ENSURING COMMUTATIVITY OF FINITE GROUPS

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To Laci Kovács on his 65th birthday

(Received 10 November 2000)

Communicated by R. A. Bryce

Abstract

Comments are made on the following question. Let \( m, n \) be positive integers and \( G \) a finite group. Suppose that for all choices of a subset of cardinality \( m \) and of a subset of cardinality \( n \) in \( G \) some member of the first commutes with some member of the second. Under what conditions on \( m, n \) is the group abelian?


This note arose out of a discussion of a paper presented at AGRAM 2000 at the University of Western Australia by Howard Bell. 'Some setwise commutativity conditions for rings': since then Professor Bell has with Professor Abraham Klein found some interesting results related to the results below [1]. The question raised at AGRAM 2000 was:

Let \( G \) be a finite group of order \( g \) and assume that however a set \( M \) of \( m \) elements and a set \( N \) of \( n \) elements of the group is chosen, at least one element of \( M \) commutes with at least one element of \( N \) (call this condition Comm). What relations between \( g, m, n \) guarantee that \( G \) is abelian?

Clearly if one of \( M, N \) contains an element of the centre of \( G \) or if \( M \) and \( N \) overlap, condition Comm is satisfied. Thus if \( m + n = g \), or even only \( m + n = g - z + 1 \), where \( z \) is the order of the centre of \( G \), Comm is satisfied without \( G \) having to be abelian. An example is every non-abelian group, the smallest being the \( S_3 \) of order 6: if \( M \) is chosen to consist of one or two elements of order 2, the two elements of order 3 together with the remaining elements or element of order 2 can be taken to form \( N \),...
showing that \( m = 1, n = 5 \) or \( m = 2, n = 4 \) are needed to ensure a group of order 6 is abelian. If we choose \( m = 1 \) [which is the most interesting case, anyway] and \( n = 5 \), Comm ensures the group is abelian, whatever \( g \).

There are of course values of \( g \) such that all groups of that order are abelian. There is a recent characterisation of such ‘abelian’ numbers in Pakianathan and Shankar [2]: for such orders \( g \) we can choose \( m = n = 1 \). For the ‘nilpotent’ numbers of [2] that are not ‘abelian’ (because they are not cube-free), \( m = 1, n = 5 \) is again best possible as exemplified by the quaternion group or the dihedral group of order 8. In this case we can do a little better: while in general \( m = 2, n = 4 \) forces a group to be abelian, whatever its order, the case of the groups of order 8 is exceptional in that we need \( m = 2, n = 5 \) to force the group to be abelian. More generally, if \( g = p^3 \) for \( p \) a prime, \( m = p, n = g - p^2 + 1 \) will ensure commutativity. It is not very difficult to compute optimal values for \( m \) and \( n \) for other values of \( g \) to ensure commutativity, but sapienti sat.

References


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