J. Austral. Math. Soc. (Series A) **46** (1989), 356–364

MAXIMAL SUBGROUPS AND THE JORDAN-HÖLDER THEOREM

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(Received 12 June 1987; revised 12 February 1988)

Communicated by H. Lausch

Abstract

In this note we present a general Jordan-Hölder type theorem for modular lattices and apply it to obtain various (old and new) versions of the Jordan-Hölder Theorem for finite groups.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 D 30.

Isbell [10] has observed that the Jordan-Hölder Theorem may be derived from the Zassenhaus Theorem, and that this yields a uniqueness statement for the correspondence given by the Jordan-Hölder Theorem. This result, however, does not give the various versions of the Jordan-Hölder Theorem for finite groups that have received some interest more recently, for example, the one that states that for any two chief series of a finite group a correspondence can be found associating Frattini chief factors with Frattini chief factors and non-Frattini ones with non-Frattini ones. Such a theorem was first published by Carter, Fischer and Hawkes [4] for finite soluble groups, and for finite groups in general in the author's [12], with a different approach by Förster in [7] (see also Chapter 1 of [2]). Further, Barnes proved that in soluble groups corresponding complemented (which, for finite soluble groups, means non-Frattini) chief factors have a common (maximal) complement. On the other hand, for arbitrary finite groups the number of complemented chief factors in a given chief series can depend on the series (see Baer and Förster [2] or Kovács and Newman [11] for examples).

https://doi.org/10.1017/S1446788700030846 Published online by Cambridge University Press

Here we will obtain a Jordan-Hölder correspondence for chief series of an arbitrary finite group G which not only respects the Frattini or non-Frattini nature of chief factors, but also the property of being complemented by a maximal subgroup; in fact, corresponding chief factors have a common maximal complement, if complemented at all by a maximal subgroup. (However, for such a correspondence, corresponding chief factors are not normally G-isomorphic, but only G-connected as defined by the author in [13] and, independently, by Förster in [7] (G-related) and [2] (G-verwandt).)

Our result will emerge as a corollary to a Jordan-Hölder type theorem for modular lattices, in an approach inspired by unpublished notes [11] of Kovács and Newman.

1. A general Jordan-Hölder Theorem in modular lattices

Throughout this section, $\mathscr{L}$ will denote a modular lattice, $\mathscr{M}$ a subset of the set $\mathscr{P}$ of its prime intervals (that is, those pairs A, B of elements of $\mathscr{L}$ such that $B < A$, and $C \in \{A, B\}$ whenever $B \leq C \leq A$; we shall adopt the notation A/B for such pairs), and $K = Y_0 < Y_1 < \cdots < Y_n = H$ will denote a chain in $\mathscr{L}$ such that $Y_i/Y_{i-1} \in \mathscr{P}$, $i = 1, \ldots, n$. We set $\mathscr{L}_{K,H} = \{X \in \mathscr{L} | K \leq X \leq H\}$ and $\mathscr{P}_{K,H} = \{X/Y \in \mathscr{P} | K \leq Y \text{ and } X \leq H\}$.

Further, we write $A/B \gg X/Y$ (or $X/Y \ll A/B$), if $A/B, X/Y \in \mathscr{P}$ are such that $A = X \vee B$ and $X \wedge B = Y$. If $X^*/X \ll Z^*/Z \gg Y^*/Y$ or $X^*/X \gg Z^*/Z \ll Y^*/Y$ for some Z^*/Z, we say that X^*/X and Y^*/Y are under the Zassenhaus correspondence: X^*/X **Zsh** Y^*/Y. (General notation and terminology will be taken from [9].)

The following observation (and its dual, which we omit) is well known.

1.1 Lemma. *For any $X^*/X \in \mathscr{P}_{K,H}$ there exists some $j \in \{1, \ldots, n\}$ such that $X^* \vee Y_k = X \vee Y_k$ for $k = j, \ldots, n$, $X^* \vee Y_k > X \vee Y_k$ for $k = 0, \ldots, j-1$ and*

$$X^* \vee Y_{j-1}/X \vee Y_{j-1} \gg X^* \vee Y_{j-2}/X \vee Y_{j-2} \gg \cdots \gg X^* \vee Y_0/X \vee Y_0 = X^*/X.$$

1.2 Definitions. (a) Two prime intervals R_i/S_i, $i = 1, 2$, are said to be *of the same $\mathscr{M}$-type*, if either both are in $\mathscr{M}$ or both are in $\mathscr{M}' = \mathscr{P} \backslash \mathscr{M}$.

(b) If $\mathscr{M} \ni C/D \ll A/B \in \mathscr{M}'$ and $A/C \in \mathscr{P}$, then $(A/B, C/D)$ is an *$\mathscr{M}$-crossing*.

(c) $\mathscr{M}$ is called an *M-set in $\mathscr{L}$*, if it satisfies the following two conditions.

(M1) If $A/B \gg C/D$, then $A/B \in \mathscr{M}$ implies that $C/D \in \mathscr{M}$.

(M2) If $(A/B, C/D)$ is an $\mathscr{M}$-crossing, then so is $(A/C, B/D)$.

Note that $\mathcal{M}$ *is an* M-set in $\mathcal{L}$ if and only if $\mathcal{M}'$ is an M-set in the dual of $\mathcal{L}$. Trivial examples of M-sets are given by $\mathcal{P}$ and $\varnothing$. We record a simple property of M-sets, leaving the verification (as well as the statement of the dual) to the reader.

1.3 LEMMA. *Let* $X^*/X \in \mathcal{M} \subseteq \mathcal{P}_{K,H}$ *and set* $Y^* = Y_j$, $Y = Y_{j-1}$ *where* $j = \max\{i \in \{1,\dots,n\} | X^* \vee Y_{i-1}/X \vee Y_{i-1} \in \mathcal{M}\}$. *Then one of the following holds.*

(i) $X^* \vee Y^* = X \vee Y^* = X^* \vee Y$, $X^*/X \ll X^* \vee Y^*/X \vee Y \gg Y^*/Y$, $X \wedge Y = X^* \wedge Y = X \wedge Y^*$ *and* $X^*/X \gg X^* \wedge Y^*/X \wedge Y \ll Y^*/Y$.

(ii) $(X^* \vee Y^*/X \vee Y^*, X^* \vee Y/X \vee Y)$ *is an* $\mathcal{M}$*-crossing,* $X^*/X \ll X^* \vee Y/X \vee Y$ *and* $Y^*/Y \ll X \vee Y^*/X \vee Y$.

In particular, if $\mathcal{M}$ *is an* M*-set, then in both cases* $Y^*/Y \in \mathcal{M}$, *and the same holds for* $X^* \vee Y^*/X \vee Y$ *and* $X \vee Y^*/X \vee Y$.

In the remainder of this section, $\mathcal{M}$ will always denote an M-set in $\mathcal{L}$.

1.4 DEFINITION. Two prime intervals X^*/X and Y^*/Y are *$\mathcal{M}$-related*, if one of the following holds.

(1) $X^*/X \ll R^*/R \gg Y^*/Y$ for some $R^*/R \in \mathcal{M}$.
(2) $X^*/X \ll B/D$ and $C/D \gg Y^*/Y$ for some $\mathcal{M}$-crossing $(A/B, C/D)$.
(3) $X^*/X \gg S^*/S \ll Y^*/Y$ for some $S^*/S \in \mathcal{M}'$.
(4) $X^*/X \gg A/B$ and $A/C \ll Y^*/Y$ for some $\mathcal{M}$-crossing $(A/B, C/D)$.

1.5 THEOREM. *Let* $\mathcal{L}$ *be a modular lattice and* $\mathcal{M}$ *an* M*-set in* $\mathcal{L}_{K,H}$. *Assume that*

$$K = X_0 < X_1 < \cdots < X_n = H \quad \text{and} \quad K = Y_0 < Y_1 < \cdots < Y_m = H$$

are two maximal chains in $\mathcal{L}$ *between* H *and* K. *Then* $n = m$, *and there exists a unique* $\pi \in S_n$ *such that* X_i/X_{i-1} *and* $Y_{i^\pi}/Y_{i^\pi-1}$ *are* $\mathcal{M}$*-related for* $i = 1,\dots,n$.

In fact,

$$i^\pi = \max\{j \in \{1,\dots,n\} | X_i \vee Y_{j-1}/X_{i-1} \vee Y_{j-1} \in \mathcal{M}\}, \quad \text{if } X_i/X_{i-1} \in \mathcal{M},$$
$$i^\pi = \min\{j \in \{1,\dots,n\} | X_i \wedge Y_j/X_{i-1} \wedge Y_j \in \mathcal{M}'\}, \quad \text{if } X_i/X_{i-1} \in \mathcal{M}'.$$

PROOF. Without loss of generality, $m \le n$. Let the map $\pi\colon \{1,\dots,n\} \to \{1,\dots,m\}$ be defined by the equations in the statement of the theorem.

First note that applying 1.3 and its dual to the definition of π one sees that X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ are $\mathcal{M}$-related for $i = 1,\dots,n$ and, therefore, have the same $\mathcal{M}$-type.

https://doi.org/10.1017/S1446788700030846 Published online by Cambridge University Press

In order to prove injectivity, and hence bijectivity, of π, write $X^*/X = X_i/X_{i-1}$, $Y^*/Y = Y_j/Y_{j-1}$, and $Z^*/Z = X_k/X_{k-1}$, where $i < k$, but $i^\pi = j = k^\pi$. Suppose that $X_i/X_{i-1} \in \mathscr{M}$; thus, all three intervals are $\mathscr{M}$-intervals. Now apply Lemma 1.3.

In the first case, $X^* \vee Y^* = X^* \vee Y$. From $X^* \leq Z$ we get that $Z \vee Y^* = Z \vee Y$. Since $Z^*/Z \in \mathscr{M}$ and $k^\pi = j$, Lemma 1.3 applies and yields the contradiction that $Z \vee Y^* > Z \vee Y$.

Hence $(X^* \vee Y^*/X \vee Y^*, X^* \vee Y/X \vee Y)$ is an $\mathscr{M}$-crossing; so $X^* \vee Y^*/X^* \vee Y \in \mathscr{M}'$. Since $Z \vee Y \neq Z^* \vee Y$ by Lemma 1.3, $Z \vee Y^*/Z \vee Y \in \mathscr{P}$. As $X^* \leq Z$ gives $X^* \vee Y \leq Z \vee Y$ as well as $X^* \vee Y^* \leq Z \vee Y^*$, we may use (M1) to deduce that $Z \vee Y^*/Z \vee Y \in \mathscr{M}'$, contrary to the conclusion of Lemma 1.3.

We have shown that the restriction of π to $I = \{i \in \{1, \dots, n\} | X_i/X_{i-1} \in \mathscr{M}\}$ is injective. Application of this conclusion to the dual of $\mathscr{L}$, with $\mathscr{M}'$ instead of $\mathscr{M}$, shows that the restriction of π to $\{1, \dots, n\} \backslash I$ is injective. As mentioned above, π leaves these two sets invariant, and we may conclude that π is injective.

Finally, if ψ is any permutation with the above properties, then the definition of π requires that $i^\psi \leq i^\pi$ $(i^\psi \geq i^\pi)$ for all $i \in I$ $(i \in \{1, \dots, n\} \backslash I)$. Consequently, $\psi = \pi$.

Taking $\mathscr{M} = \mathscr{P}$ gives Isbell's result, with the Zassenhaus correspondence: here Condition 1.4(1) always applies, and there are no $\mathscr{M}$-crossings. Somewhat more general, under the following hypothesis $(*)$, conditions (2) and (4) in Definition 1.4 are redundant:

$(*)$ $(A/B, C/D)$ is an $\mathscr{M}$-crossing in $\mathscr{L}$ implies $A/E \in \mathscr{M}$ for some $E \in \mathscr{L}$ with $D < E < A$.

Observe that, for A, B, C, D, E as in $(*)$, by Theorem 1.5, applied to $\mathscr{L}_{D,A}$, $A/E \in \mathscr{M}$ implies that $E/D \in \mathscr{M}'$; furthermore, if $X^*/X \ll B/D$, $C/D \gg Y^*/Y$, then we have $X^*/X \ll A/E \gg Y^*/Y$ (and, of course, the dual statement also holds).

It is easy to see that, under the hypothesis of Theorem 1.5, to a given X_i/X_{i-1} there may exist more than one Y_k/Y_{k-1} $\mathscr{M}$-related to X_i/X_{i-1}. However, one always has

1.6 Proposition. *Assume the hypothesis of Theorem* 1.5 *and let* Π *be any theoretical property on* $\mathscr{P}_{K,H}$ *which is preserved under the relation of being* $\mathscr{M}$*-related.*

Then for any X_i/X_{i-1} *with* Π, *there exists at least one* X_j/X_{j-1} *with* Π *and of the same* $\mathscr{M}$*-type as* X_i/X_{i-1}, *which is* $\mathscr{M}$*-related to only one* Y_k/Y_{k-1}.

Proof. We consider the case $X_i/X_{i-1} \in \mathscr{M}$. Let us define

$$k = \min\{k' \in \{1, \dots, n\} | Y_{k'}/Y_{k'-1} \text{ has } \Pi \text{ and is in } \mathscr{M}\},$$

and write $k = j^\pi$ with π given by Theorem 1.5. From Theorem 1.5, X_j/X_{j-1} is $\mathscr{M}$-related to Y_k/Y_{k-1}; in particular, it has Π and belongs to $\mathscr{M}$.

Assume that X_j/X_{j-1} is $\mathscr{M}$-related to $Y_{k'}/Y_{k'-1}$. Then the latter, like X_j/X_{j-1}, has Π and is in $\mathscr{M}$; so $k \leq k'$ by choice of k. On the other hand, $k = j^\pi$ is maximal with respect to $X_j \vee Y_{k-1}/X_{j-1} \vee Y_{k-1} \in \mathscr{M}$, so that $k \geq k'$ is a consequence of the following general observation (which is easily derived from Definition 1.4, using (M1)): if X_j/X_{j-1} is $\mathscr{M}$-related to $Y_{k'}/Y_{k'-1}$, and is in $\mathscr{M}$, then $X_j \vee Y_{k'-1}/X_{j-1} \vee Y_{k'-1} \in \mathscr{M}$.

2. Applications to finite groups

In this section we consider chief series of a finite group G (and $K, H \trianglelefteq G$, $K \leq H$).

(1) Since the lattice $\mathscr{L}$ of normal subgroups of a group is modular, we may apply Theorem 1.5 to deduce Isbell's version of the Jordan-Hölder Theorem for finite groups (namely, by taking $\mathscr{M} = \mathscr{P}_{K,H}$). This yields a correspondence π between the chief factors of the two series such that for all i the corresponding factors X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ satisfy $X_i/X_{i-1}\,\mathbf{Zsh}\,Y_{i^\pi}/Y_{i^\pi-1}$ and, in particular, are G-isomorphic.

(2) To get the Carter, Fischer and Hawkes version mentioned in the introduction (but for not necessarily soluble finite groups), with a correspondence π_Φ respecting the Frattini or non-Frattini nature of corresponding chief factors, one considers the set $\mathscr{M}_\Phi$ of all non-Frattini chief factors between K and H (the chief factors supplemented in G by a proper subgroup of G). This is an M-set: indeed, condition (M1) is trivial, while (M2) follows from two basic properties of the Frattini subgroup (see, for example, 1.25 in [2], for the less well-known one of them); in fact, the latter property also proves the validity of hypothesis (*) from Section 1; so $\mathscr{M}_\Phi$-related chief factors A/B and C/D always satisfy $A/B\,\mathbf{Zsh}\,C/D$, and hence are G-isomorphic.

(3) Let $\mathscr{S}$ be any set of maximal subgroups of G and consider the set $\mathscr{M}_S$ of all those chief factors X/Y of G complemented in G by at least one element U of $\mathscr{S}$:

$$G = UX \quad \text{and} \quad U \cap X = Y.$$

Again $\mathscr{M}_S$ satisfies (M1), but (M2) does not hold generally; for example, if G is elementary abelian of order r, $\{A, B, C\}$ the set of its maximal subgroups and $\mathscr{S} = \{A, B\}$, then all chief factors of G except G/C are complemented by some U in $\mathscr{S}$; thus $(G/C, B/1)$ is an $\mathscr{M}_S$-crossing, but $(G/B, C/1)$ is not.

A similar example, but with the relevant chief factors being non-abelian, is given by $G = E_1 \times E_2 \times E_3$ where E_1, E_2, E_3 are any three isomorphic non-abelian simple groups, with $\mathscr{S} = \{D_{12} \times E_3, D_{23} \times E_1\}$ where D_{ij} is a diagonal subgroup of $E_i \times E_j$. Here $(E_2 \times E_3/E_2, E_3/1)$ is an $\mathcal{M}_S$-crossing, but $(E_2 \times E_3/E_3, E_2/1)$ is not.

Yet another type of counterexample is obtained as follows. Let $G \in \mathcal{P}'_\Pi$, the class of all groups G with a maximal subgroup U such that $\mathbf{Core}_G(U)$, the normal core of U in G, is 1 and, $\mathrm{S}(G)$, the socle in G, is a non-abelian minimal normal subgroup of G complemented by U. (For examples of such groups see Förster [6]; a description of all groups in $\mathcal{P}'_\Pi$ can be found in Förster [8].) Let $S \cong_G \mathrm{S}(G)$ and form the semidirect product $H = GS$. This has precisely two minimal normal subgroups: S, and a diagonal subgroup T of $\mathrm{S}(G) \times S$, and these are complemented by G (see, for example, the first sections in Baer [1], Förster [5]). Now let $\mathcal{S} = \{G, UT\}$. Then all chief factors of G below $T \times S$ except $(T \times S)/S$ are complemented.

These three examples suggest the hypothesis (#) on $\mathcal{S}$ stated below. This hypothesis is not necessary for $\mathcal{M}_\mathcal{S}$ to satisfy (M2) in the lattice $\mathcal{L}_{1,G}$ (it is satisfied, though, by those $\mathcal{M}_\mathcal{S}$ we are interested in), but it appears to be difficult to formulate in a satisfactory manner the precise condition on $\mathcal{S}$ for $\mathcal{M}_\mathcal{S}$ to satisfy (M2). Before stating (#), we recall from Baer and Förster [2], Förster [7], Lafuente [13], the definition of the crown C/R of a group G associated with its non-Frattini chief factor X/Y:

> $C = X\mathbf{C}_G(X/Y)$, $R = \bigcap_{U \in \mathcal{T}} \mathbf{Core}_G(U)$, $\mathcal{T}$ the set of all maximal subgroups U of G such that X/Y is G-isomorphic to a minimal normal subgroup of $G/\mathbf{Core}_G(U)$.

(#) For each crown C/R of G and any two chief factors X_i/Y_i of G such that $R \leq Y_i$ and $X_i \leq C$ $(i = 1, 2)$, if X_1/Y_1 has a complement in G from $\mathcal{S}$, then so does X_2/Y_2, except, perhaps, when $X_i = C \neq X_{3-i}$ for some $i \in \{1, 2\}$.

(We do not require that the chief factors have a common complement.)

Basic properties of crowns are described in [2, 7, 13], and will be used without further reference. From such properties the following is immediate.

(+) Let X/Y be a chief factor of G and C/R the crown of G associated with it, and let U be a maximal subgroup of G. Then U complements X/Y if and only if U complements XR/YR.

Using (+), we will deduce a Jordan-Hölder Theorem for general $\mathcal{M}_\mathcal{S}$ from the special case where the lattice $\mathcal{L}_{K,H}$ involved is $\mathcal{L}_{R,C}$. So we now assume that $\mathcal{S}$ consists of maximal subgroups U of G complementing a chief factor of G between R and C; in view of the structure of crowns (cf. 2.4 in [7]), this means that U complements a minimal normal subgroup of $G/\mathbf{Core}_G(U)$ and $R \leq \mathbf{Core}_G(U) \leq C$.

Assume hypothesis (#). In order to verify condition (M2) for $\mathscr{M}_{\mathscr{S}}$, let $(A/B, E/F)$ be an $\mathscr{M}_{\mathscr{S}}$-crossing. Then some $U \in \mathscr{S}$ complements E/F, while A/B does not have a complement in $\mathscr{S}$; in particular, U cannot complement A/B. Now $U \cap B > F$ would easily lead to the contradiction that $U \cap A = U \cap B = B$. Thus $U \cap B = F$ and $B \nleq U$; so U complements B/F. (In fact, we could have infered from (#) the existence of a complement of U from $\mathscr{S}$.) It remains to observe that from (#) it follows that A/E is not complemented by an element of $\mathscr{S}$: since A/B is not complemented by an element of $\mathscr{S}$, but $B/F < C/F$ is, (#) requires that $A = C$; and then, if A/E were complemented by an element of $\mathscr{S}$, the same should apply to A/B.

Next, let $\mathscr{S}_{C/R}$ be the set of all maximal subgroups complementing a chief factor between R and C. Recall that all chief factors X/Y of G between R and C are isomorphic (although all of them are G-isomorphic only if C/R is abelian or is itself a chief factor of G; however, they are always similar in the sense of 53.11 of [15], and G-connected/G-related in the sense of [13] and [7]. Observe that all these chief factors X/Y are complemented in G by a maximal subgroup, except those for which $X = C$ and $G/Y \notin \mathscr{P}'_{\Pi}$. Actually, in [14] we have pointed out that each non-soluble finite group G has a crown C/R such that the [pairwise isomorphic] groups G/T, $T \trianglelefteq G$ with $R \leq T \leq C$ and C/T a chief factor, are not in $\mathscr{P}'_{\Pi}$). Evidently, the set $\mathscr{S}_{C/R}$ satisfies hypothesis (#) irrespective of whether or not the crown is complemented (that is, the G/T, $T \trianglelefteq G$ with $R \leq T \leq C$ and C/T a chief factor, are in $\mathscr{P}'_{\Pi}$ or not). Hence the above discussion together with Theorem 1.5 yields a Jordan-Hölder correspondence $\pi_{C/R}$, and a uniqueness statement for this.

(4) To get the general result, note that each chief factor X/Y of G is either Frattini or has a unique crown C/R associated with it. The latter is determined by the requirement that $XR < YR$ (and then $X/Y \ll XR/YR$; in fact, $X/Y \cong_G XR/YR$). Therefore, given any chief series of G, multiplying by R induces a bijection between those factors in the series whose associated crown is C/R and the factors in the chief series of G between R and C obtained by taking the images of the former chief factors under such multiplication. Now put $\mathscr{C}_{\Phi} = \mathscr{M}_{\Phi}$, the set of all Frattini chief factors of G and, for each crown C/R of G, let $\mathscr{C}_{C/R}$ comprise all non-Frattini chief factors of G with C/R as their associated crown. Define $\mathscr{M}_{C/R} = \mathscr{M}_{\mathscr{S}}$ where $\mathscr{S} = \mathscr{S}_{C/R}$, and note that $\mathscr{M}_{C/R} \subseteq \mathscr{C}_{C/R}$. Also, say that two chief factors are $\mathscr{M}$-related, if both belong to the same $\mathscr{C}_x$ and are $\mathscr{M}_x$-related, where x is Φ or some C/R. Finally, given two chief series of G of lengths n, m, define $\pi \in S_n$ by requiring that the restriction of π to $I_x = \{i \in \{1, \dots, n\} | H_i/K_i \in \mathscr{C}_x\}$, where $x = \Phi$ or $x = C/R$ for some crown C/R of G, be π_x. Then from (2) and (3) we obtain (most of) our main result

(which we formulate for $\mathscr{L}_{K,H}$, although our proof here dealt only with the special case $K = 1$ and $H = G$).

2.1 THEOREM. *Let $K = X_0 < X_1 < \cdots < X_n = H$ and $K = Y_0 < Y_1 < \cdots < Y_m = H$ be two chief series of G between H and K. Then $n = m$, and there exists a unique $\pi \in S_n$ such that X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ are $\mathscr{M}$-related for $i = 1, \dots, n$. This means the following.*

(i) *$Y_{i^\pi}/Y_{i^\pi-1} \leq \Phi(G/Y_{i^\pi-1}) \Leftrightarrow X_i/X_{i-1} \leq \Phi(G/X_{i-1}) \Leftrightarrow X_i/X_{i-1} \cong_G Y_{i^\pi}/Y_{i^\pi-1}$; in fact, there is a Frattini factor A/B such that $X_i/X_{i-1} \gg A/B \ll Y_{i^\pi}/Y_{i^\pi-1}$.*

(ii) *$X_i/X_{i-1} \nleq \Phi(G/X_{i-1}) \Leftrightarrow X_i/X_{i-1}$ is G-connected to $Y_{i^\pi}/Y_{i^\pi-1}$.*

(iii) *$Y_{i^\pi}/Y_{i^\pi-1}$ is complemented in G by a maximal subgroup $\Leftrightarrow X_i/X_{i-1}$ is complemented in G by a maximal subgroup $\Rightarrow X_i/X_{i-1}$ and $Y_{i^\pi}/Y_{i^\pi-1}$ have a common maximal complement in G, and for the crown C/R of G associated with X_i/X_{i-1}, either $X_i/X_{i-1} \ll A/B \gg Y_{i^\pi}/Y_{i^\pi-1}$ for some chief factor A/B of G between R and C (in particular, $X_i/X_{i-1} \cong_G Y_{i^\pi}/Y_{i^\pi-1}$), or $X_i/X_{i-1} \ll C/T_i \neq C/S_i \gg Y_{i^\pi}/Y_{i^\pi-1}$ where $T_i, S_i \trianglelefteq G$ contain R and are such that C/T_i and C/S_i are non-complemented chief factors of G.*

(iv) *X_i/X_{i-1} is non-Frattini, but not complemented by a maximal subgroup $\Rightarrow X_iR = C = Y_{i^\pi}R$ and $C/X_{i-1}R$ and $C/Y_{i^\pi-1}R$ are non-complemented chief factors of G, where C/R is the crown of G associated with X_i/X_{i-1}. Moreover, for each $x \in \{\Phi\} \cup \{C/R | C/R$ a crown of $G\}$ and all $i \in \{1, \dots, n\}$,*

$$i^\pi = \max\{j \in \{1, \dots, n\} | X_iY_{j-1}/X_{i-1}Y_{j-1} \in \mathscr{M}_x\}, \quad \text{if } X_i/X_{i-1} \in \mathscr{M}_x,$$
$$i^\pi = \min\{j \in \{1, \dots, n\} | X_i \cap Y_j/X_{i-1} \cap Y_j \in \mathscr{C}_x \backslash \mathscr{M}_x\}, \quad \text{if } X_i/X_{i-1} \in \mathscr{C}_x \backslash \mathscr{M}_x.$$

To check the above conditions (iii) and (iv), apply Definition 1.4 (here only cases (1,2) can be relevant) together with statement (+) above and the description of the structure of G/R for a crown C/R given in [7, 24].

Acknowledgement

I am grateful to P. Förster, L. G. Kovács and M. F. Newman for the opportunity to see [2, 7, 8, 11], and to the referee for the helpful comments.

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