Proceedings of the Edinburgh Mathematical Society (1989) 32, 371–375 ©

ON PSEUDO-FINITE NEAR-FIELDS WHICH HAVE FINITE DIMENSION OVER THE CENTRE

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(Received 12th October 1987)

1. Introduction

In [1] J. Ax studied a class of fields with similar properties as finite fields called pseudo-finite fields. One can prove that pseudo-finite fields are precisely the infinite models of the first-order theory of finite fields. Similarly a near-field F is called pseudo-finite if F is an infinite model of the first-order theory of finite near-fields. The structure theory of these near-fields has been initiated by U. Felgner in [**5**].

In this paper we characterize all pseudo-finite near-fields having finite dimension over the centre. We prove that these near-fields are precisely the derivations of pseudo-finite fields with finite cyclic Dickson groups.

Apart from the fact that we use right near-fields we mainly follow the terminology of Wähling [**8**]. For a field $(K, +, \cdot)$ and a coupling map $\chi: K\backslash\{o\}\to\mathrm{Aut}(K)$ with Dickson group Δ_χ we let $\mathrm{Fix}(\Delta_\chi)=\{k\in K \mid \gamma(k)=k\ \forall\gamma\in\Delta_\chi\}$, $U_\chi=\{k\in K\backslash\{o\}\mid \chi(k)=\mathrm{id}\}$ and K^χ be the χ-derivation of K. If $(F, +, o)$ is a near-field, then $Z(F)$, $K(F)$ shall denote the centre and the kernel of F, respectively. If (q,n) is a Dickson pair, where $q=p^l$ for some prime p and n is a positive integer, let $F(q,n)$ denote a finite Dickson near-field of order q^n. For an index set A, an ultrafilter U on A and a collection $\{F_\alpha \mid \alpha\in A\}$ of near-fields F_α we can form the ultraproduct $\prod_U F_\alpha$. Elements of $\prod_U F_\alpha$ shall be denoted by $(f_\alpha)_U$. If $F_\alpha=F$ for all $\alpha\in A$, we write F^A/U instead of $\prod_U F_\alpha$.

Ultraproducts yield an alternative definition of pseudo-finite near-fields. A near-field F is pseudo-finite if and only if F is infinite and elementarily equivalent to an ultraproduct of finite Dickson near-fields. In particular the class of all pseudo-finite near-fields is closed under ultraproducts. For ultraproducts consult Chang–Keisler [**2**]. To denote elementary equivalence we shall use the symbol $\equiv$.

2.

We shall make frequent use of the following result. A proof can be found in Trautvetter [**7**].

Proposition 2.1. *Let $\{F_\alpha \mid \alpha\in A\}$ be a collection of Dickson near-fields, where $F_\alpha=K_\alpha^{\chi_\alpha}$ for a field K_α, $\alpha\in A$. For any ultrafilter U on A, $\prod_U F_\alpha$ is again a Dickson near-field and $\prod_U F_\alpha=(\prod_U K_\alpha)^\chi$ where $\chi:(\prod_U K_\alpha)\backslash\{o\}\to\mathrm{Aut}(\prod_U K_\alpha)$, $\chi((k_\alpha)_U)=(\chi_\alpha(k_\alpha))_U$. Here the action* *of* $(\chi_\alpha(k_\alpha))_U$ *on* $\prod_U K_\alpha$ *is component-wise. Moreover,* $\Delta_\chi = \prod_U \Delta_{\chi\alpha}$ *and* $\mathrm{Fix}(\Delta_\chi) = \prod_U \mathrm{Fix}(\Delta_{\chi\alpha})$.

If F is pseudo-finite, then $Z(F) = K(F)$ (3.1 in [**5**]), thus $Z(F)$ is a subfield of F. The dimension of F as a vector-space over $Z(F)$ shall be denoted by $[F{:}Z(F)]$.

Proposition 2.2 *Let F be a pseudo-finite near-field with $[F{:}Z(F)]$ finite. Then*

(a) *F is a Dickson near-field and there exists a commutative field K such that $F = K^\chi$ for some coupling map χ on K.*

(b) $Z(F) = Fix(\Delta_\chi) \subseteq U_\chi \cup \{o\}$.

Proof.

(a) Has been mentioned by Felgner [**5**].

(b) By ([**8**, III.5.7]) $Z(F) \subseteq \mathrm{Fix}(\Delta_\chi)$. On the other hand $\mathrm{Fix}(\Delta_\chi) \subseteq Z(F)$ since $\mathrm{Fix}(\Delta_\chi) \subseteq K(F)$ and $K(F) = Z(F)$. Moreover $Z(F) \backslash \{o\} \subseteq U_\chi$ by ([**8**, III.5.5.(b)]).

For a field K, a subfield L of K and $l_1, \ldots, l_n \in K$ let $L(l_1, \ldots, l_n)$ denote the subfield generated by $L \cup \{l_1, \ldots, l_n\}$. If G is a group and $g \in G$ then $\langle g \rangle$ shall denote the subgroup generated by g.

Proposition 2.3. *Let E be a commutative field and let χ be a coupling map on E such that $\Delta_\chi = \{\gamma, \gamma^2, \ldots, \gamma^{n-1}, id\}$ is cyclic of finite order n. If $Fix(\Delta_\chi) \cong \prod_U GF(q_\alpha)$ where U is an ultrafilter on a set A and $q_\alpha = p_\alpha^{l_\alpha}$ for a collection $\{p_\alpha \mid \alpha \in A\}$ of prime numbers and positive integers $\{l_\alpha \mid \alpha \in A\}$, then E^χ is isomorphic to an ultraproduct of finite Dickson near-fields and $[E^\chi{:}Z(E^\chi)] = n$.*

Proof. Let $L_1 = \mathrm{Fix}(\Delta_\chi)$, $L_2 = \prod_U GF(q_\alpha)$ and $\sigma{:}L_1 \to L_2$ be an isomorphism. Since $|\Delta_\chi| = n$ we have that $[E{:}L_1] = n$. If $K = \prod_U GF(q_\alpha^n)$, then $[K{:}L_2] = n$ since $[GF(q_\alpha^n){:}GF(q_\alpha)] = n$ for all $\alpha \in A$. Both L_1, L_2 are perfect fields. Let $l_1 \in E$ such that $L_1(l_1) = E$ and let $p_1(x) = x^n + a_1 x^{n-1} + \cdots + a_n, a_1, \ldots, a_n \in L_1$, be the minimal polynomial of l_1. Clearly all zeros of $p_1(x)$ are given by $l_1, \gamma(l_1), \ldots, \gamma^{n-1}(l_1)$. Let $p_2(x) = x^n + \sigma(a_1)x^{n-1} + \cdots + \sigma(a_n)$ where $\sigma(a_i) = (b_{\alpha i})_U$, $i \in \{1 \ldots n\}$. Since $p_2(x)$ is irreducible over L_2 there exists an element $w \in U$ such that $p_2^\alpha(x) = x^n + b_{\alpha_1} x^{n-1} + \cdots + b_{\alpha_n}$ is irreducible over $GF(q_\alpha)$ for all $\alpha \in w$. By ([**6**, Th. 3.46]) we can find elements $l_\alpha \in GF(q_\alpha^n)$ such that l_α is a zero of $p_2^\alpha(x)$ for all $\alpha \in w$. If $l_\alpha \in GF(q_\alpha^n)$ is chosen arbitrarily for $\alpha \in Cw$, then $l_2 = (l_\alpha)_U$ is a zero of $p_2(x)$. Let $\xi{:}K \to K$, $\xi((x_\alpha)_U) = (x_\alpha^{q_\alpha})_U$. ξ is an element of $\mathrm{Aut}(K)$ and $L_2 = \mathrm{Fix}(\{\xi, \ldots, \xi^{n-1}, id\})$. All roots of $p_2(x)$ are given by $l_2, \xi(l_2), \ldots, \xi^{n-1}(l_2)$ and $K = L_2(l_2, \xi(l_2), \ldots, \xi^{n-1}(l_2))$. Extend σ to an isomorphism $\sigma^*{:}E \to K$ such that $\sigma^*(l_1) = l_2$ and $\sigma^*(\gamma^i(l_1)) = \xi^i(l_2)$, $i \in \{l \ldots n-1\}$. From this we get $\sigma^*(\gamma^i(x)) = \xi^i(\sigma^*(x))$ for all $x \in E$. By ([**8**, II.5.2]) we can define a coupling map $\psi{:}K\backslash\{0\} \to \mathrm{Aut}(K)$ by $\psi(\sigma^*(x)) = \sigma^*\chi(x)\sigma^{*-1}$, $x \in E\backslash\{0\}$. Then σ^* becomes an isomorphism from E^χ onto K^ψ. Let $x \in E\backslash\{0\}$ and $\chi(x) = \gamma^i$ for some $i \in \{1 \ldots n\}$. If $y \in K$, then $\psi(\sigma^*(x))(y) = \sigma^*(\gamma^i(\sigma^{*-1}(y)) = \xi^i(\sigma^*(\sigma^{*-1}(y))) = \xi^i(y)$. Thus $\Delta_\psi = \{\xi, \xi^2, \ldots, \xi^{n-1}, id\}$ and $\mathrm{Fix}(\Delta_\psi) = L_2$.

It remains to show that K^ψ is an ultraproduct of finite Dickson near-fields. Let $k\in K\backslash\{0\}$ such that $\psi(k)=\xi$. For a positive integer m let $\underline{k^m}$ denote the mth power of k in K^ψ. Since $K^\psi\backslash\{0\}/U_\psi\cong\Delta_\psi$, $k\circ U_\psi\cup \underline{k^2}\circ U_\psi\cup\cdots\cup \underline{k^{n-1}}\circ U_\psi\cup U_\psi$ is a partition of $K^\psi\backslash\{0\}$ into cosets, such that $\psi(\underline{k^i}\circ U_\psi)=\{\xi^i\}$.

Let ω_α be a fixed generator of $GF(q_\alpha^n)$ for $\alpha\in A$. U_ψ is also a subgroup of $K\backslash\{0\}$, (II.3.3. in [8]) and by definition of U_ψ we have that $\underline{k^i}\circ U_\psi=\underline{k^i}\cdot U_\psi$ for $i\in\{1\ldots n-1\}$. Consequently $|K\backslash\{0\}/U_\psi|=n$, hence $((\omega_\alpha^n)^{l_\alpha})_U=((\omega_\alpha^{l_\alpha})_U)^n\in U_\psi$ for all sequences of non-negative integers (l_α). Thus $\prod_U\langle\omega_\alpha^n\rangle\subseteq U_\psi$. For $\alpha\in A$, $|GF(q_\alpha^n)\backslash\{0\}/\langle\omega_\alpha^n\rangle|=n$, hence $|K\backslash\{0\}/\prod_U\langle\omega_\alpha^n\rangle|=n$. But then $\prod_U\langle\omega_\alpha^n\rangle=U_\psi$.

Let $\underline{k^i}=(\omega_\alpha^{k_{\alpha i}})_U$, $i\in\{1\ldots n-1\}$. Since U is an ultrafilter there exists an element $w\in U$ such that $\omega_\alpha^{k_{\alpha 1}}\langle\omega_\alpha^n\rangle\cup\cdots\cup\omega_\alpha^{k_{\alpha n-1}}\langle\omega_\alpha^n\rangle\cup\langle\omega_\alpha^n\rangle$ is a partition of $GF(q_\alpha^n)\backslash\{0\}$ for all $\alpha\in w$. Let $\xi_\alpha:GF(q_\alpha^n)\to GF(q_\alpha^n)$, $\xi_\alpha(x)=x^{q_\alpha}$, $\alpha\in A$. For $\alpha\in A$ we define a map $\psi_\alpha:GF(q_\alpha^n)\backslash\{0\}\to \mathrm{Aut}(GF(q_\alpha^n))$ as follows. If $\alpha\in w$, let $\psi_\alpha(k)=\mathrm{id}$ for $k\in\langle\omega_\alpha^n\rangle$ and $\psi_\alpha(k)=\xi_\alpha^i$ if $k\in\omega_\alpha^{k_{\alpha i}}\langle\omega_\alpha^n\rangle$. For $\alpha\in Cw$ let $\psi_\alpha(k)=id$ for all $k\in GF(q_\alpha^n)\backslash\{0\}$. Let $k_\alpha\in GF(q_\alpha^n)\backslash\{0\}$, $\alpha\in A$, and $(\psi_\alpha(k_\alpha))_U:K\to K$, $(\psi_\alpha(k_\alpha))_U((f_\alpha)_U)=(\psi_\alpha(k_\alpha)(f_\alpha))_U$. Then $\psi((k_\alpha)_U)=(\psi_\alpha(k_\alpha))_U$ for all $(k_\alpha)_U\in K\backslash\{0\}$ and since ψ is a coupling map $v=\{\alpha\in A|\psi_\alpha$ is a coupling map on $GF(q_\alpha^n)\}\in U$. Thus $GF(q_\alpha^n)^{\psi_\alpha}$ is a finite Dickson near-field for $\alpha\in v$. If we define $\psi_\alpha(k)=id$ for $\alpha\in Cv$, $k\in GF(q_\alpha^n)\backslash\{0\}$ it is easy to verify that $K^\psi=\prod_U GF(q_\alpha^n)^{\psi_\alpha}$.

We are now ready to establish our major result:

Theorem 2.4. *The following are equivalent:*

(1) *F is a pseudo-finite near-field with $[F:Z(F)]=n$ for some positive integer n.*

(2) $F=E^\chi$ where E is a pseudo-finite field and χ *is a coupling map on E such that the Dickson group Δ_χ is cyclic of order n and Fix(Δ_χ) is a pseudo-finite field.*

Proof. **(1)⇒(2).** Since F is pseudo-finite there exists an ultraproduct $D=\prod_U F_\alpha$ of finite Dickson near-fields $F_\alpha=F(q_\alpha,n_\alpha)$ with coupling maps η_α and Dickson groups Δ_α, $\alpha\in A$, such that $F\equiv D$. By Proposition 2.1 $D=K^\eta$ where $K=\prod_U GF(q_\alpha^{n_\alpha})$, $\eta:K\backslash\{0\}\to\mathrm{Aut}(K)$, $\eta((k_\alpha)_U)=(\eta_\alpha(k_\alpha))_U$ and $\Delta_\eta=\prod_U\Delta_\alpha$. The centre of a near-field can be described by a first-order sentence, hence the property of having finite dimension n over the centre is expressible by a first-order sentence. Consequently $[D:Z(D)]=n$. Since $Z(F_\alpha)\cong GF(q_\alpha)$ and $[F_\alpha:Z(F_\alpha)]=n_\alpha$ an application of Łoś' theorem ([2, Th. 4.1.9]) yields $\{\alpha|n_\alpha=n\}\in U$. We may therefore assume that $n_\alpha=n$ for all $\alpha\in A$. By Proposition 2.2 $F=E^\chi$ for some coupling map on a commutative field E. We show that E is a pseudo-finite field. Since $F\equiv D$ there exists by the Theorem of Keisler–Shelah ([2, Th. 6.1.15]) a set I and an ultrafilter $\mathscr{F}$ on I such that $(E^\chi)^I/\mathscr{F}\cong D^I/\mathscr{F}$. Again by Proposition 2.1 $(E^\chi)^I/\mathscr{F}=(E^I/\mathscr{F})^\varphi$ where $\varphi:(E^I/\mathscr{F})\backslash\{0\}\to\mathrm{Aut}(E^I/\mathscr{F})$, $\varphi((e_i)_\mathscr{F})=(\chi(e_i))_\mathscr{F}$ with Dickson group $\Delta_\varphi=\Delta_\chi^I/\mathscr{F}$ and $D^I/\mathscr{F}=(K^I/\mathscr{F})^\psi$ where $\psi:(K^I/\mathscr{F})\backslash\{0\}\to\mathrm{Aut}(K^I/\mathscr{F})$, $\psi((k_i)_\mathscr{F})=(\eta(k_i))_\mathscr{F}$ with Dickson group $\Delta_\psi=\Delta_\eta^I/\mathscr{F}$. Thus $(E^I/\mathscr{F})^\varphi\cong(K^I/\mathscr{F})^\psi$ by some isomorphism σ. By Proposition 2.2 $Z(E^\chi)=\mathrm{Fix}(\Delta_\chi)\subseteq U_\chi\cup\{0\}$, hence $[E^\chi:\mathrm{Fix}(\Delta_\chi)]=[E:\mathrm{Fix}(\Delta_\chi)]=n$ and by Łoś' theorem $[E^I/\mathscr{F}:\mathrm{Fix}(\Delta_\varphi)]=[E^I/\mathscr{F}:\mathrm{Fix}(\Delta_\chi)^I/\mathscr{F}]=n$. Similarly $[K^I/\mathscr{F}:\mathrm{Fix}(\Delta_\psi)]=n$, hence σ is an isomorphism from $E^I/\mathscr{F}$ onto $K^I/\mathscr{F}$ ([8, III.4.4]). By ([8, II.5.2]) $\Delta_\varphi\cong\Delta_\psi$ and

$\text{Fix}(\Delta_\chi)^I/\mathscr{F} = \text{Fix}(\Delta_\varphi) \cong \text{Fix}(\Delta_\psi) = (\prod_U GF(q_\alpha))^I/\mathscr{F}$. Thus $E \equiv K = \prod_U GF(q_\alpha^n)$ and $\text{Fix}(\Delta_\chi) \equiv \prod_U GF(q_\alpha)$. Since $n_\alpha = n$ for all $\alpha \in A$, $|\Delta_\alpha| = n$, hence $\prod_U \Delta_\alpha \cong \Delta_\beta$ for all $\beta \in A$. Consequently Δ_η, Δ_ψ, Δ_φ, Δ_χ are all cyclic of order n.

(2)$\Rightarrow$(1). Since $\text{Fix}(\Delta_\chi)$ is pseudo-finite there exists an ultraproduct $L = \prod_U GF(q_\alpha)$ of finite fields such that $\text{Fix}(\Delta_\chi) \equiv L$. Let I be an index set and $\mathscr{F}$ be an ultrafilter on I such that $\text{Fix}(\Delta_\chi)^I/\mathscr{F} \cong L^I/\mathscr{F}$ by some isomorphism σ. $\text{Fix}(\Delta_\chi)^I/\mathscr{F} = \text{Fix}(\Delta_\varphi)$ for the coupling map $\varphi:(E^I/\mathscr{F})\backslash\{0\} \to \text{Aut}(E^I/\mathscr{F})$, $\varphi((e_i)_\mathscr{F}) = (\chi(e_i))_\mathscr{F}$. Since Δ_χ is cyclic of order n we have that Δ_φ is cyclic of order n. Let $K = \prod_U GF(q_\alpha^n)$. In a way similar to the first part of the proof of Proposition 2.3 we can extend σ to an isomorphism $\sigma^*: E^I/\mathscr{F} \to K^I/\mathscr{F}$ such that σ^* is an isomorphism from $(E^I/\mathscr{F})^\varphi$ onto $(K^I/\mathscr{F})^\eta$ for some coupling map η on $K^I/\mathscr{F}$, where $\Delta_\eta = \langle(\gamma_i)_\mathscr{F}\rangle$, $\gamma_i = \gamma: K \to K$, $\gamma((k_\alpha)_U) = (k_\alpha^{q_\alpha})_U$ for all $i \in I$, $U_\eta = (\prod_U \langle\omega_\alpha^n\rangle)^I/\mathscr{F}$ and $\text{Fix}(\Delta_\eta) = L^I/\mathscr{F}$. Proceeding as in the second part of the proof of Proposition 2.3 we eventually find coupling maps η_i, $i \in I$, on K with $\Delta_{\eta_i} = \langle\gamma\rangle$, $\text{Fix}(\Delta_{\eta_i}) = L$ and $(K^I/\mathscr{F})^\eta \cong \prod_\mathscr{F} K^{\eta_i}$. By Proposition 2.3 each K^{η_i} is pseudo-finite and $[K^{\eta_i}:Z(K^{\eta_i})] = n$, hence $[\prod_\mathscr{F} K^{\eta_i}:Z(\prod_\mathscr{F} K^{\eta_i})] = n$.

Since $E^\chi \equiv (E^\chi)^I/\mathscr{F} = (E^I/\mathscr{F})^\varphi \cong (K^I/\mathscr{F})^\eta \cong \prod_\mathscr{F} K^{\eta_i}$, E^χ is pseudo-finite and $[E^\chi:Z(E^\chi)] = n$.

It follows from Theorem 2.4 that some locally finite near-fields are also pseudo-finite. For information on Steinitz numbers see for example [1].

Example 2.5. Let F_0 be a finite Dickson near-field of order q^n, $q = p^l$, where $Z(F_0) \cong GF(q)$. Let P be the set of all prime numbers π such that $\pi \equiv 1 \pmod n$. It is known that $\mathscr{P}$ is an infinite set, say $\mathscr{P} = \{p_i | i \geqq 1\}$. By ([5, Lemma 2.2]) we can therefore construct an infinite chain of finite Dickson near-fields $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq \cdots$ such that $|F_i| = q^{n\prod_{k=1}^i p_k}$, $|Z(F_i)| = q^{\prod_{k=1}^i p_k}$ for $i \geqq 1$ and $Z(F_0) \subseteq Z(F_1) \subseteq \cdots \subseteq Z(F_k) \subseteq \cdots$. If $E_0 = GF(q^n)$ and $E_i = GF(q^{n \cdot \prod_{k=1}^i p_k})$ for $i \geqq 1$, we may assume that $F_i = E_i^{\chi_i}$ for some coupling map χ_i on E_i, $i \geqq 0$, such that $\Delta_{\chi_i} = \langle\delta_i\rangle$, where $\delta_0: E_0 \to E_0$, $\delta_0(x) = x^q$ and $\delta_i: E_i \to E_i$, $\delta_i(x) = x^{q\prod_{k=1}^i p_k}$ for $i \geqq 1$.

Let $F = \bigcup_{k=0}^\infty F_k$ and $E = \bigcup_{k=0}^\infty E_k$. By ([4, Th. 2.2]) it can be shown that F is a Dickson near-field. We briefly recall this construction. For $e \in E\backslash\{0\}$ let k be the least non-negative integer such that $e \in E_k$. Let $\chi(e) = (\gamma_{i,e})_{i \geqq 0}$, where $\gamma_{i,e} = \chi_i(e)$ for $i \geqq k$ and $\gamma_{i,e}$ is the restriction of $\chi_k(e)$ to E_i for $0 \leqq i < k$. For $f \in E$ define $\chi(e)(f) = \gamma_{l,e}(f)$, where l is the least non-negative integer such that $f \in E_l$. Then $\chi(e) \in \text{Aut}(E)$ for $e \in E\backslash\{0\}$ and $\chi: E\backslash\{0\} \to \text{Aut}(E)$ is a coupling map on E such that $E^\chi \cong F$. From the construction of the F_i's it now easily follows that $\Delta_\chi = \{(\delta_i^j)_{i \geqq 0} | 0 < j \leqq n\}$. Thus Δ_χ is cyclic of order n with generator $(\delta_i)_{i \geqq 0}$ and $\text{Fix}(\Delta_\chi) = GF(q) \cup \bigcup_{i=1}^\infty GF(q^{\prod_{k=1}^i p_k})$.

$\text{Fix}(\Delta_\chi)$ is an algebraic extension of $GF(q)$. Let $S = \prod_{p\ prime} p^{n(p)}$ denote the Steinitz number of $\text{Fix}(\Delta_\chi)$. Clearly $n(p) \neq \infty$ for all primes p, $n(p) = 1$ if $p \in \mathbb{P}$ and $n(p) = 0$ otherwise. Since $\mathbb{P}$ is infinite, $\text{Fix}(\Delta_\chi)$ is a pseudo-finite field ([1, §6, Cor.). Similarly it follows that E is pseudo-finite. Thus $F \cong E^\chi$ is a pseudo-finite near-field by Theorem 2.4.

For a pseudo-finite near-field F of characteristic $p \neq 0$ let $L(F)$ denote the union of all finite sub-near-fields of F (the locally-finite socle of F, see [5]). It has been shown by J. Ax ([1, §8, Th. 4]) that two pseudo-finite fields K_1, K_2 of characteristic $p \neq 0$ are elementarily equivalent if and only if their locally-finite socles are isomorphic. The following example shows that this result does not continue to hold for pseudo-finite near-fields.

Example 2.6. Let $F=\bigcup_{k=0}^{\infty} F_k$ be the pseudo-finite near-field constructed in Example 2.5. By ([**3**, Lemma 2.1]) we can find an infinite set $\{\sigma_i | i \geqq 2\}$ of prime numbers σ_i such that σ_i divides $p^{l\prod_{k=1}^{i} p_k}-1$, but σ_i does not divide $p^{l\prod_{k=1}^{i-1} p_k}-1$. For $i \geqq 2$ let $q_i=p^{l\prod_{k=1}^{i} p_k}$ and $n_i=n\sigma_i$. There exists a positive integer $j \geqq 2$ such that $\sigma_i \neq 2$ for all $i \geqq j$. Consequently, if $4|n_i$ for some $i \geqq j$, then $4|n$. Since (q_i, n) is a Dickson pair for all $i \geqq 2$, it follows that (q_i, n_i) is a Dickson pair for all $i \geqq j$. By ([**4**, Lemma 1.3]) there exists for $i \geqq j$ a finite Dickson near-field D_i such that $|D_i|=q_i^{n_i}$ and $Z(D_i) \cong GF(q_i)$ which contains F_i as a sub-near-field. Let U be a nonprincipal ultrafilter on $A=\{i | i \geqq j\}$ and $D=\prod_U D_i$. Clearly $[D:Z(D)]$ is infinite since $\{i | n_i \leqq l\}$ is finite for every positive integer l. Similarly, as in ([**5**, Th. 4.3]) we can prove that $L(F)=F=L(D)$, but F is not elementarily equivalent to D since $[F:Z(F)]=n$ and the property of having finite dimension n over the centre is first-order.

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