. Also, $ab \in Z(G)$, contradicts the fact that c commutes with b but not with $a = (ab)b^{-1}$. Thus $ab \in G \setminus Z(G)$ forms a triangle with $\{a, b\}$. □

Recall that a bridge in a connected component Γ_1 of a graph Γ is an edge whose deletion (without removing vertices) makes Γ_1 disconnected. Also, a leaf is a vertex of valency one, and thus the unique edge containing it is a bridge.

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Corollary 3.5. *Let Γ be the commuting graph of a finite group G. If $\{u, v\} \in \Gamma$ is a bridge in some connected component, then $\{u, v\}$ is an isolated edge.*

Corollary 3.6. *Let Γ be a commuting graph of a group and suppose that u is a leaf of Γ with unique neighbour v. Then $\{u, v\}$ is an isolated edge.*

The lexicographic product of graphs Γ_1 and Γ_2 is the graph $\Gamma_1[\Gamma_2]$, with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, where (x_1, y_1) and (x_2, y_2) form an edge if $(x_1, x_2) \in E(\Gamma_1)$ or if $x_1 = x_2$ and $(y_1, y_2) \in E(\Gamma_2)$. We note the following result of Vahidi and Talebi [17].

Lemma 3.7 [17]. *Let G be a group with nontrivial centre of size k. Then $\Gamma(G)$ is the lexicographic product $\Gamma_1[K_k]$, where Γ_1 is the subgraph of Γ induced on a set of representatives of the nontrivial cosets of $Z(G)$ in G.*

The structure of regular commuting graphs was essentially obtained by Itô [12] and follows from Lemma 3.7. Recall that a finite group whose order is a power of a prime p is called a p-group, and a finite group whose order is not divisible by a prime p is called a p'-group.

Lemma 3.8. *Let Γ be the commuting graph of a finite group G and suppose that Γ is regular. Then $G \cong P \times A$ for some p-group P and abelian p'-group A. Moreover, Γ is the lexicographic product $\Gamma_1[K_{p^a|A|}]$, where $|P| = p^{a+b}$ with $a, b \geq 1$ and Γ_1 has order $p^b - 1$.*

Proof. Let $g \in G$ and let d be the valency of Γ. If $g \notin Z(G)$, then g is a vertex of Γ and so has centralizer of order $|Z(G)| + d + 1$. Thus the conjugacy classes of G have size $|G|/(|Z(G)| + d + 1)$ or one (for elements in $Z(G)$). The structure of groups with only two conjugacy class sizes was determined by Itô [12], from which we deduce that G is as in the statement of the lemma. Then $Z(G) = Z(P) \times A$. The result follows from the fact that p-groups have nontrivial centre and Lemma 3.7. □

We can now state the first main result of the present section. We say that a group automorphism ϕ is *fixed-point-free* if $x\phi = x$ implies $x = 1$. If $x\phi = x$ for $x \neq 1$, then x is a nontrivial fixed point. An automorphism of a group A is referred to as *inversion* if it maps each $a \in A$ to a^{-1}.

Theorem 3.9. *Suppose that Γ is a finite graph with an isolated vertex. Then the following are equivalent.*

(i) *Γ is the commuting graph of a group.*
(ii) $1 \neq |V(\Gamma)| \equiv 1 \bmod 4$ *and Γ has exactly $(|V(\Gamma)| + 1)/2$ isolated vertices, while the remaining $(|V(\Gamma)| - 1)/2$ vertices form a complete graph.*

Moreover, the commuting graph of a group G has an isolated vertex if and only if G is the semidirect product $A \rtimes \mathbb{Z}_2$ for some abelian group A of odd order, where $\mathbb{Z}_2$ acts on A by inversion.

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Proof. Let Γ be a finite graph with isolated vertex v and suppose that Γ is the commuting graph of a group G. By Lemma 3.1, $|Z(G)| = 1$ and so G is a finite group with $V(\Gamma) = G\backslash\{1\}$. Since v commutes with v^2 but v has no neighbours, $v^2 \in Z(G) = 1$. As conjugation by an element of G induces an automorphism of Γ, every element in the conjugacy class of v corresponds to an isolated vertex in Γ. Since the conjugacy class of v contains $|G|/|C_G(v)|$ elements, and since $C_G(v) = \{1, v\}$, we see that $|G| = 2n$ is even and at least n vertices are isolated. We label them so that the conjugacy class of v consists of $\{v_1, \ldots, v_n\}$, where $v_1 = v$. In particular, $v_i^2 = 1$ for each i.

Now, for distinct indices i, j, if v_iv_j is isolated, then, as above, $(v_iv_j)^2 = 1$, which we rewrite as $v_iv_j = v_j^{-1}v_i^{-1} = v_jv_i$, so v_i and v_j commute, which contradicts the fact that v_i is isolated. Hence, fixing i, $\{v_iv_1, \ldots, v_iv_n\}\backslash\{v_i^2\}$ consists of $n-1$ pairwise distinct and nonisolated vertices. Since $|G\backslash\{v_1, \ldots, v_n, 1\}| = n - 1$, it follows that every vertex in the set $\hat{N} := G\backslash\{v_1, \ldots, v_n, 1\}$ can be written as v_iv_j for some j. Let $N := \hat{N} \cup \{1\}$. Then, for $u, w \in N$, $u = v_1v_i$ for some i, and $w = v_iv_k$ for some k, and hence $uw = v_1v_iv_iv_k = v_1v_k \in N$. Also, $u^{-1} = v_iv_1 \in N$, so N is a subgroup of index two in G and, therefore, is a normal subgroup.

The map $n \mapsto v^{-1}nv$ is an automorphism of N of order two. Moreover, it is fixed-point-free because $C_G(v) = \{1, v\}$. This implies (see [11, Theorem 1.4, page 336]) that N is abelian and $v^{-1}nv = n^{-1}$ for all $n \in N$. Therefore, $|N| = n = 2k + 1$ must be odd (otherwise, $n^{-1} = n$ for some $n \in N\backslash\{1\}$). Hence $|V(\Gamma)| = |G| - 1 = 2n - 1 = 4k + 1$ for some integer k, and Γ has exactly $n = (|V(\Gamma)| + 1)/2$ isolated vertices, while the remaining $|N| - 1 = (|V(\Gamma)| - 1)/2$ vertices lie in the abelian subgroup N, so they form a complete induced subgraph, as claimed in (ii). Moreover, since $v_1 \notin N$ and has order two, it follows that $G = N \rtimes \langle v_1 \rangle$.

Conversely, if A is an abelian group of odd order $n = 2k + 1 \geq 3$, then A has no elements of order two, so inversion is a fixed-point-free automorphism of order two. This implies that $G := A \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on A by inversion, has trivial centre. It is easily seen that the commuting graph $\Gamma(G)$ has an isolated vertex, corresponding to the generator of $\mathbb{Z}_2$ and, by the first part of the proof, $\Gamma(G)$ is a graph with the properties stated in (ii). This also shows that (ii) implies (i), because if Γ is a graph with properties as in (ii) of order $4n + 1 \geq 5$, then there exists an abelian group A of odd order $2n + 1$. □

Corollary 3.10. *If Γ has an isolated vertex and $|V(\Gamma)| = 2p - 1$ for some odd prime p, then there exists, up to isomorphism, exactly one group such that Γ is its commuting graph.*

We next study groups whose commuting graph has an isolated edge. We will not be able to give as precise a description of the groups in this case as we are only aware of a structural description of nilpotent groups admitting fixed-point-free automorphisms of order three, rather than their complete classification.

Lemma 3.11. *Let G be a finite group whose commuting graph $\Gamma = \Gamma(G)$ contains an isolated edge $\{v, w\}$. Then $Z(G)$ contains at most two elements. Moreover:*

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(i) *if* $|Z(G)| = 1$, *then* $|v| = 3$ *(also,* $w = v^2$ *and* $C_G(v) = \langle v\rangle$*);*

(ii) *if* $|Z(G)| = 2$, *then either:*

(a) $|v| = 4$ *(also,* $Z(G) = \{1, v^2\}$, $w = v^3$ *and* $C_G(v) = \langle v\rangle \cong \mathbb{Z}_4$*); or*

(b) $|v| = 2$ *(also,* $Z(G) = \{1, z\}$, $w = zv$ *and* $C_G(v) = \langle v, z\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$*).*

Proof. Each vertex in an isolated edge has valency one, so the claim about the size of the centre of G follows from Lemma 3.1. First, assume that $Z(G)$ is trivial. Then $\langle v\rangle \leqslant C_G(v) = \{v, w\} \cup Z(G) = \{1, v, w\}$. Thus v has order three, and (i) follows.

Next, assume that $Z(G) = \{1, z\}$ has two elements. Then $z^2 = 1$ and $|G| = |V(\Gamma)| + 2$ and, clearly, zv is noncentral, but commutes with v. Hence $w = zv$. Also, v^2 commutes with v, so either $v^2 = w = zv$ or $v^2 \in Z(G)$. The former possibility contradicts the fact that v is noncentral, so

$$v^2 \in \{1, z\}.$$

If $v^2 = 1$, then $C_G(v) = \{1, v, z, zv\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. However, if $v^2 = z$, then $C_G(v) = \{1, v, v^2, v^3\} \cong \mathbb{Z}_4$ and so $w = v^3 = zv$. □

We can now combine Lemma 3.11 with several group-theoretic results to obtain a characterization of centrefree groups whose commuting graph has an isolated edge. Note that, as additive groups, $\mathbb{Z}_2^4$ is isomorphic to the $\mathrm{GF}(4)^2$ and thus has a natural $\mathrm{SL}(2, 4)$ module structure.

Lemma 3.12. *Let G be a finite group with trivial centre whose commuting graph has an isolated edge. Then one of the following holds.*

(i) $G \cong N \rtimes \mathbb{Z}_3$ *or* $G \cong N \rtimes S_3$, *where N is a nilpotent group of nilpotency class at most two with* $|N| \equiv 1 \bmod 3$, *and each element of G of order three acts as a fixed-point-free automorphism of N.*

(ii) $G \cong N \rtimes A_5$, *where N is the direct product of copies of* $\mathbb{Z}_2^4$, *each viewed as the natural module for* $\mathrm{SL}(2, 4) \cong A_5$.

(iii) $G \cong \mathrm{PSL}(2, 7)$.

Proof. Let $\{v, w\}$ be an isolated edge. Then, by Lemma 3.11, $\langle v\rangle$ is a self-centralizing subgroup of G of order three. Thus, by Feit and Thompson [8], we have one of the following:

(a) G has a normal nilpotent subgroup N such that $G/N \cong \mathbb{Z}_3$ or S_3;

(b) G has a normal 2-subgroup N such that $G/N \cong A_5$;or

(c) $G \cong \mathrm{PSL}(2, 7)$.

Note that $C_G(v) = \langle v\rangle \cong \mathbb{Z}_3$ implies that its normalizer, $\mathcal{N}_G(\langle v\rangle)$, either fixes elements in $C_G(v)$ or swaps v and v^2, so $\mathcal{N}_G(\langle v\rangle) \cong \mathbb{Z}_3$ or S_3.

In case (a), we only need to consider $N \neq 1$. Then N has a nontrivial centre and, as $\langle v\rangle$ is self-centralizing, we must have either $N = \langle v\rangle$ or $N \cap \langle v\rangle = 1$. The first option is not possible, as it would imply that a Sylow 3-subgroup of G has order nine, which would contradict the fact that $\langle v\rangle$ is self-centralizing. Thus $N \cap \langle v\rangle = 1$

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and so conjugation by v is a fixed-point-free automorphism of N. Then, considering the orbits of this action, $|N| \equiv 1 \bmod 3$. Consequently, $\langle v\rangle$ is a Sylow 3-subgroup of G and hence all elements of G of order three are conjugate to either v or v^{-1}, and so act fixed-point-freely on N. Moreover, as $\mathcal{N}_G(\langle v\rangle) \cap N \trianglelefteq \mathcal{N}_G(\langle v\rangle)$ and $v \notin N$ it follows that $\mathcal{N}_G(\langle v\rangle) \cap N = 1$. By Mazurov [13, Theorem, page 29], $G = N\mathcal{N}_G(\langle v\rangle)$, so either $G \cong N \rtimes \mathbb{Z}_3$ or $G \cong N \rtimes S_3$. The additional structure for N and G in cases (a) and (b) follows from Mazurov [13, Theorem, page 29 and Lemma 9, page 33]. □

We note that not every group appearing in (a) and (b) of the proof of Lemma 3.12 has a self-centralizing subgroup of order three. For example, the dihedral group D_{12} of order 12 contains a normal cyclic (and hence nilpotent) subgroup N of order two such that G/N is isomorphic to S_3. However, the unique subgroup of order three in D_{12} is not self-centralizing.

We now investigate the structure of commuting graphs in each of the cases given by Lemma 3.12. We denote by $nK_i + mK_j$ a disjoint union of n copies of the complete graph K_i and m copies of the complete graph K_j.

Theorem 3.13. *Let Γ be the commuting graph of a finite group G with trivial centre and suppose that Γ has an isolated edge. Then one of the following holds:*

(i) $G \cong S_3$ *and* $\Gamma = 3K_1 + K_2$;
(ii) $G \cong A_5$ *and* $\Gamma = 10K_2 + 5K_3 + 6K_4$;
(iii) $G \cong \mathrm{PSL}(2,7)$ *and* $\Gamma = 28K_2 + 8K_6 + \Delta$, *where* Δ *is a connected component on* 63 *vertices with diameter five;*
(iv) $G \cong N \rtimes \mathbb{Z}_3$ *and* $\Gamma = ((|V(\Gamma)| + 1)/3)\,K_2 + \Delta$, *where* Δ *is a connected component of size* $|V(\Gamma)| - 2/3$ *containing a vertex adjacent to all other vertices in* Δ;
(v) $G \cong N \rtimes S_3$ *and* $\Gamma = ((|V(\Gamma)| + 1)/6)\,K_2 + \Delta$, *where* Δ *is a connected component of diameter three; or*
(vi) $G \cong N \rtimes A_5$ *and* $\Gamma = ((|V(\Gamma)| + 1)/6)\,K_2 + ((|V(\Gamma)| + 1)/10)\,K_4 + \Delta$, *where* Δ *is a connected component of diameter three which contains a clique* C *of size* $(|V(\Gamma)| + 1)/60$ *and each element of* Δ *is adjacent to an element of* C.

In the last three cases the structure of N is given in Lemma 3.12.

Proof. Let $\{v, w\}$ be an isolated edge. By Lemma 3.11, $|v| = 3$ and $w = v^2$. The possibilities for G are listed in Lemma 3.12. If $G \cong S_3, A_5$ or $\mathrm{PSL}(2,7)$, then we have one of the first three cases.

Suppose that $G \cong N \rtimes \mathbb{Z}_3$ or $N \rtimes S_3$ for some nontrivial nilpotent group N with properties as in (1) of Lemma 3.12. By Lemma 3.12 and its proof, we know that $C_G(v) = \langle v\rangle \cong \mathbb{Z}_3$ is a Sylow 3-subgroup of G and so there are precisely

$$|G|/|\mathcal{N}_G(\langle v\rangle)|$$

isolated edges. Moreover, $\mathcal{N}_G(\langle v\rangle) \cong \mathbb{Z}_3$ or S_3 and $\mathcal{N}_G(\langle v\rangle) \cap N = 1$. Thus $G = N \rtimes \mathcal{N}_G(\langle v\rangle)$ and the number of isolated edges equals $|G|/|\mathcal{N}_G(\langle v\rangle)| = (|V(\Gamma)| + 1)/3$, when $G/N \cong \mathbb{Z}_3$, and $|G|/|\mathcal{N}_G(\langle v\rangle)| = (|V(\Gamma)| + 1)/6$, when $G/N \cong S_3$.

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If $G/N \cong \mathbb{Z}_3$, then these cover all the elements of G not in N. Since N is nilpotent, it has a nontrivial centre and so the elements of $N\backslash\{1\}$ all have at least one common neighbour. This gives us case (iv).

Let $G/N \cong S_3$. Conjugation by v acts fixed-point-freely on N and hence also on $Z(N)$. All involutions of $\mathcal{N}_G(\langle v\rangle) \cong S_3$ are conjugate and so if one acts fixed-point-freely (by conjugation) on $Z(N)$ all three of them do. Moreover, it follows that each such involution acts as inversion [11, Theorem 1.4, page 336] and so the product of any two acts trivially on $Z(N)$. This contradicts v acting fixed-point-freely on $Z(N)$ and so each involution $g \in \mathcal{N}_G(\langle v\rangle)$ centralizes a nontrivial element of $Z(N)$. Thus each $ng \in Ng$ centralizes some $z \in Z(N)\backslash\{1\}$ and so

$$\Delta = \Bigg(N \cup \bigcup_{\substack{g\in\mathcal{N}_G(\langle v\rangle)\\ |g|=2}} Ng\Bigg) \Bigg\backslash \{1\}$$

forms a connected component of diameter at most three with $|\Delta| = 4|N| - 1 = 4((|V(\Gamma)| + 1)/6) - 1$. Let g_1, g_2 be distinct involutions in $\mathcal{N}_G(\langle v\rangle)$ and suppose that there exists $x \in C_G(g_1) \cap C_G(g_2)$. Then $x = nh$ for some $n \in N$ and $h \in \mathcal{N}_G(\langle v\rangle)$. Now $nh = (nh)^{g_i} = n^{g_i}h^{g_i}$ with $n^{g_i} \in N$ and $h^{g_i} \in \mathcal{N}_G(\langle v\rangle)$. Thus both h and n are centralized by g_1 and g_2. Hence $h = 1$ and n is centralized by g_1g_2. However, g_1g_2 is a nontrivial element of $\langle v\rangle$, which acts fixed-point-freely on N, which is a contradiction. Thus the diameter of Δ is three.

Finally, suppose that $G \cong N \rtimes H$ with $H \cong A_5 \cong \mathrm{SL}(2,4)$ and with $N = \bigoplus_1^t \mathbb{Z}_2^4$ the direct sum of t copies of the natural module $\mathbb{Z}_2^4 \cong \mathrm{GF}(4)^2$ for $\mathrm{SL}(2,4)$. Then $\langle v\rangle$ is a Sylow 3-subgroup of G and so there are precisely $|G|/|\mathcal{N}_G(\langle v\rangle)| = (|V(\Gamma)| + 1)/6$ isolated edges (use that $S_3 \subseteq H$, so v and v^2 are conjugate). Let $g \in G$ have order five. By Sylow's theorem, g is conjugate to an element of H, so we may assume that $g \in H$. Since no matrix of order five in $\mathrm{SL}(2,4)$ acting on its natural module has one as an eigenvalue, it follows that $C_G(g) \cap N = 1$. Thus $C_G(g) \cong C_G(g)N/N \leqslant C_{G/N}(gN) \cong C_H(g) = \langle g\rangle$. Hence $C_G(g) = \langle g\rangle$ and $\langle g\rangle\backslash\{1\}$ is a connected component of $\Gamma(G)$ isomorphic to K_4. Since $\mathcal{N}_G(\langle g\rangle) \cap N$ is a normal subgroup of $\mathcal{N}_G(\langle g\rangle)$ that is disjoint from $\langle g\rangle$, it must centralize $\langle g\rangle$ and so $\mathcal{N}_G(\langle g\rangle) \cap N = 1$. Thus $\mathcal{N}_G(\langle g\rangle) \cong N\mathcal{N}_G(\langle g\rangle)/N$, which is isomorphic to a subgroup of $\mathcal{N}_{G/N}(\langle Ng\rangle) \cong \mathcal{N}_H(\langle g\rangle) \cong D_{10}$. Hence $\mathcal{N}_G(\langle g\rangle) = \mathcal{N}_H(\langle g\rangle) \cong D_{10}$, which gives $|G|/|\mathcal{N}_G(\langle g\rangle)| = |G|/10$ isolated copies of K_4 in Γ. It remains to consider the set

$$\Delta = \Bigg(N \cup \bigcup_{\substack{g_i\in H\\ |g_i|=2}} g_iN\Bigg) \Bigg\backslash \{1\},$$

which has size $|G| - 2|G|/6 - 4|G|/10 - 1 = 16|G|/60 - 1$ (note that A_5 contains 15 involutions). Since N is abelian, the elements of $N\backslash\{1\}$ form a clique. Moreover, since $|N|$ is even, each involution in H centralizes a nontrivial element of N and so every element of Δ is adjacent to some element of $N\backslash\{1\}$. Hence the elements of Δ form a single connected component of diameter at most three. To show that its diameter is at least three, choose involutions $g_1, g_2 \in N_G(\langle g\rangle) \cong D_{10}$. Since $\langle g_1, g_2\rangle = N_G(\langle g\rangle)$ and $C_N(g) = 1$, it follows that g_1 and g_2 do not centralize the same nontrivial element of N. Consequently, if $g_1 \sim x \sim g_2$ is a path in $\Gamma(G)$, then $x \notin N$ and, as such, $g_1N \sim xN \sim g_2N$ would be a path in $\Gamma(G/N) = \Gamma(A_5)$. However, in $\Gamma(A_5)$, the involutions lie in cliques of size three, which contradicts the fact that $\langle g_1N, g_2N\rangle \cong \langle g_1, g_2\rangle \cong D_{10}$ is nonabelian. Hence Δ has diameter three. □

We now investigate groups with nontrivial centre. First we define some groups.

Definition 3.14.

(i) Let $J = \langle a, b, c, \gamma \mid a^3 = b^3 = c^2 = abc = \gamma^2, a^\gamma = b\rangle$ be a nonsplit extension of $\mathrm{SL}(2, 3) = \langle a, b, c \mid a^3 = b^3 = c^2 = abc\rangle$ by $\langle\gamma\rangle \cong \mathbb{Z}_4$; this is SmallGroup (48, 28) in GAP [9]. We refer to [18, pages 104 and 105] for a realization of J as the subgroup of semilinear transformations $\Gamma\mathrm{L}(2, 9)$.

(ii) Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle$, for $n \geq 2$, be a dihedral group. We remark that $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Let $SD_{2n} = \langle a, b \mid a^n = b^2 = 1, a^b = a^{n/2-1}\rangle$, for $n = 2^k \geq 8$, be a semidihedral group.

(iv) Let $Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1, a^n = b^2, a^b = a^{-1}\rangle$, for $n \geq 2$, be a generalized quaternion group.

Lemma 3.15. *Let G be a finite group with nontrivial centre whose commuting graph contains an isolated edge $\{v, w\}$. Then one of the following holds:*

(i) $G \cong \mathrm{SL}(2, 3)$ *or* $G \cong \mathrm{SL}(2, 5)$;

(ii) *G has an abelian normal subgroup N of odd order with $G/N \cong \mathrm{GL}(2, 3)$ or J, the preimage of* $\mathrm{SL}(2, 3)$ *in G, centralizes N and v acts on N by inversion; or*

(iii) *$G \cong N \rtimes H$, where N is an abelian group of odd order and H is isomorphic to one of $\mathbb{Z}_4$, D_4, D_8, Q_8 or D_{2^k}, SD_{2^k}, Q_{2^k}, with $k \geq 4$. Furthermore, if $N \neq 1$, then the group induced by the action of H on N via conjugation is $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and v acts on N by inversion. Moreover, if $N = 1$, then $H \not\cong \mathbb{Z}_4$ or D_4.*

Proof. Let $\{v, w\}$ be an isolated edge in the commuting graph $\Gamma(G)$. By Lemma 3.11, $|Z(G)| = 2$ and either $|v| = 4$ with $w = v^3$ and $Z(G) = \langle v^2\rangle$ or $|v| = 2 = |w| = |vw|$ with $Z(G) = \langle vw\rangle$. Thus $C_G(v) = C_G(w) = C_G(\langle v, w\rangle) = \langle v, w\rangle$ and $\langle v, w\rangle$ is a self-centralizing subgroup of G of order four. Let N be the largest odd order normal subgroup of G. Since $|Z(G)| = 2$, we have $Z(G) \cap N = 1$ and so $Z(G/N) \geq 2$. Thus, by Wong [19, Theorems 1 and 2], G/N is isomorphic to one of $\mathbb{Z}_4$, D_4, D_8, Q_8, $\mathrm{SL}(2, 3)$, $\mathrm{SL}(2, 5)$, $\mathrm{GL}(2, 3)$, J or D_{2^k}, SD_{2^k} or Q_{2^k}, for some $k \geq 4$. Moreover, if $G/N \cong \mathbb{Z}_4$ or D_4, then $N \neq 1$, since G is nonabelian.

Since $|C_G(v)| = 4$ while $|N|$ is odd, $C_G(v) \cap N = 1$. Combined with $v^2 \in Z(G)$, it follows that v induces a fixed-point-free automorphism of N of order two. This implies (see [11, Theorem 1.4, page 336]) that N is abelian and v acts on N by inversion.

Suppose that $N \neq 1$ and let $\phi\colon G \to \mathrm{Aut}(N)$ be the homomorphism induced by the action of G on N by conjugation. Let $M = \ker\phi$. Then $\mathbb{Z}_2 \cong Z(G) \leqslant M$ and

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$Z(G) \cap N = 1$. Thus $N < M \trianglelefteq G$ and M/N contains a central subgroup of G/N of order two. Moreover, $v \notin \ker(\phi)$ and, since $v\phi$ is inversion, $1 \neq v\phi \in Z((G)\phi) \cong Z(G/M)$. In particular, letting R be the preimage of $Z((G)\phi)$ in G, we have the chain of subgroups

$$N < NZ(G) \leqslant M < R \leqslant G,$$

each normal in G, with $R/M = Z(G/M) \neq 1$ and $NZ(G)/N$ being central in G/N and of order two. This is impossible when $G/N \cong \mathrm{SL}(2,3)$ or $\mathrm{SL}(2,5)$. Thus, if $G/N \cong \mathrm{SL}(2,3)$ or $\mathrm{SL}(2,5)$, then $N = 1$ and (i) holds.

Next, assume that $G/N \cong \mathrm{GL}(2,3)$ or J. Since $\mathrm{GL}(2,3)$ and J have a unique normal subgroup of order two, we must have $G/NZ(G) \cong S_4$. Using the normal structure of S_4 and the fact that R/M is a nontrivial central subgroup of G/M, it follows that $R = G$ and $M/NZ(G) \cong A_4$. In particular, $M/N \cong \mathrm{SL}(2,3)$ and, by definition, M centralizes N. Thus (ii) holds.

Suppose finally that $G/N \cong \mathbb{Z}_4$, D_4, D_8, Q_8 or D_{2^k}, SD_{2^k} or Q_{2^k}, with $k \geq 4$. Since N is odd, Sylow's theorems imply that $G = N \rtimes H$ for some Sylow 2-subgroup H of G containing v. Clearly, $H \cong G/N$. Now v acts on N by inversion and, by Lemma 3.11, $|C_G(v)| = 4$. Thus, if $H \cong \mathbb{Z}_4$ or D_4, then $H = C_G(v)$ and $(H)\phi = \mathbb{Z}_2$ as $Z(G) \leqslant H$ and acts trivially on N. In the rest of the cases, take standard generators a and b for H, as given in Definition 3.14, with b having order four when $H \cong Q_{2^k}$ and order two otherwise. Suppose that $|H| \geq 16$. Since $|C_G(v)| = 4$, it follows that $v = a^i b$ for some integer i. Moreover, as $a^v = a^{-1}$ or $a^{2^{k-2}-1}$ but the inversion map $(v)\phi$ commutes with $(a)\phi$, it follows that $(a^2)\phi = 1$. Hence $(H)\phi = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The same argument holds when $|H| = 8$ and $v = a^i b$. Thus it remains to consider the case where $H = D_8$ or Q_8, and $v = a$. However, since $v^b = v^{-1}$ and $(v)\phi$ commutes with $(b)\phi$, it once again follows that $(a^2)\phi = 1$. Thus (iii) holds. $\square$

We now determine the graphs that arise from the groups listed in Lemma 3.15. Commuting graphs of dihedral groups and generalized quaternion groups were studied by [17].

THEOREM 3.16. *Let Γ be the commuting graph of a group G with nontrivial centre and suppose that Γ has an isolated edge. Then one of the following holds.*

(i) $G \cong \mathrm{SL}(2,3)$ *and* $\Gamma = 3K_2 + 4K_4$.
(ii) $G \cong \mathrm{SL}(2,5)$ *and* $\Gamma = 15K_2 + 10K_4 + 6K_8$.
(iii) $G \cong \mathrm{GL}(2,3)$ *or* J *and* $\Gamma = 6K_2 + 4K_4 + 3K_6$.
(iv) $G \cong Q_8$ *or* D_8 *and* $\Gamma = 3K_2$.
(v) $G \cong D_{2^k}$, SD_{2^k} *or* Q_{2^k} *with* $k \geq 4$, *and* $\Gamma = 2^{k-2}K_2 + K_{2^{k-1}-2}$.
(vi) *G is as in part (ii) of Lemma 3.15 with* $N \neq 1$ *and* $\Gamma = ((|V(\Gamma)| + 2)/8)K_2 + \Delta$ *with Δ connected of diameter four.*
(vii) *G is as in part (iii) of Lemma 3.15 with* $N \neq 1$, $|V(\Gamma)| + 2$ *is not a power of two and either:*
    (a) $\Gamma = ((|V(\Gamma)| + 2)/4)K_2 + K_k$ *where* $k = (|V(\Gamma)| - 2)/2$; *or*
    (b) $\Gamma = ((|V(\Gamma)| + 2)/8)K_2 + \Delta$ *with Δ connected of diameter at most three.*

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Proof. Let $\{v, w\}$ be an isolated edge. By Lemma 3.11, $|Z(G)| = 2$ and $\langle v, w\rangle \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The possibilities for G are given in Lemma 3.15. If $G \cong \mathrm{SL}(2, 3)$, we have case (i), if $G = \mathrm{SL}(2, 5)$, we have case (ii) while, if $G = \mathrm{GL}(2, 3)$ or J, we have case (iii).

To continue, suppose first that $G/N \cong \mathrm{GL}(2, 3)$ or J, where N is a nontrivial abelian normal subgroup of odd order. We will use various properties of $\mathrm{GL}(2, 3)$ and J that can be verified using Magma [5] or GAP [9]. By Lemma 3.15, the preimage H of $\mathrm{SL}(2, 3)$ in G centralizes N and v acts on N by inversion. Since $|Z(G) \cap \langle v, w\rangle| = 2$, any element of G that normalizes $\langle v, w\rangle$ either centralizes $\langle v, w\rangle$ or interchanges v and w. Thus $|\mathcal{N}_G(\langle v, w\rangle)| = 4$ or 8. A Sylow 2-subgroup of G is isomorphic to a Sylow 2-subgroup of G/N, and so, looking in $\mathrm{GL}(2, 3)$ or J, we see that $|\mathcal{N}_G(\langle v, w\rangle)| = 8$. Moreover, in $\mathrm{GL}(2, 3)$ and J all self-centralizing subgroups of order four are conjugate and so G has only one conjugacy class of self-centralizing subgroups of order four. Thus there are $|G|/8$ isolated edges in Γ. Since N is abelian, the elements of $NZ(G)\backslash Z(G)$ form a clique and each element of $H\backslash NZ(G)$ is adjacent to each element of N. As $H/N \cong \mathrm{SL}(2, 3)$ is nonabelian, the graph induced on $H\backslash Z(G)$ is of diameter two and contains $|G|/2 - 2$ vertices. If $g \in \mathrm{GL}(2, 3)\backslash \mathrm{SL}(2, 3)$ (respectively, $J\backslash \mathrm{SL}(2, 3)$) does not lie in a self-centralizing subgroup of order four, then g has order eight, $g^2 \in \mathrm{SL}(2, 3)$ and $C_{\mathrm{GL}(2,3)}(g^2) = \langle g\rangle$ (respectively, $C_J(g^2) = \langle g\rangle$). Hence, given two elements $g_1, g_2 \in G\backslash H$ that are not in an isolated edge, we have that $g_1^2, g_2^2 \in H\backslash Z(G)$ and so, for arbitrary $n \in N\backslash\{1\}$, $g_1 \sim g_1^2 \sim n \sim g_2^2 \sim g_2$ is a path of length four in Γ. Choose g_1, g_2 so that $\langle g_1N\rangle$, $\langle g_2N\rangle$ are distinct self-centralizing subgroups of order eight in G/N whose intersection is $Z(G/N)$. If $g_1 \sim a \sim b \sim g_2$ were a path of length three in Γ, then $g_1N \sim aN \sim bN \sim g_2N$ would be a path in the commuting graph of G/N, where we now allow elements in the centre. Since $\langle g_1N\rangle$ and $\langle g_2N\rangle$ are self-centralizing in G/N, it follows that $aN \in \langle g_1N\rangle$ and $bN \in \langle g_2N\rangle$. Since the only elements of $\langle g_1N\rangle$ which commute with elements not in $\langle g_1N\rangle$ are those in $Z(G/N)$, it follows that $aN \in Z(G/N)$, which contradicts the fact that $g_1 \in G\backslash H = vH$ acts on N as a fixed-point-free inversion. Thus the set of vertices of Γ not in an isolated edge forms a connected subgraph of diameter four and Γ is as in part (vi).

Next, suppose that $G = N \rtimes \mathbb{Z}_4$ or $N \rtimes D_4$, with N a nontrivial abelian group of odd order. Since $|Z(G) \cap \langle v, w\rangle| = 2$, any element of G that normalizes $\langle v, w\rangle$ either centralizes $\langle v, w\rangle$ or interchanges v and w. Since $\langle v, w\rangle$ is self-centralizing and is a Sylow 2-subgroup of G, it follows that $\mathcal{N}_G(\langle v, w\rangle) = \langle v, w\rangle$ and there are $|G|/4 = (|V(\Gamma)| + 2)/4$ isolated edges in Γ. This covers $|G|/2$ of the vertices in G and consists of all elements not in the index-two normal subgroup $NZ(G)$. Since N is abelian, so is $NZ(G)$, and hence the vertices not in an isolated edge form a clique of size $(|V(\Gamma)| - 2)/2$. This gives case (vii)(a).

Next, suppose that $G = N \rtimes H$ with $H \cong D_{2^k}$, SD_{2^k} or Q_{2^k} and $k \geq 3$ (if $H \cong SD_{2^k}$, then $k \geq 4$). Take standard generators a and b for H as in Definition 3.14. Let $z = a^{2^{k-2}}$ such that $Z(G) = \langle z\rangle$. If $N = 1$ and $k = 3$, then $G \cong D_8$ or Q_8 and $\Gamma(G) = 3K_2$. This is case (iv). If $N = 1$ and $k \geq 4$, then the elements of $\langle a\rangle\backslash\langle z\rangle$ form a clique of size $2^{k-1} - 2$. Elements of the form a^ib have order two when H is dihedral and order four

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when H is quaternion. When H is semidihedral, $a^i b$ has order two, when i is even, and order four, when i is odd. In all three cases, $C_H(a^i b) = \langle a^i b, z\rangle$ has order four. Thus $\Gamma(G)$ is as in part (v).

Now suppose that $N \neq 1$. By Sylow's theorems, we may assume that $v \in H$, and, by Lemma 3.15, $\langle a^2\rangle$ centralizes N. Hence $(N \times \langle a^2\rangle)\backslash\langle z\rangle$ is a clique of size $|N|2^{k-2} - 2$. Since v acts on N by inversion and $|C_G(v)| = 4$, we have for $|H| > 8$, that $v \notin \langle a\rangle$ and so the kernel M_1 of the action of H on N is either $\langle a^2\rangle$, $\langle a\rangle$ or $\langle a^2, av\rangle$. For $|H| = 8$, it is possible to have $v = a$ or a^3, in which case $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$ are also possibilities for M_1.

Suppose, first, that $M_1 = \langle a\rangle$. Then $N \times \langle a\rangle$ is an abelian group and so $(N \times \langle a\rangle)\backslash\langle z\rangle$ is a clique of size $|G|/2 - 2$. For $g \in G\backslash(N \times \langle a\rangle)$, g induces inversion on N and conjugates a to a^{-1} or $a^{2^{k-2}-1}$. Thus $C_G(g) \cong \langle g, z\rangle$ and so $\Gamma(G)$ contains precisely $|G|/4$ isolated edges. Hence we have case (vii)(a).

Next, suppose that $M_1 = \langle a^2\rangle$. Since v acts on N by inversion, each of the $|H|/2$ elements of $H\backslash\langle a^2, v\rangle$ induces an automorphism of N of order two that is not an inversion (see (iii) of Lemma 3.15). Hence it centralizes a nontrivial element of the abelian group N. Therefore, as $N \neq 1$, the elements from $[N \times (\langle a^2\rangle \cup (H\backslash\langle a^2, v\rangle))] \setminus \langle z\rangle$ form a connected component Δ of diameter at most three on $|G|/4 + |G|/2 - 2$ vertices. In addition, $|\mathcal{N}_G(\langle v, z\rangle)| = 8$ implies there are $|G|/8$ isolated edges in $\Gamma(G)$ conjugate to $\{v, w\}$ and $\Gamma(G) = (|G|/8)K_2 + \Delta$, where Δ is a connected graph of diameter at most three. Thus we have case (vii)(b).

Next, suppose that $M_1 = \langle a^2, av\rangle$. Then, $(N \times M_1)\backslash\langle z\rangle$ has diameter at most two. Also, we may assume $v = b$. Moreover, the $|G|/4$ conjugates of v provide $|G|/8$ isolated edges. The elements of Na^i for each odd i act on N by inversion. For $|H| > 8$, these elements commute with $a^2 \notin Z(G)$ and so $\Gamma(G) = (|G|/8)K_2 + \Delta$, where Δ is a connected graph of diameter two as in (vii)(b). When $|H| = 8$, such elements provide another $|G|/8$ isolated edges and so $\Gamma(G) = (|G|/4)K_2 + K_{|G|/2-2}$, as in (vii)(a).

Finally suppose that $|H| = 8$, $v = a$ or a^3, and that $M_1 = \langle a^2, b\rangle$ or $\langle a^2, ab\rangle$. Then elements of $G\backslash(N \times M_1)$ act on N by inversion and we obtain $|G|/4$ isolated edges. Moreover, $(N \times M_1)\backslash\langle z\rangle$ is a clique, since M_1 is abelian, so we are in case (vii)(a). □

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