KNOT SINGULARITIES OF HARMONIC MORPHISMS

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Abstract A harmonic morphism defined on $\mathbb{R}^3$ with values in a Riemann surface is characterized in terms of a complex analytic curve in the complex surface of straight lines. We show how, to a certain family of complex curves, the singular set of the corresponding harmonic morphism has an isolated component consisting of a continuously embedded knot.

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1. Introduction

In complex variable theory the polynomial equation

$$P(w, z) \equiv z^2 - w = 0,$$

determines $z$ as a 2-valued analytic function of $w$. The point $w = 0$ is called a singular point. It is a branch point of the function $z$, i.e. a point where the equation $P = 0$ has a multiple root. More generally, if $P(w, z)$ is analytic in both variables, a point $w \in \mathbb{C}$ is said to be singular if there exists $z \in \mathbb{C}$ such that

$$P(w, z) = 0,$$

$$\frac{\partial P}{\partial z}(w, z) = 0.$$

The notion of harmonic morphism was introduced in the 1960s as a natural generalization of analytic functions in the plane [4,5]. In general terms, a harmonic morphism is a mapping that preserves harmonic functions. The study of these mappings in the context of Riemannian manifolds began with the work of Fuglede [7] and Ishihara [10]. Thus, let $\phi : M \to N$ be a continuous mapping of Riemannian manifolds. Then $\phi$ is called a harmonic morphism if, for every real-valued function $f$, harmonic on an open subset $V \subset N$ such that $\phi^{-1}(V)$ is non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$.
in $M$. The fundamental characterization \[7, 10\] asserts that $\phi : M \rightarrow N$ is a harmonic morphism if and only if $\phi$ is both horizontally conformal and harmonic. (For background on harmonic mappings, see \[6\]. A map $\phi$ is said to be horizontally conformal if, for each $x \in M$, where $d\phi_x \neq 0$, the restriction $d\phi_x|_{(\ker d\phi_x)\perp} : (\ker d\phi_x)\perp \rightarrow T_{\phi(x)}N$ is conformal and surjective.)

Consequently, if $\phi : M \rightarrow N$ is a harmonic morphism between Riemannian manifolds \[7\], then

(i) $\phi$ is smooth, i.e. $C^\infty$; and

(ii) $\phi$ is an open mapping (in particular $\dim M \geq \dim N$).

Hence, such mappings possess properties enjoyed by analytic functions.

Harmonic morphisms from domains in $\mathbb{R}^3$ with values in a Riemann surface were characterized by Baird and Wood \[3\] in the following way. Let $\phi : U \subset \mathbb{R}^3 \rightarrow N$ be a harmonic morphism to a Riemann surface $N$. Consider a point $x_0 \in U$, and let $\psi : V \subset N \rightarrow \mathbb{C}$ be a local chart about $\phi(x_0) \in N$. Set $z = \psi \circ \phi$. Then, in a neighbourhood of $x_0$, $z = z(x)$ is determined implicitly by

\[(1 - g(z)^2)x_1 + i(1 + g(z)^2)x_2 - 2g(z)x_3 = 2h(z) \quad (x = (x_1, x_2, x_3)), \quad (1.1)\]

where $g(z), h(z)$ are meromorphic or anti-meromorphic functions of $z$. Conversely, any local solution $z$ to (1.1) defined on an open subset $U \subset \mathbb{R}^3$ determines a harmonic morphism $z : U \rightarrow \mathbb{C}$. In particular,

(i) the fibres are always line segments (this is a consequence of a more general result of \[2\]); and

(ii) the foliation by line segments extends to critical points of $\phi$.

By changing orientation on the codomain, we can always assume that $g$ and $h$ are meromorphic functions of $z$. If in addition we suppose they are rational functions, then equation (1.1) becomes a polynomial equation in $z$ of the form

\[P(x, z) \equiv a_n(x)z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x) = 0, \quad (1.2)\]

affine linear in $x$, which can be thought of as defining a multivalued harmonic morphism.

By analogy with analytic function theory, the singular set $K \subset \mathbb{R}^3$ is defined to be those points $x \in \mathbb{R}^3$ simultaneously satisfying

\[P = 0, \quad \frac{\partial P}{\partial z} = 0.\]

Singular points are points where the polynomial $P$ has a multiple root (i.e. the discriminant vanishes). They occur as the envelope points of the congruence of lines determined by (1.2). At such points, a branch $z$ of the solution to (1.2) becomes singular (i.e.
Knot singularities of harmonic morphisms

\[ |dz|^2 \to \infty \). They can also occur as the inverse image of a critical point of a weakly conformal transformation of the codomain: since the composition of a harmonic morphism with a weakly conformal mapping is also a harmonic morphism; if, for example, we set \( z = (w - a)^p \), \( p \geq 2 \), then

\[
\frac{\partial P}{\partial w} = \frac{\partial P}{\partial z} \frac{dz}{dw} = p \frac{\partial P}{\partial z} (w - a)^{p-1}
\]

vanishes when \( w = a \), and the fibre over \( w = a \) determined by \( P(x, w) = 0 \) is singular.

Multivalued harmonic morphisms were considered in [1] in the compact case (replacing \( \mathbb{R}^3 \) by \( S^3 \), where a similar representation to (1.1) holds, see [3]). By taking an \( r \)-valued harmonic morphism defined on \( S^3 \), branched over two linked circles, it was shown how to construct the Lens spaces \( L(r, 1) \) together with a single-valued harmonic morphism \( \phi : L(r, 1) \to S^2 \), by cutting and gluing in a similar fashion to the procedure for constructing compact Riemann surfaces from multivalued analytic functions [12].

This construction was put on a formal footing, at least in the \( \mathbb{R}^3 \) case (indeed \( \mathbb{R}^m \)) by Gudmundsson and Wood [8]. They showed how an equation of the type (1.2) determines a smooth submanifold \( M \subset \mathbb{R}^3 \times \mathbb{C} \), which is a branched covering of \( \mathbb{R}^3 \), branched over the singular set \( K \), together with a single-valued harmonic morphism \( \phi : M \to N \). They proved that the singular set \( K \) is real analytic, and, apart from exceptional cases, consists of arcs of curves, joining points where the multiplicity of the roots of (1.2) increases.

A simple example is given by taking \( g(z) = z \), \( h(z) = iz \). Then (1.2) becomes the polynomial equation

\[
z^2(x_1 - ix_2) + 2z(x_3 + i) - (x_1 + ix_2) = 0.
\]

The solution \( z \) is a 2-valued harmonic morphism branched along the singular set \( K \) which is given by \( x_3 = 0 \), \( x_1^2 + x_2^2 = 1 \), i.e. the unit circle in the \((x_1, x_2)\)-plane. The manifold \( M \) is diffeomorphic to \( S^2 \times \mathbb{R} \), which double covers \( \mathbb{R}^3 \), branched over the circle \( K \). There are very few other examples where the singular set has been constructed and in all such cases, it has a very simple structure.

Following the well-known topological construction, obtaining 3-manifolds as branched coverings over knots (see [13]), it becomes interesting to know whether it is possible to obtain a knot singularity to equation (1.2). In this paper, we prove that such singularities exist. Precisely, we give examples of \( g \) and \( h \) such that the singular set \( K \) has an isolated component consisting of a continuously embedded knotted curve. In fact, we exhibit a family of such parametrized by the odd integers \( p = 3, 5, 7, \ldots \), beginning with the trefoil knot \((p = 3)\).

We consider a particular holomorphic curve \( C \) in \( \mathbb{C}^2 \) parametrized in the form

\[
g = z^p, \quad h = z^{p+2} + i \beta z^p,
\]

for \( \beta \) a real positive number. By estimating roots of a certain polynomial equation, we are able to demonstrate that a connected component of the singular set in \( \mathbb{R}^3 \) is a knot.
2. The singular set of a harmonic morphism

Let \( z : U \to \mathbb{C}, U \text{ open in } \mathbb{R}^3 \) be implicitly defined by equation (1.1). For ease of exposition we write \( q = x_1 + ix_2 \), identifying \( \mathbb{R}^3 \) with \( \mathbb{C} \times \mathbb{R} \), so that (1.1) can now be written

\[
P(x, z) = g^2 \bar{q} + 2gx_3 - q - 2h = 0,
\]

for meromorphic functions \( g \) and \( h \).

The singular set of \( z \) is defined to be the solution set in \( \mathbb{R}^3 \) of the simultaneous equations

\[
\begin{align*}
P &= 0, \\
\frac{\partial P}{\partial z} &= 0.
\end{align*}
\]

Note that this represents four real constraints in five real variables. In general, the solution in \( x \) will be a real analytic subset of \( \mathbb{R}^3 \) of codimension 2 \([8]\).

There is also a solution set in the \( z \)-plane, which we refer to as the singular values of \( z \).

In terms of the representation (2.1), equation (2.2) has the form

\[
\begin{align*}
g^2 \bar{q} + 2gx_3 - q - 2h &= 0, \\
gg' \bar{q} + g'x_3 - h' &= 0.
\end{align*}
\]

Solving for \( q \) and \( x_3 \) gives

\[
\begin{align*}
q &= \frac{2\{gh' - h'g\} - g^2 \bar{g}^2 (\bar{h} - \bar{h}')}{|g|^2 (1 - |g|^4)}, \\
x_3 &= -g\bar{q} - \frac{h'}{g}.
\end{align*}
\]

The requirement that \( x_3 \) be real yields the condition

\[
2|g|^2 (\bar{g}h - gh) = (1 + |g|^2)(g\bar{h}' - g'h').
\]

Equation (2.4) determines the singular values \( \Sigma \) of \( z \). The image of \( \Sigma \) under the mapping (2.3) gives the singular set \( K \subset \mathbb{R}^3 \).

If we holomorphically reparametrize \( z \) by a transformation \( z = z(w), z'(w) \neq 0 \), then \( g'(z) = g'(w)w'(z) \) and we see that equation (2.4) remains invariant as well as the image \( K \) determined by (2.3). In particular, the singular set \( K \) depends only on the holomorphic curve \( C \subset \mathbb{C} \times \mathbb{C} \) determined by \( g \) and \( h \). (More precisely, the space of lines in \( \mathbb{R}^3 \) is identified with the complex surface \( TS^2 \) (cf. [9]), and \( C \) should be thought of as lying in \( TS^2 \). Then \( g \) and \( h \) may admit poles. However, for our purposes, it suffices to work in a trivialization: \( T(S^2 \setminus \text{point}) \cong \mathbb{C} \times \mathbb{C} \).)

3. Knot examples

Let \( g(z) = z^p \), where \( p \) is an odd integer \( \geq 3 \) and set \( h(z) = z^{p+2} + i\beta z^p \), where \( \beta \) is a real positive number. Equation (2.4) becomes

\[
2p|z|^{4p-2}(z^2 - z^2 - 2i\beta) = (1 + |z|^{2p})|z|^{2p-2}((p + 2)(z^2 - z^2) - 2ip\beta).
\]
Thus, either $z = 0$, or, incorporating constants into $\beta$,

$$
(z^2 - z^2)((p - 2)|z|^{2p} - (p + 2)) - i\beta(|z|^{2p} - 1) = 0.
$$

(3.1)

We will prove the solution has the form sketched in Figure 1.

**Theorem 3.1.** For $\beta > 0$, the solution set to (3.1) inside the circle $|z| = a$, $a = ((p+2)/(p-2))^{1/2p}$, consists of a smoothly embedded closed curve. The curve is symmetric under reflection in the lines $y = \pm x$ and may be parametrized in the form $z = \sqrt{R}e^{i\theta}$, where $R = R(\theta)$ is a smooth positive function satisfying $R > 1$ for $\theta \in (0, \pi/2) \cup (\pi, 3\pi/2)$; $R < 1$ for $\theta \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$; $R$ is monotone increasing on the interval $(-\pi/4, \pi/4)$ and has Taylor expansion about $\theta = 0$ given by

$$
R(\theta) = 1 + \frac{8\theta}{p\beta} - \frac{16(p^2 - 6)\theta^2}{p^2\beta^2} + O(\theta^3).
$$

(3.2)

**Remark 3.2.** The estimates on $R$ provide essential information in establishing winding numbers, which will confirm that the singular set is a knot.

Note that the point $z = 1$ satisfies (3.1). It is the solution passing through this point that we isolate. By the reflectional symmetry of (3.1), we restrict our attention to the quadrant $-\pi/4 \leq \theta \leq \pi/4$.

**Lemma 3.3.** The solution set to equation (3.1) consists of smoothly embedded curves.

**Proof.** This is a simple application of the Implicit Function Theorem. \(\square\)
Let \( z = re^{i\theta} \) and set \( R = r^2 \), then (3.1) becomes
\[
P_\beta(R, \theta) \equiv R\{ (p - 2)R^p - (p + 2) \} \sin 2\theta - \beta(1 - R^p) = 0.
\] (3.3)

For \( \theta = 0 \), this has the unique solution \( R = 1 \).

**Lemma 3.4.**

(i) For \( \theta \in (-\pi/4, 0) \), there is precisely one positive root \( R = R(\theta) \) to (3.3) satisfying \( R^p < (p + 2)/(p - 2) \). This root lies in the interval
\[
I = \left( 0, \min \left\{ \frac{\beta}{(p - 2)\sin 2\theta}, 1 \right\} \right).
\]

(ii) For \( \theta \in (0, \pi/4) \), there is precisely one positive root \( R = R(\theta) \) to equation (3.3); this lies in the interval
\[
J = \left( \max \left\{ 1, \left( \frac{p + 2}{p - 2} - \frac{4\beta}{(p - 2)^2\sin 2\theta} \right)^{1/p}, \left( \frac{p + 2}{p - 2} \right)^{1/p} \right\} \right).
\]

**Proof.** (i) Define \( f \) on the interval
\[
(0, a^2), \quad a = \left( \frac{p + 2}{p - 2} \right)^{1/2p},
\]
by
\[
f(R) = \frac{\beta(1 - R^p)}{R\{ (p - 2)R^p - (p + 2) \}}.
\]
Then \( P_\beta(R, \theta) = 0 \) if and only if \( \sin 2\theta = f(R) \). Now,
\[
\lim_{R \to 0} f(R) = -\infty \quad \text{and} \quad \lim_{R \to a^2} f(R) = +\infty,
\]
so for each \( \theta \), there exists \( R(\theta) \in (0, a^2) \) such that \( f(R(\theta)) = \sin 2\theta \). A calculation of the derivative,
\[
f'(R) = \frac{\beta\{2(p - 2)R^{2p} + (p + 2)R^p + (p + 2)\}}{R^2\{(p - 2)R^p - (p + 2)\}^2} > 0,
\]
shows that \( f \) is monotone over \((0, a^2)\), and the root \( R = R(\theta) \) is unique.

To locate the root more accurately for \( \theta \in (-\pi/4, 0) \), we note that
\[
P_\beta(0, \theta) = -\beta < 0,
\]
\[
P_\beta \left( \frac{\beta}{-(p - 2)\sin 2\theta}, \theta \right) = 4\beta/(p - 2) > 0,
\]
\[
P_\beta(1, \theta) = -4\sin 2\theta > 0,
\]
so the root \( R(\theta) \) lies in \( I \).
(ii) As for part (i), for \( \theta \in (0, \pi/4) \), there is a unique root \( R = R(\theta) \) lying in the interval \((0, a^2)\). There can be no other root, since \( P_\beta(R, \theta) > 0 \) for \( R \geq a^2 \).

For \( R = ((p + 2)/(p - 2))^{1/p} \), \( P_\beta(R, \theta) = 4\beta/(p - 2) > 0 \).

For \( R = 1 \), \( P_\beta(R, \theta) = -4 \sin 2\theta < 0 \).

For

\[
R = \left( \frac{p + 2}{p - 2} \right)^{1/p} - \frac{4\beta}{(p - 2)^2 \sin 2\theta} \quad (\geq 1),
\]

\[
P_\beta(R, \theta) = -\frac{4\beta R}{(p - 2)} - \frac{4\beta^2}{(p - 2)^2 \sin 2\theta} + \frac{4\beta}{(p - 2)} < 0,
\]

so the root lies in \( J \).

For each \( \theta \in [0, 2\pi] \), let \( R(\theta) \) denote the unique root to \( P_\beta(R, \theta) = 0 \) lying in the interval \((0, ((p + 2)/(p - 2))^{1/p}) \). Then by Lemma 3.3, \( R = R(\theta) \) is a smooth function of \( \theta \).

**Lemma 3.5.** \( R(\theta) \) is monotone increasing on the interval \((-\pi/4, \pi/4)\).

**Proof.** Differentiating equation (3.3) implicitly yields

\[
R' = \frac{2R(p + 2) - (p - 2)R^p \cos 2\theta}{((p + 1)(p - 2)R^p \sin 2\theta - (p + 2) \sin 2\theta + p\beta R^{p-1})}.
\]

Expressing \( \sin 2\theta \) in terms of \( R \), again using equation (3.3) yields

\[
R' = \frac{2R^2(p + 2) - (p - 2)R^p \cos 2\theta}{((p - 2)R^p + (p + 2))(1 + R^p)\beta},
\]

which is strictly positive on \((-\pi/4, \pi/4)\). \( \square \)

Setting \( \theta = 0 \) in (3.4) yields \( R'(0) = 8/p\beta \). Differentiating once more establishes \( R''(0) = -32(p^2 - 6)/p^2 \beta^2 \) and the Taylor expansion to order two for \( R(\theta) \) about \( \theta = 0 \) follows. This completes the proof of Theorem 3.1. \( \square \)

We now consider the map (2.3) into \( \mathbb{R}^3 \), sending \( \Gamma_\beta \) to the singular set. We establish the following theorem.

**Theorem 3.6.** The singular set of the multivalued harmonic morphism determined by equation (1.1), with \( q(z) = z^p, h(z) = z^{p+2} + i\beta z^p, p = 3, 5, 7, \ldots, \) where \( \beta \) is a real positive number, has a compact component \( K^p_0 \), consisting of a continuously embedded knotted curve. For each \( p = 3, 5, 7, \ldots, \) all of the knots \( K^p_0 \) are distinct, i.e. non-isotopic.

By equation (2.3)

\[
q(z) = \left\{ \begin{array}{ll}
4z^{p-2}(-z^4 + |z|^{2p+4}) \\
p(1 - |z|^{4p})
\end{array} \right.,
\]

\[
x_3(z) = \left\{ \begin{array}{ll}
4z|z|^{2p} + (p - 2)z^2|z|^{4p} - (p + 2)z^2 \\
p(1 - |z|^{4p}) - i\beta.
\end{array} \right.
\]

\( \square \)
Lemma 3.7. The maps $q$ and $x_3$ are continuous on the set $\Gamma_\beta$. The image of the point $z = 1$ is given by $q(1) = (-2 + i\beta)/p$, $x_3(1) = -1$.

Proof. By Lemma 3.4 (showing $|z| \neq 1$ for $z \in \Gamma_\beta$, $-\pi/4 \leq \arg z < 0$ and $0 < \arg z \leq \pi/4$), we need only check continuity at the points $z = \pm 1, \pm i$ and indeed by reflectional symmetry, the point $z = 1$ will suffice.

Note that, if $\xi : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ is the map $\xi(z) = (q(z), x_3(z))$ given by (2.3), then $\xi$ is not continuous at $z = 1$. Setting $z = 1 + \rho e^{i\alpha}$, we calculate

$$
q(z) = -\frac{2}{p} + \frac{4i \tan \alpha}{p^2},
$$

whose limit as $\rho \to 0$ depends on $\alpha$. However, along $\Gamma_\beta$ we have continuity.

From equation (3.5),

$$
q(\theta) = \frac{4R^{(p+2)/2}}{p(1 - R^2p)} \{-e^{i(p+2)\theta} + R^p e^{i(p-2)\theta}\},
$$

$$
x_3(\theta) = -\frac{R\{(p - 2)R^p + (p + 2)\}}{p(1 + R^p)} \cos 2\theta.
$$

Substituting the Taylor approximation to first order for $R(\theta)$ about $\theta = 0$,

$$
R(\theta) = 1 + \frac{8\theta}{p\beta} + O(\theta^2),
$$

we see that, as $\theta \to 0$, $q \to (-2 + i\beta)/p$, $x_3 \to -1$. □

Note that $q$ obeys the same reflectional symmetry (now in the planes $x_1 = \pm x_2$) as the $R$-curve, and, as a consequence (or by direct observation), $q(\theta) = -q(\theta + \pi)$. Also, the height function $x_3$ satisfies $x_3(\theta) = x_3(\theta + \pi)$.

A useful picture to have in mind is that of a horizontal rotating rod that moves up and down the $x_3$-axis. Imagine two points at opposite ends of the rod, equidistant from the $x_3$-axis, that move in and out. These two points will trace out the curve $K_0^p$.

Sketches of $K_0^p$ for $p = 3$ and $p = 5$ are given in Figure 2.

In order to establish this form, we calculate the winding numbers of the inner and outer curves about the $x_3$-axis. However, to be sure no local knotting, unknotting or self-intersection points occur, we must study the functions $|q(\theta)|$ and arg $q(\theta)$.

Lemma 3.8. The argument arg $q(\theta)$ is a $C^1$ function of $\theta$ such that $(d/d\theta) \arg q(\theta) > 0$ for all $\theta$. In particular, $\arg q(\theta)$ is a strictly increasing function of $\theta$.

Note. The continuous function $q(\theta)$ is not even differentiable.

Proof.

$$
\arg q = \arg \{-e^{i(p+2)\theta} + R^p e^{i(p-2)\theta}\}
= \arg \{X + iY\},
$$
Figure 2. The trefoil and cinqufoil knots arise as the compact components $K_3^0$ and $K_5^0$, respectively.

where $X = -\cos(p + 2)\theta + R^p \cos(p - 2)\theta$ and $Y = -\sin(p + 2)\theta + R^p \sin(p - 2)\theta$. Then

$$\frac{d}{d\theta} \arg q = \frac{XY' - YX'}{X^2 + Y^2}.$$ 

Setting $N(\theta) = XY' - YX'$ and $D(\theta) = X^2 + Y^2$, a lengthy calculation verifies

$$N = p(1 - R^p)^2 + 2pR^2(1 - \cos 4\theta) + 2(1 - R^{2p}) + pR^{p-1}R' \sin 4\theta,$$

$$D = (1 - R^p)^2 + 2R^p(1 - \cos 4\theta).$$

(3.8) (3.9)

Clearly $D \geq 0$ and $D = 0$ if and only if $R = 1$ and $\theta = 0, \pi/2, \pi, 3\pi/2$. 

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Substituting the expression for $R'$ given by equation (3.4) and using (3.3) to eliminate $\theta$, we calculate

$$N = \frac{(1 - R^p)^2 \{(p + 2) - (p - 2)R^2\}^2}{(1 + R^p)^2 \{(p + 2) + (p - 2)R\}^2}$$

$$+ \frac{8p\beta^2 R^p - 2(p + 2)(p - 2)R^2}{(1 + R^p)^2 \{(p + 2) - (p - 2)R^p\}^2 \{(p + 2) + (p - 2)R^p\}}.$$  

(3.10)

which again is $\geq 0$ and $= 0$ if and only if $R = 1$ (and so $\theta = 0, \pi/2, \pi, 3\pi/2$).

In order to evaluate the limit, $\lim_{\theta \to 0} N(\theta)/D(\theta)$, we evaluate the Taylor expansions to order 2 of $N$ and $D$, using the expansion for $R$ given by (3.2). This is most simply done by substituting into equations (3.8) and (3.9) and we find

$$N(\theta) = \frac{16}{p\beta^2} (16 + p^2\beta^2)\theta^2 + O(\theta^3),$$

$$D(\theta) = \frac{16}{p^2\beta^2} (4 + p^2\beta^2)\theta^2 + O(\theta^3),$$

so that $(d/d\theta) \arg q$ is continuous and positive for all $\theta$ with

$$\frac{d}{d\theta} \arg q|_{\theta=0} = \frac{p(16 + p^2\beta^2)}{4 + p^2\beta^2}. $$

Thus, there can be no ‘local’ knotting or self-intersection, i.e. for $\theta$ in a sufficiently small interval. Knotting or self-intersection can only occur as a consequence of winding about the $x_3$-axis.

**Lemma 3.9.** The function $|q|^2$ is monotone increasing on the interval $-\pi/4 < \theta < \pi/4$.

**Proof.** Now

$$|q|^2 = \frac{16R^{p+2}}{p^2(1 - R^p)^2} \{1 + R^{2p} - 2R^p \cos 4\theta\},$$

which, after setting $\cos 4\theta = 1 - 2\sin^2 2\theta$ and substituting for $\sin 2\theta$ from equation (3.3), becomes

$$|q|^2 = \frac{16R^{p+2}}{p^2(1 + R^p)^2} \left\{1 + \frac{4R^{2p-2}\beta^2}{\{(p - 2)R^p - (p + 2)\}^2}\right\}.$$ 

Differentiating with respect to $\theta$ yields

$$\frac{d}{d\theta} |q|^2 = \frac{16R^{p+1}}{p^2(1 + R^p)^2} \left\{(p + 2) - (p - 2)R^2\right\} \left(1 + \frac{4\beta^2 R^{p-2}}{\{(p - 2)R^p - (p + 2)\}^2}\right)$$

$$+ \frac{4(p - 2)(p + 2)R^{p-2}\beta^2}{\{(p + 2) - (p - 2)R^p\}^2} R'.$$

The coefficient of $R'$ in this expression is always $> 0$, so that the sign of $(d/d\theta)|q|^2$ equals that of $R'$ and $|q|^2$ increases and decreases with $R$. The result now follows from Theorem 3.1. \qed
Lemma 3.10.

(i) As $\theta$ varies from $-\pi/2$ to 0 (the inner curve), $q(\theta)$ rotates about the origin through an angle $(p + 2)\pi/2 - 2\tan^{-1}(\beta/2)$, beginning at the point $(-1)^{(p-1)/2}(\beta - 2i)/p$ and ending at the point $(-2 + i\beta)/p$.

(ii) As $\theta$ varies from 0 to $\pi/2$ (the outer curve), $q(\theta)$ rotates about the origin through an angle $(p - 2)\pi/2 + 2\tan^{-1}(\beta/2)$, beginning at the point $(-2 + i\beta)/p$ and ending at the point $(-1)^{(p-1)/2}(-\beta + 2i)/p$.

Remark 3.11. Because of reflectional symmetry of $q$ in the lines $\theta = \pm\pi/4$, the above lemma describes the rotation of $q$ completely. For example, as $\theta$ varies from $-\pi/4$ to $\pi/4$, $q$ rotates through $(p + 2)\pi/4 - \tan^{-1}(\beta/2) + (p - 2)\pi/4 + \tan^{-1}(\beta/2) = p\pi/2$ about the origin.

Proof. The index about the origin,

$$\text{Ind}_{\gamma}(0) = \int_{\gamma} \frac{\gamma'}{\gamma} \, d\theta,$$

of a curve $\gamma$ not passing through 0, measures the change in argument about 0. Noting that $\text{Ind}_{\gamma \mu}(0) = \text{Ind}_{\gamma}(0) + \text{Ind}_{\mu}(0)$, we deduce

$$\text{Ind}_{(\gamma+\mu)}(0) = \text{Ind}_{(\gamma)(1+(\mu/\gamma))}(0) = \text{Ind}_{\gamma}(0) + \text{Ind}_{(1+(\mu/\gamma))}(0).$$

(3.11)

Suppose $-\pi/2 < \theta < 0$. Let $\gamma(\theta) = -e^{i(p+2)\theta}$, $\mu(\theta) = R^p e^{i(p-2)\theta}$. Then $\mu/\gamma = -R^p e^{-4i\theta}$. Noting that $R < 1$ in the interval $(-\pi/2, 0)$, we apply equation (3.11) and
study the change in argument of the curve \( \alpha(\theta) = 1 - R^p e^{-4i\theta} \), which begins and terminates at 0 and has the tear drop shape indicated in Figure 3.

Now,

\[
\arg \alpha = \tan^{-1} \left( \frac{R^p \sin 4\theta}{1 - R^p \cos 4\theta} \right).
\]

We evaluate \( \lim_{\theta \to 0} \tan \arg \alpha \). From Theorem 3.1, the Taylor expansion to first order of \( R(\theta) \) is given by

\[
R(\theta) \sim 1 + \frac{8\theta}{\beta^2},
\]

so that

\[
R^p \sin 4\theta \sim 4\theta \left( 1 + \frac{8\theta}{\beta} \right)
\]

and

\[
1 - R^p \sin 4\theta \sim -\frac{8\theta}{\beta}.
\]

Thus

\[
\lim_{\theta \to 0} \tan \arg \alpha = -\frac{\beta}{2}.
\]

On the other hand, \( \text{Ind}_\gamma(0) = (p + 2)\pi/2 \). The result now follows from (3.11) and Lemma 3.7. Similar arguments give part (ii).

Finally, we note the behaviour of the height function \( x_3 \).

**Lemma 3.12.** The sign of \( x_3 \) is equal to the sign of \( -\cos 2\theta \).

**Proof.** This is a simple consequence of equation (3.7).
Figure 5. The shadow of the trefoil knot with $x_3 < 0$ depicted by the continuous curve and $x_3 > 0$ depicted by the dashed curve.

**Proof of Theorem 3.6.** As $\theta$ varies from $-\pi/4$ to $\pi/4$ and from $3\pi/4$ to $5\pi/4$ ($x_3$ negative), the shadows in the $q$-plane of the corresponding arcs of $K_0^p$ trace out two interlacing spirals, which, by Remark 3.11, rotate through $p\pi/2$ as indicated in Figure 4.

These spirals can never intersect on account of Lemma 3.9, since $|q|$ is strictly increasing over these intervals.

A similar pair of interlacing spirals occur on the complementary arcs $(\pi/4, 3\pi/4)$ and $(5\pi/4, 7\pi/4)$, now with $x_3$ positive. In $\mathbb{R}^3$, the curves are joined at the points $(q(\theta_k), 0)$, $\theta_k = (2k + 1)\pi/4$, $k = 0, 1, 2, 3, \ldots$, giving a simple continuous closed curve $K_0^p$ with no self-intersection points, i.e. a knot. This is sketched in Figure 5, where the continuous curve corresponds to $x_3 < 0$ and the broken curve to $x_3 > 0$.

To establish the topological nature of the knot $K_0^p$, we can ‘pull back’ the outer curve $(0 \leq \theta \leq \pi/2$ and $\pi \leq \theta \leq 3\pi/2)$, like winding a spring and increase the rotation of the inner curve by $(p - 2)\frac{1}{2}\pi + 2\tan^{-1}(\frac{1}{2}\beta)$. Then each ‘inner curve’ now rotates through $p\pi$.

It is well known (cf. [11]) that these knots all have distinct Alexander Polynomials, and, hence, are non-isotopic. This establishes Theorem 3.6. □

4. Comments and further problems

Similar considerations apply to the holomorphic curve $g = z^p$, $h = z^{p+1} + iz^p$, for $p = 2, 3, 4, \ldots$, which yields a closed loop of singular values as sketched in Figure 6. However, the corresponding closed loop singularity in $\mathbb{R}^3$ is never knotted.

For more general holomorphic curves, the singularity becomes difficult to compute,
however, it is to be expected that knots and links of various kinds should occur. Also, more general deformations of the type

\[ g = z^p, \quad h = z^q + \varepsilon_1 z^{r_1} + \varepsilon_2 z^{r_2} + \cdots, \]

following a Puiseux expansion, should yield interesting behaviour. It would be useful to have an effective computer method to sketch the singular set in \( \mathbb{R}^3 \).

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**References**


