UNIFORM DISTRIBUTION OF SEQUENCES IN RINGS OF INTEGRAL QUATERNIONS

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1. Introduction. Let $\mathbb{Z}$ and $\mathbb{Z}[i]$ have their usual meaning. Let Y_0 denote the noncommutative ring of integral quaternions, that is the set of all elements $a+bi+cj+dk$ with $a, b, c, d \in \mathbb{Z}$ and where i, j and k together with the number 1 are the four units of the system of quaternions.

Let $\mathcal{N}(\mu)=a^2+b^2+c^2+d^2$ be the norm of the element $\mu=a+bi+cj+dk \in Y_0$. The nontrivial ideals in Y_0 are exactly the principal ideals (μ) generated by elements $\mu \in Y_0$ with $\mathcal{N}(\mu) \geqslant 2$.

Analogous to the definition of uniformly distributed sequences in $\mathbb{Z}$ due to Niven [**4**] (see also Kuipers and Niederreiter [**1**, Chapter 5]) and that in $\mathbb{Z}[i]$ due to Kuipers, Niederreiter and Shiue [**2**] we consider sequences of integral quaternions and ask how they are distributed modulo an arbitrary nontrivial left ideal in Y_0. We introduce the following definition.

Definition 1. Let (μ_n) $(n=1,2,\ldots)$ be a sequence of integral quaternions. Let $\beta, \mu \in Y_0$ with $\mathcal{N}(\mu) \geqslant 2$. For a positive integer N, let $A(\beta, \mu, N)$ denote the number of n $(1 \leqslant n \leqslant N)$ such that $\mu_n \equiv \beta (\text{mod } \mu)$. The sequence (μ_n) is said to be *uniformly distributed modulo* μ (abbreviated *u.d.* mod μ) in Y_0 if

$$\lim_{N\to\infty} A(\beta, \mu, N)/N = (\mathcal{N}(\mu))^{-2}, \ldots \tag{1}$$

for all $\beta \in Y_0$. (The choice of β may be restricted to all elements of a complete residue system modulo μ (abbreviated c.r.s. mod μ)). The sequence (μ_n) is said to be *u.d.* in Y_0 if it is u.d. mod μ for all $\mu \in Y_0$ with $\mathcal{N}(\mu) \geqslant 2$.

When investigating specific sequences it is useful to have available an explicit description of a c.r.s. mod μ.

Let $H(\gamma)$ denote a c.r.s. mod γ $(\gamma \in \mathbb{Z}[i])$ in $\mathbb{Z}[i]$. If $\mu=a+bi+cj+dk \in Y_0$, then we denote the g.c.d. (or one of its associates) of the elements $a+bi$ and $c-di \in \mathbb{Z}[i]$ by $\delta=(a+bi, c-di)$. The g.c.d. of the integers $a, b, c, d \in \mathbb{Z}$ is denoted by $g=(a, b, c, d)$. The conjugate of $\mu=a+bi+cj+dk \in Y_0$ is denoted by $\mu^*=a-bi-cj-dk$. We have

$$\mu\mu^*=\mu^*\mu=a^2+b^2+c^2+d^2=\mathcal{N}(\mu).$$

The following description of a c.r.s. modulo q was given by Shiue and Hwang [**5**].

Theorem 1. *A c.r.s. mod q, where $q=a+bi+cj+dk \in Y_0$ $(q \neq 0)$ is given by the set*

$$G(q)=\{r_0+r_1i+r_2j+r_3k \mid r_0+r_1i \in H(\delta), r_2+r_3i \in H(\mathcal{N}(q)\delta^{-1})\}.$$

Here $\delta=(a+bi, c-di)$.

COROLLARY. *A c.r.s.* mod *q, where* $q = a + bi + cj + dk$ $(q \neq 0)$, *is given by the set*

$$G(q) = \{r_0 + r_1 i + r_2 j + r_3 k \mid 0 \leq r_0 < \mathcal{N}(\delta)g^{-1}, 0 \leq r_1 < g, \\ 0 \leq r_2 < \mathcal{N}(q)\mathcal{N}(\delta)^{-1}g, 0 \leq r_3 < \mathcal{N}(q)g^{-1}\}.$$

The cardinality of $G(q)$ *is* $\mathcal{N}(q)^2$.

2. Uniform distribution modulo an ideal. The elements of a c.r.s. modulo a nontrivial (left) ideal form an additive group. With regard to this group we have the following result.

THEOREM 2. *For* $\mu = a + bi + cj + dk \in Y_0$ *with* $\mu \neq 0$, $g = (a, b, c, d)$, $\delta = (a + bi, c - di)$, *the characters of the group* $Y_0/(\mu)$ *are given by*

$$\chi(x + yi + zj + uk) = \exp\{(r_0(ax + by + cz + du) + r_1(-bx + ay + dz - cu) \\ + r_2(-cx - dy + az + bu) + r_3(-dx + cy - bz + au))/\mathcal{N}(\mu)\},$$

where $0 \leq r_0 < \mathcal{N}(\delta)g^{-1}$, $0 \leq r_1 < g$, $0 \leq r_2 < \mathcal{N}(\mu)\mathcal{N}(\delta)^{-1}g$, $0 \leq r_3 < \mathcal{N}(\mu)g^{-1}$, *and where* $\exp t$ *stands for* $e^{2\pi i t}$ *for real t.*

Proof. First, these expressions are really characters of $Y_0/(\mu)$, for assume that $x + yi + zj + uk \equiv x_0 + y_0 i + z_0 j + u_0 k \pmod{\mu}$. Then we have $\chi(x + yi + zj + uk) = \chi(x_0 + y_0 i + z_0 j + u_0 k)$ if and only

$$r_0(ax + by + cz + du) + \ldots + r_3(-dx + cy - bz + au) \\ \equiv r_0(ax_0 + by_0 + cz_0 + du_0) + \ldots + r_3(-dx_0 + cy_0 - bz_0 + au_0) \pmod{\mathcal{N}(\mu)},$$

a congruence which can be written in the form:

$$r_0\{a(x - x_0) + b(y - y_0) + c(z - z_0) + d(u - u_0)\} + \ldots \\ + r_3\{-d(x - x_0) + c(y - y_0) - b(z - z_0) + a(u - u_0)\} \equiv 0 \pmod{\mathcal{N}(\mu)}.$$

Now we have for some $A, B, C, D \in \mathbb{Z}$ the relation

$$(x - x_0) + (y - y_0)i + (z - z_0)j + (u - u_0)k = (a + bi + cj + dk)(A + Bi + Cj + Dk),$$

which implies the system

$$x - x_0 = aA - bB - cC - dD, \qquad y - y_0 = aB + bA + cD - dC,$$
$$z - z_0 = aC - bD + cA + dB, \qquad u - u_0 = aD + bC - cB + dA,$$

and therefore we have

$$a(x - x_0) + b(y - y_0) + c(z - z_0) + d(u - u_0) = A\mathcal{N}(\mu),$$
$$-b(x - x_0) + a(y - y_0) + d(z - z_0) - c(u - u_0) = B\mathcal{N}(\mu),$$
$$-c(x - x_0) - d(y - y_0) + a(z - z_0) + b(u - u_0) = C\mathcal{N}(\mu),$$
$$-d(x - x_0) + c(y - y_0) - b(z - z_0) + a(u - u_0) = D\mathcal{N}(\mu),$$

and thus the above congruence is satisfied.

Second, for $\xi, \eta \in Y_0$, one has $\chi(\xi + \eta) = \chi(\xi)\chi(\eta)$.

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Third, the abelian group $Y_0/(\mu)$ has $\mathcal{N}(\mu)^2$ characters, so that it suffices to show that the $\mathcal{N}(\mu)^2$ characters χ given above are distinct.

Suppose that

$$r_0(ax+by+cz+du)+\ldots+r_3(-dx+cy-bz+au) \equiv s_0(ax+by+cz+du)+\ldots+s_3(-dx+cy-bz+au) \pmod{\mathcal{N}(\mu)}$$

for all $x+yi+zj+uk \in Y_0$, and where r_0, r_1, r_2, r_3, s_0, s_1, s_2, s_3 satisfy the above-mentioned inequalities.

Now we substitute $x=-b/g$, $y=a/g$, $z=d/g$, $u=-c/g$. Then we have

$$ax+by+cz+du=0,\ -cx-dy+az+bu=0,\ -dx+cy-bz+au=0$$

and the last mentioned congruence reduces to $(r_1-s_1)\mathcal{N}(\mu)/g \equiv 0 \pmod{\mathcal{N}(\mu)}$, and since $|r_1-s_1|<g$ we must have $r_1=s_1$. Hence according to our assumption we have for all $x+yi+zj+uk \in Y_0$ the congruence

$$r_0(ax+by+cz+du)+r_2(-cx-dy+az+bu)+r_3(-dx+cy-bz+au)$$
$$\equiv s_0(ax+by+cz+du)+s_2(-cx-dy+az+bu)+s_3(-dx+cy-bz+au) \pmod{\mathcal{N}(\mu)}.$$

Now there are infinitely many numbers $x, y, z, u \in \mathbb{Z}$ such that

$$ax+by+cz+du=g\mathcal{N}(\mu)/\mathcal{N}(\delta), \qquad -cx-dy+az+bu=0, \qquad -dx+cy-bz+au=0.$$

Upon substitution we obtain

$$(r_0-s_0)\mathcal{N}(\mu)g/\mathcal{N}(\delta) \equiv 0 \pmod{\mathcal{N}(\mu)},$$

which is possible only if $r_0=s_0$ since $|r_0-s_0|<\mathcal{N}(\delta)/g$. Hence the congruence we are investigating reduces to

$$(r_2-s_2)(-cx-dy+az+bu)+(r_3-s_3)(-dx+cy-bz+au) \equiv 0 \pmod{\mathcal{N}(\mu)},$$

valid for all $x+yi+zj+uk \in Y_0$. Now there are infinitely many integers $x, y, z, u \in \mathbb{Z}$ such that

$$-cx-dy+az+bu=\mathcal{N}(\delta)/g, \qquad -dx+cy-bz+au=0.$$

Then upon substitution one obtains the congruence

$$(r_2-s_2)\mathcal{N}(\delta)/g \equiv 0 \pmod{\mathcal{N}(\mu)}.$$

Since $|r_2-s_2|<\mathcal{N}(\mu)g/\mathcal{N}(\delta)$, we must have $r_2=s_2$.

Finally assume that

$$r_3(-dx+cy-bz+au) \equiv s_3(-dx+cy-bz+au) \pmod{\mathcal{N}(\mu)}$$

for all $x+yi+zj+uk \in Y_0$. Now choose $\xi, \eta, \phi, \psi \in \mathbb{Z}$ such that $g=-d\xi+c\eta-b\phi+a\psi$. Then we obtain

$$(r_3-s_3)(-d\xi+c\eta-b\phi+a\psi) \equiv 0 \pmod{\mathcal{N}(\mu)},$$

or

$$(r_3-s_3)g \equiv 0 \pmod{\mathcal{N}(\mu)}.$$

But since $|r_3-s_3|<\mathcal{N}(\mu)/g$ we have $r_3=s_3$.

COROLLARY. *The characters of* $Y_0/(\mu)$ *are given by* $\chi_\theta(\xi) = \exp \operatorname{Re}(\xi\theta^*/\mu)$, *where*

$$\xi = x + yi + zj + uk, \qquad \theta = r_0 + r_1 i + r_2 j + r_3 k \quad (\theta \in G(\mu)).$$

THEOREM 3. (Weyl criterion). *The sequence* (x_n) $(n = 1, 2, \ldots, x_n \in Y_0)$ *is u.d.* mod μ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp \operatorname{Re}(x_n\theta^*/\mu) = 0$$

for all $\theta \in G(\mu)$.

Proof. By viewing $Y_0/(\mu)$ as a compact abelian group with the discrete topology the assertion follows immediately from the Weyl criterion on such groups (see [**1**, Chapter 4]) and from the above corollary.

THEOREM 4. *Let* $\mu \in Y_0$ *with* $\mathcal{N}(\mu) \geq 2$. *Assume* $\lambda \mid \mu$ *where* $\lambda \in Y_0$ *with* $\mathcal{N}(\lambda) \geq 2$. *Then a sequence in* Y_0 *is u.d.* mod λ *whenever it is u.d.* mod μ.

Proof. For any $\nu \in Y_0$ we have $A(\nu, \lambda, N) = \sum_{\omega + (\mu) \in S} A(\omega, \mu, N)$ where S is the set of all distinct $\omega + (\mu)$ such that $\omega \equiv \nu \pmod{\lambda}$. The cardinality of S is equal to the cardinality of $(\lambda)/(\mu)$. From the group isomorphism theorem we know that

$$Y_0/(\lambda) \cong (Y_0/(\mu))/((\lambda)/(\mu)),$$

and hence $\mathcal{N}(\lambda)^2 = \mathcal{N}(\mu)^2/\operatorname{card}(\lambda)/(\mu)$, or

$$\operatorname{card} S = \operatorname{card}(\lambda)/(\mu) = \mathcal{N}(\mu)^2/\mathcal{N}(\lambda)^2.$$

Hence

$$\begin{aligned} \lim_{N\to\infty} A(\nu, \lambda, N)/N &= \lim_{N\to\infty} \sum_{\omega+(\mu)\in S} (A(\omega, \mu, N)/N) \\ &= \sum_{\omega+(\mu)\in S} \lim_{N\to\infty} (A(\omega, \mu, N)/N) \\ &= \operatorname{card} S \,.\, \mathcal{N}(\mu)^{-2} \\ &= \mathcal{N}(\lambda)^{-2}. \end{aligned}$$

THEOREM 5. *Let* $\mu_1, \mu_2, \ldots$ *be a finite or denumerable collection of integral quaternions with* $\mathcal{N}(\mu_n) \geq 2$ $(n = 1, 2, \ldots)$. *Then there exists a sequence in* Y_0 *which is u.d.* mod μ_n *for all n, but which is not u.d.* mod λ *for* $\lambda \in Y_0$, $\mathcal{N}(\lambda) \geq 2$ *with* $\mu_n \notin (\lambda)$ *for all n.*

Proof. We apply the following theorem of Zame [**6**]: Let G be a locally compact abelian group with countable base, and let $\varphi \neq \varnothing$ and $\mathcal{T}$ be countable collections of closed subgroups of G such that (i) finite intersections of members of $\varphi \cup \mathcal{T}$ are of compact index, (ii) for each $S \in \varphi$ and $T \in \mathcal{T}$ we have $S \not\subseteq T$, (iii) for each $T \in \mathcal{T}$ there exists a character χ_T of G such that χ_T is trivial on T but is nontrivial on each $S \in \varphi$. Then there is a sequence (g_n) $(n = 1, 2, \ldots)$ in G such that (g_n) is u.d. mod S for all $S \in \varphi$, but not u.d. mod T for $T \in \mathcal{T}$.

Now let $G = Y_0$ with the discrete topology, and let φ consist of the left ideals (μ_1),

$(\mu_2), \dots$ whereas $\mathcal{T}$ consists of all nonzero left ideals (λ), $\mathcal{N}(\lambda) \geq 2$ with $\mu_n \notin (\lambda)$ for all n. Then conditions (i) and (ii) are easily checked. (Note that every nonzero left ideal in Y_0 has finite, thus compact index.) As to condition (iii), let $(\lambda) \in \mathcal{T}$ be given, and choose $\theta = 1$ in the character formula for $Y_0/(\lambda)$. Then we get a character χ with $\chi(\xi) = \exp \operatorname{Re}(\xi/\lambda)$ for $\xi \in Y_0$. This character is trivial on (λ). For any n we have

$$\mu_n/\lambda = a_n + b_n i + c_n j + d_n k \notin Y_0,$$

and the following statements hold:

$$\begin{array}{llll} \text{if} & a_n \notin \mathbb{Z} & \text{then} & \chi(\mu_n) = \exp \operatorname{Re}(\mu_n \lambda^{-1}) = \exp a_n \neq 1; \\ \text{if} & b_n \notin \mathbb{Z} & \text{then} & \chi(i\mu_n) = \exp \operatorname{Re}(i\mu_n \lambda^{-1}) = \exp(-b_n) \neq 1; \\ \text{if} & c_n \notin \mathbb{Z} & \text{then} & \chi(j\mu_n) = \exp \operatorname{Re}(j\mu_n \lambda^{-1}) = \exp(-c_n) \neq 1; \\ \text{if} & d_n \notin \mathbb{Z} & \text{then} & \chi(k\mu_n) = \exp \operatorname{Re}(k\mu_n \lambda^{-1}) = \exp(-d_n) \neq 1. \end{array}$$

In all cases χ is nontrivial on (μ_n). Thus χ satisfies the conditions in (iii) and the theorem follows immediately.

LEMMA 1. *Let $\mu = a + bi + cj + dk \in Y_0$, $\mu \neq 0$, $g = (a, b, c, d)$, $h \in \mathbb{Z}$. Then $\mu \mid h$ if and only if $(\mathcal{N}(\mu)/g) \mid h$.*

Proof. Write $a = ga_1, \dots, d = gd_1$. Assume $(\mathcal{N}(\mu)/g) \mid h$, or $(a + bi + cj + dk) \,.\, (a_1 - b_1 i - c_1 j - d_1 k) \mid h$. Then $\mu \mid h$. Conversely let $\mu \mid h$. Then for some $C, D, E, F \in \mathbb{Z}$ we have

$$h = (a + bi + cj + dk) \,.\, (C + Di + Ej + Fk) = aC - bD - cE - dF$$
$$+ (aD + bC + cF - dE)i + (aE - bF + cC + dD)j + (aF + bE - cD + dC)k.$$

Hence

$$\begin{cases} bC + aD - dE = -cF, \\ cC + dD + aE = bF, \\ dC - cD + bE = -aF, \end{cases} \quad \text{and} \quad \det \begin{bmatrix} b & a & -d \\ c & d & a \\ d & -c & b \end{bmatrix} = d\mathcal{N}(\mu).$$

We obtain

$$dC = -Fa, \qquad dD = Fb, \qquad dE = Fc.$$

Since $h = aC - bD - cE - dF$, we have $dh = -\mathcal{N}(\mu)F$. Now there are three more similar relations. Since $g = (a, b, c, d)$ there are integers $\xi, \eta, \zeta, \varepsilon \in \mathbb{Z}$ such that $g = \xi a + \eta b + \zeta c + \varepsilon d$. By addition we find that $(\mathcal{N}(\mu)/g) \mid h$.

THEOREM 6. *Let $\mu = a + bi + cj + dk$, $\mathcal{N}(\mu) \geq 2$. Let (h_n) $(n = 1, 2, \dots)$ be a sequence in $\mathbb{Z}$, and let $\alpha \in Y_0$. Then the sequence $(h_n\alpha)$ $(n = 1, 2, \dots)$ is not u.d.* mod μ.

Proof. Let $(h_n\alpha)$ by u.d. mod μ. Then the residue classes of $(h_n\alpha)$ mod μ must run through all elements of $Y_0/(\mu)$. In particular, the residue class of α mod μ generates $Y_0/(\mu)$, so that $Y_0/(\mu)$ is cyclic. By Lemma 1, we have $\beta\mathcal{N}(\mu)/g \equiv 0 \pmod{\mu}$ for all $\beta \in Y_0$, and so the order of any residue class mod μ in $Y_0/(\mu)$ is at most $\mathcal{N}(\mu)/g$. Since $Y_0/(\mu)$ is cyclic of order $\mathcal{N}(\mu)^2$ we obtain a contradiction.

LEMMA 2. *Let* $\mu = a+bi+cj+dk \in Y_0$, $g=(a,b,c,d)$. *Let* $C+Di \in \mathbb{Z}[i]$. *Then* $(\mathcal{N}(\mu)/g) \mid C+Di$ *implies* $\mu \mid C+Di$. *If moreover* $(a_1-b_1i, c_1-d_1i)=1$, *where* $a=ga_1, \ldots, d=gd_1$, *then* $\mu \mid C+Di$ *implies* $(\mathcal{N}(\mu)/g) \mid C+Di$.

Proof. The first implication to be shown is evident since

$$\mathcal{N}(\mu)/g = \mu(a_1-b_1i-c_1j-d_1k).$$

Now assume $\mu \mid C+Di$. Hence for some $X, Y, Z, U \in \mathbb{Z}$ we have

$$\begin{aligned} C+Di &= (a+bi+cj+dk)(X+Yi+Zj+Uk) \\ &= \{a+bi+j(c-di)\}.\{X+Yi+j(Z-Ui)\} \\ &= (a+bi)(X+Yi)-(c+di)(Z-Ui)+j\{(c-di)(X+Yi)+(a-bi)(Z-Ui)\}. \end{aligned}$$

Hence

$$(c_1-d_1i)(X+Yi)+(a_1-b_1i)(Z-Ui)=0.$$

Let a_1-b_1i and c_1-d_1i be relatively prime. Then $a_1-b_1i \mid X+Yi$. Hence for some $s, t \in \mathbb{Z}$ we have $X+Yi=(a_1-b_1i)(s+ti)$. This implies that $Z-Ui=-(c_1-d_1i)(s+ti)$. Substitution yields now that

$$\begin{aligned} C+Di &= (a+bi)(a_1-b_1i)(s+ti)+(c+di)(c_1-d_1i)(s+ti) \\ &= \mathcal{N}(\mu)(s+ti)/g \end{aligned}$$

or $(\mathcal{N}(\mu)/g) \mid C+Di$.

THEOREM 7. *Let* $\mu = a+bi+cj+dk \in Y_0$ *with* $\mathcal{N}(\mu) \geq 2$, *and let* $(c-di, \mathcal{N}(\mu))=1$ *in* $\mathbb{Z}[i]$. *Let* $p, q \in \mathbb{Z}$ *be chosen such that in* $\mathbb{Z}[i]$ $(c-di)(p+qi) \equiv 1 \pmod{\mathcal{N}(\mu)}$. *Then the sequence* $(x_n+y_ni+z_nj+u_nk)$ $(n=1,2,\ldots)$ *is u.d.* mod μ *in* Y_0 *if and only if the sequence* (χ_n), *with*

$$\chi_n = x_n+y_ni-(p+qi)(a+bi)(z_n-u_ni)$$

is u.d. mod $\mathcal{N}(\mu)$ *in* $\mathbb{Z}[i]$.

Proof. Since $(c-di, \mathcal{N}(\mu))=1$, we have $g=(a,b,c,d)=1$, and also $(c-di, a-bi)=1$. Hence we may apply Lemma 2. Assume that the sequence (χ_n) is u.d. mod $\mathcal{N}(\mu)$ in $\mathbb{Z}[i]$; then (χ_n) is also u.d. mod μ in Y_0. For according to Lemma 2 we have the equivalence of the congruences $X \equiv Y \pmod{\mathcal{N}(\mu)}$ and $X \equiv Y \pmod{\mu}$, where X and Y are in $\mathbb{Z}[i]$. Now we make use of the congruences

$$\begin{aligned} (c-di)(p+qi) &\equiv 1 \pmod{\mathcal{N}(\mu)} \text{ (our assumption)}, \\ (c-di)(p+qi) &\equiv 1 \pmod{\mu} \text{ (Lemma 2)}, \\ -(a+bi) &\equiv j(c-di) \pmod{\mu}, \end{aligned}$$

and obtain

$$-(a+bi)(p+qi) \equiv j \pmod{\mu}.$$

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So the conclusion that

$$(x_n + y_n i + j(z_n - u_n i)) = (x_n + y_n i + z_n j + u_n k)$$

is u.d. mod μ is true. The converse can be shown similarly.

REMARK. The following statement is closely related to Theorem 7 and can be shown along the same lines. Let $\mu = a + bi + cj + dk \in Y_0$, $\mathcal{N}(\mu) \geq 2$. Assume $\delta = (a + bi, c - di) = 1$ in $\mathbb{Z}[i]$. Choose $\gamma \in \mathbb{Z}[i]$ such that

$$\gamma(c - di) \equiv 1 \pmod{\mathcal{N}(\mu)}$$

in $\mathbb{Z}[i]$. Then for $\alpha_n, \beta_n \in \mathbb{Z}[i]$ the sequence $(\alpha_n + \beta_n j)$ $(n = 1, 2, \ldots)$ is u.d. mod μ if and only if the sequence $(\alpha_n - \beta_n \gamma^*(a + bi))$ is u.d. mod $\mathcal{N}(\mu)$ in $\mathbb{Z}[i]$.

3. Uniform distribution in Y_0. We recall the following definition: The sequence $((x_n, y_n, z_n, u_n))$ is u.d. in $\mathbb{Z}^4$ if it is u.d. modulo all subgroups of $\mathbb{Z}^4$ of finite index.

THEOREM 8. *The sequence* $(x_n + y_n i + z_n j + u_n k) \in Y_0$ $(n = 1, 2, \ldots)$ *is u.d. in* Y_0 *if and only if the sequence* $((x_n, y_n, z_n, u_n))$ *of lattice points is u.d. in* $\mathbb{Z}^4$.

Proof. Let $\psi : Y_0 \to \mathbb{Z}^4$ be the group isomorphism given by

$$\psi(x + yi + zj + uk) = (x, y, z, u)$$

for $x + yi + zj + uk \in Y_0$. The isomorphism ψ maps the nontrivial ideals of Y_0 into subgroups of $\mathbb{Z}^4$ of finite index. Now let the sequence $((x_n, y_n, z_n, u_n))$ be u.d. in $\mathbb{Z}^4$. In particular, the sequence $((x_n, y_n, z_n, u_n))$ is then u.d. modulo all subgroups of $\mathbb{Z}^4$ corresponding to the nontrivial ideals of Y_0 under the isomorphism ψ, and therefore the sequence $(x_n + y_n i + z_n j + u_n k)$ is u.d. in Y_0.

To show the converse it suffices to prove that every subgroup of $\mathbb{Z}^4$ of finite index contains a subgroup corresponding to a nonzero ideal in Y_0 under the isomorphism ψ. So let H be an arbitrary subgroup of $\mathbb{Z}^4$ of finite index. Let h be the exponent of the factor group $\mathbb{Z}^4/H$, that is the smallest positive integer h for which $h(a, b, c, d) \in H$ for all $(a, b, c, d) \in \mathbb{Z}^4$. Then we have

$$h(a, b, c, d) = (ha, hb, hc, hd) \in H$$

for all $(a, b, c, d) \in \mathbb{Z}^4$, so that

$$H \supset h\mathbb{Z} \oplus h\mathbb{Z} \oplus h\mathbb{Z} \oplus h\mathbb{Z}.$$

Since the subgroup $h\mathbb{Z} \oplus h\mathbb{Z} \oplus h\mathbb{Z} \oplus h\mathbb{Z}$ corresponds to the principal ideal (h) of Y_0 under the isomorphism ψ, the proof is complete.

THEOREM 9. *Let* (x_n) *be a sequence in* Y_0 *with*

$$x_n = a_n + b_n i + c_n j + d_n k \quad (n = 1, 2, \ldots).$$

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Then (x_n) *is u.d. in* Y_0 *if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(r_1a_n+r_2b_n+r_3c_n+r_4d_n)=0$$

for all $r_1, r_2, r_3, r_4\in\mathbb{Q}$, *not all four being integers.*

Proof. This follows from Theorem 8 and the Weyl criterion for u.d. in $\mathbb{Z}^4$ given by Niederreiter [**3**].

THEOREM 10. *The sequence* (α_n) *in* Y_0 *is u.d. in* Y_0 *if and only if for any* $\xi=a+bi+cj+dk\in Y_0$ *with* $(a, b, c, d)=1$ *in* $\mathbb{Z}$, *the sequence* $(\mathrm{Re}(\alpha_n\xi))$ *is u.d. in* $\mathbb{Z}$.

Proof. Necessity. If $\alpha_n=x_n+y_ni+z_nj+u_nk$, we have

$$\mathrm{Re}(\alpha_n\xi)=ax_n-by_n-cz_n-du_n.$$

Assume that the sequence (α_n) is u.d. in Y_0. Then we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(r_1x_n+r_2y_n+r_3z_n+r_4u_n)=0$$

for all $r_1, r_2, r_3, r_4\in\mathbb{Q}$, not all four of them in $\mathbb{Z}$. Choose $r_1=ah/m$, $r_2=-bh/m$, $r_3=-ch/m$, $r_4=-dh/m$ $(m\in\mathbb{Z}, m\geqslant 2, h=1, 2, \ldots, m-1.)$ So

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp((ax_n-by_n-cz_n-du_n)h/m)=0.$$

Hence the sequence $(ax_n-by_n-cz_n-du_n)$ is u.d. in $\mathbb{Z}$.

Sufficiency. Let r_1, r_2, r_3, r_4 be four rational numbers, not all in $\mathbb{Z}$. We write $r_1=R_1/S_1$, $r_2=R_2/S_2$, $r_3=R_3/S_3$, $R_4=R_4/S_4$, where $R_1, \ldots, R_4, S_1, \ldots, S_4\in\mathbb{Z}$, $S_1, \ldots, S_4\geqslant 1$, $(R_1, S_1)=1, \ldots, (R_4, S_4)=1$. Let $S=[S_1, S_2, S_3, S_4]$. Then $r_1=T_1/S, \ldots, r_4=T_4/S$. Hence

$$r_1x_n+r_2y_n+r_3z_n+r_4u_n=(T_1x_n+T_2y_n+T_3z_n+T_4u_n)/S.$$

Let $T_1\equiv V_1 \pmod S, \ldots, T_4\equiv V_4 \pmod S$ $(0\leqslant V_1, \ldots, V_4\leqslant S-1)$. Let $h=(V_1, V_2, V_3, V_4)$ and write $V_i=hU_i$. Then $1\leqslant h\leqslant S-1$ and

$$\exp(r_1x_n+\ldots+r_4u_n)=\exp((V_1x_n+\ldots+V_4u_n)/S)$$

and moreover

$$\begin{aligned}V_1x_n+\ldots+V_4u_n&=\mathrm{Re}(x_n+y_ni+z_nj+u_nk)(U_1-U_2i-U_3j-U_4k)\\&=\mathrm{Re}(\alpha_n\xi\mid\mu)\end{aligned}$$

with $\mu=1$, $\xi=U_1-U_2i-U_3j-U_4k$. According to our assumption we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp(\mathrm{Re}(\alpha_n\xi)/S)=0$$

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and so

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(r_1 x_n + r_2 y_n + r_3 z_n + r_4 u_n) = 0.$$

EXAMPLE. Let $\theta_1, \theta_2, \theta_3, \theta_4$ be real numbers such that $1, \theta_1, \theta_2, \theta_3, \theta_4$ are linearly independent over the rationals. Set $\alpha_n = [n\theta_1]$, $\beta_n = [n\theta_2]$, $\gamma_n = [n\theta_3]$, $\delta_n = [n\theta_4]$. According to Niederreiter [3] the sequence $((\alpha_n, \beta_n, \gamma_n, \delta_n))$ is u.d. in $\mathbb{Z}^4$, and hence the sequence $(\alpha_n + \beta_n i + \gamma_n j + \delta_n k)$ is u.d. in Y_0.

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