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Abstract

We calculate equivariant elliptic cohomology of the partial flag variety G/H, where $H \subseteq G$ are compact connected Lie groups of equal rank. We identify the RO(G)-graded coefficients $\mathcal{E}ll_G^*$ as powers of Looijenga's line bundle and prove that transfer along the map

$$\pi : G/H \longrightarrow \mathrm{pt}$$

is calculated by the Weyl–Kac character formula. Treating ordinary cohomology, K-theory and elliptic cohomology in parallel, this paper organizes the theoretical framework for the elliptic Schubert calculus of [N. Ganter and A. Ram, *Elliptic Schubert calculus*, in preparation].

Contents

1. Introduction

The topological aspects of representation theory are captured by the generalized cohomology theory known as equivariant K-theory. Applied to a point, the K-group

$$K_G(\mathrm{pt}) = R(G)$$

is the representation ring of the structure group G. Applied to other spaces, it yields rings related to the representation rings. For instance, the K_G-theoretic transfer along the map

$$\pi\colon G/H \to \mathrm{pt}$$

gives the induction map

$$\mathrm{ind}\colon R(H) \longrightarrow R(G).$$

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In [AB68], Atiyah and Bott use this point of view to derive Weyl's famous formula for the character of induced representations. For this purpose, they consider an inclusion $H \subseteq G$ of compact, connected Lie groups of equal rank and a joint maximal torus T. Then T acts on G/H from the left, allowing the authors to calculate the equivariant transfer along π as an application of their fixed point formula for the T-equivariant transfer $\pi_!$.

Schubert calculus, originally concerned with the cohomology of the partial flag varieties, has long been extended to include the analogous K-theoretic picture. Essential ingredients in both theories are pull-backs and transfers (push-forwards) along maps between partial flag varieties.

In [BE90], Bressler and Evens formulate Schubert calculus in broad generality, replacing cohomology and K-theory with any generalized multiplicative cohomology theory possessing the relevant transfers. The universal example of such a theory is complex cobordism, and cobordism-theoretic Schubert calculus is now becoming a discipline of its own (see [BE92, CPZ13, HK11]).

We are interested in equivariant elliptic cohomology, $\mathcal{E}ll_G$. It has long been conjectured that $\mathcal{E}ll_G$ plays the same role for the representation theory of the loop group $\mathcal{L}G$ that K_G plays for the representation theory of G. This idea can already be found in Grojnowski's article on the definition of $\mathcal{E}ll_G$ (see [Gro07, p. 2 and 3.3]), it took shape in Ando's work on Euler classes [And03] and was later picked up by Lurie [Lur].

In a sense, the paper at hand is a following to Ando's [And00] and [And03], taking apart the ideas presented in those papers and rearranging them into a new picture. The most important new feature is that our derivation of the Weyl–Kac formula as a push-forward is entirely a compact-manifold argument. This tells us that whatever is infinite-dimensional about representations of the loop group $\mathcal{L}G$ is entirely encoded in the relationship between equivariant elliptic cohomology and $\mathcal{L}G$. Further, we will see in § 7.3 that

$$\Gamma\mathcal{E}ll_G^*(\mathrm{pt}) \cong \widetilde{Th}_*^W$$

is Looijenga's ring of theta functions [Loo77]. This is where the loop group characters take their values (see § 7.3). This paper is the first in a joint program with Arun Ram, studying Schubert calculus in elliptic cohomology. For this, the ring $\widetilde{Th}_*$ will play the same role as $R(T)$ plays for the K-theoretic Schubert calculus or the symmetric algebra $S(\mathfrak{t}_{\mathbb{C}}^*)$ plays in cohomology. Our work ties in with the program formulated by Bressler and Evens in [BE90], but is not a special case of their discussion; Bressler and Evens work Borel equivariantly, while Grojnowski's $\mathcal{E}ll_T$ is a genuinely equivariant theory, taking values in sheaves over a scheme $\mathcal{M}_T$. More importantly, $\mathcal{E}ll_T$ does not possess Thom isomorphisms for complex vector bundles, so that the theory of transfer maps acquires a twist by a line bundle, called the *relative Thom sheaf.*

In our main application (see § 8.3), the relative Thom sheaf for $\pi_!$ turns out to be $\mathcal{L}_{\mathrm{Lo}}^g$, the Looijenga line bundle raised to the dual Coxeter number g. This accounts for the shift of level by g occurring in the Weyl–Kac formula. The definitions of Looijenga line bundle [Loo77] and dual Coxeter number are reviewed in § 6.3.

The paper is organized as follows: treating cohomology, K-theory and elliptic cohomology simultaneously (all with complex coefficients), we view all three theories as sheaf-valued, revisiting, and to some extent reorganizing, the circle of ideas in [And03, Gep05, GKV95, Gro07, Lur, Roş01], and [Roş03].

After recalling the definitions and the general setup (§§ 2 and 3), we review a powerful calculational tool: the theory of moment graphs (§ 4). We show how to deduce the isomorphism

$$K_T(G/H) \cong R(T) \otimes_{R(G)} R(H)$$

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for torsion-free $\pi_1 G$ (see [McL79]) directly from the moment graph of G/H. Our proof does not use Pittie's theorem that $R(H)$ is free over $R(G)$, nor does it involve any explicit calculations with basis elements. Our argument is identical for K-theory and cohomology and also yields a description of $\mathcal{E}ll_T(G/H)$.

Now we are in a position to prove the axioms of [GKV95] needed in our applications, which is done in § 5. Also in § 5, we define the *Ginzburg–Kapranov–Vasserot characteristic class* c_ξ of a complex vector bundle ξ. The term 'class', borrowed from [GKV95], is slightly misleading here; c_ξ is actually a map, obtained by applying elliptic cohomology to a classifying map of ξ.

Section 6 treats the theory of Thom sheaves. We identify the Thom sheaf $\mathbb{L}^\xi$ of any complex vector bundle ξ with the pull-back along c_ξ of a universal example

$$\mathbb{L}^\xi \cong c_\xi^* \mathbb{L}^{\mathrm{univ}}.$$

Section 7 deals with Euler classes, Looijenga theta functions, relative Thom sheaves and push-forwards (also known as transfers).

Finally, in § 8, we arrive at the promised formula for the transfer $\pi_!$. In cohomology, this is a formula by Akyildiz and Carrell [AC83], in K-theory, it is the Weyl formula, and in elliptic cohomology it is the Weyl–Kac formula.

The reader is not expected to be familiar with elliptic cohomology. I have included a fair bit of expository material on various topics, hoping that this will make the paper readable for a wide audience.

The combinatorial aspects of the theory, as well as concrete examples, will be addressed in [GR1].

2. The three sheaf-valued theories

Let G be a compact Lie group, and let X be a finite G-CW-complex. We write

$$H_G(X) := \sum_{n\in\mathbb{Z}} H^{2n}(EG \times_G X; \mathbb{C})$$

for (even) Borel equivariant cohomology with complex coefficients, and

$$K_G(X) := (\mathrm{Vect}_G^{\mathbb{C}}(X)/\cong, \oplus)^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{C}$$

for equivariant K-theory (as in [Seg68]) with complex coefficients. We will also consider the relative and reduced versions of these theories. These are contravariant functors in G and in X (or pairs (X, A) or (X, x_0)). For abelian T it follows that the coefficient rings

$$H_T := H_T(\mathrm{pt}) \quad \text{and} \quad K_T := K_T(\mathrm{pt})$$

form Hopf algebras, with the comultiplication given by multiplication in T. For the circle group $U(1)$, we have

$$H_{U(1)} = \mathbb{C}[x] \quad \text{and} \quad K_{U(1)} = \mathbb{C}[z^{\pm 1}].$$

These are the Hopf algebras of regular functions on the (affine) group schemes

$$\mathbb{G}_a = \mathbb{A}^1_{\mathbb{C}} \quad \text{(additive group)}$$

and

$$\mathbb{G}_m = \mathbb{A}^1_{\mathbb{C}} \backslash \{0\} \quad \text{(multiplicative group)}.$$

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Here

$$x = c_1(\mathbb{C}_1)_{U(1)}$$

is the first Borel equivariant Chern class of the defining representation $\mathbb{C}_1$ of $U(1)$. It generates the ideal

$$x\mathbb{C}[x] = I(0)$$

of regular functions on $\mathbb{A}^1_{\mathbb{C}}$ that vanish at 0. The K-theory class z is the character of $\mathbb{C}_1$. The $K_{U(1)}$-theoretic first Chern class of $\mathbb{C}_1$ equals $1-z$, generating the ideal $I(1)$ of regular functions on $\mathbb{G}_m$ vanishing at 1.

More generally, let T be a compact abelian Lie group with Lie algebra $\mathfrak{t}$, and let

$$\widehat{T} = \mathrm{Hom}(T, U(1))$$

be the character lattice of T. Let $T_0 \subseteq T$ be the connected component of 1, and let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of the torus T_0. If $T = T_0$ is connected there is an isomorphism

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cong} & \widehat{T} \\ \lambda & \longmapsto & e^{2\pi i\lambda}. \end{array}$$

For $\lambda \in \Lambda$, let $\mathbb{C}_\lambda$ be the one-dimensional representation of T_0 with character $e^{2\pi i\lambda}$. Then the coefficients $H_T \cong H_{T_0}$ are identified by the Hopf-algebra isomorphism

$$\begin{array}{rcl} H_T & \cong & \Gamma\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}} \\ c_1(\mathbb{C}_\lambda)_{T_0} & \leftarrow\!\shortmid & \lambda_{\mathbb{C}}. \end{array}$$

Here $\lambda_{\mathbb{C}} = \lambda \otimes_{\mathbb{R}} \mathbb{C}$ is viewed as a regular function on the complex algebraic group $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. This point of view, going back to Borel, allows us to interpret $H_T(X, A)$ as the global sections of a coherent sheaf $\mathcal{H}_T(X, A)$ on $\mathfrak{t}_{\mathbb{C}}$.

In K-theory, the T-equivariant coefficients are given by the representation ring

$$K_T = R(T) = \mathbb{C}[\widehat{T}].$$

For instance,

$$K_{T_0}(\mathrm{pt}) \cong \mathbb{C}\{e^\lambda\}_{\lambda \in 2\pi i\Lambda},$$

with $e^\lambda e^\mu = e^{\lambda+\mu}$. So,

$$K_T \cong \Gamma\mathcal{O}_{T_{\mathbb{C}}}$$

is identified with the ring of regular functions of the complexification $T_{\mathbb{C}}$ of T. This allows us to view $K_T(X, A)$ as the global sections of a coherent sheaf $\mathcal{K}_T(X, A)$ on $T_{\mathbb{C}}$.

We have

$$T_{\mathbb{C}} \cong \mathrm{Hom}(\widehat{T}, \mathbb{C}^\times) \quad \text{and} \quad \mathfrak{t}_{\mathbb{C}} \cong \mathrm{Hom}(\widehat{T}, \mathbb{C}).$$

Let E be a complex elliptic curve, and let[1]

$$\mathcal{M}_T := \mathrm{Hom}(\widehat{T}, E).$$

[1] We may interpret $\mathcal{M}_T$ as the moduli scheme of certain principal T-bundles on E, see [GKV95, (1.4.2)]; note that in [GKV95, (1.4.2)], the authors denote $\mathcal{M}_G$ by $\mathcal{X}_G$.

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Grojnowski's T-equivariant elliptic cohomology takes values in coherent sheaves over $\mathcal{M}_T$, and

$$\mathcal{E}ll_T(\mathrm{pt}) = \mathcal{O}_{\mathcal{M}_T}.$$

We will see that the above theories form the degree zero parts of three $\mathrm{RO}(T)$-graded (sheaf-valued) equivariant cohomology theories. Note that the complex group $\mathcal{M}_T$ is no longer affine and the global sections $\Gamma\mathcal{E}ll_T(-)$ do not form a cohomology theory. This makes the sheaf point of view essential to the theory. We will now see how the formal properties of $\mathcal{E}ll_T$, as axiomatically postulated in [GKV95], determine the stalks of the theory. This is the motivation behind Grojnowski's construction, which we will recall in § 3.3.

2.1 Homogeneous spaces and representation spheres

Let $T' \subseteq T$ be a closed subgroup. Then we have canonical inclusions $\mathfrak{t}'_{\mathbb{C}} \subseteq \mathfrak{t}_{\mathbb{C}}$ and $T'_{\mathbb{C}} \subseteq T_{\mathbb{C}}$ and $\mathcal{M}_{T'} \subseteq \mathcal{M}_T$ and isomorphisms of coherent sheaves

$$\begin{aligned} \mathcal{H}_T(T/T') &\cong \mathcal{O}_{\mathfrak{t}'_{\mathbb{C}}} \quad (\text{over } \mathfrak{t}_{\mathbb{C}}\ ) \\ \mathcal{K}_T(T/T') &\cong \mathcal{O}_{T'_{\mathbb{C}}} \quad (\text{over } T_{\mathbb{C}}) \\ \mathcal{E}ll_T(T/T') &\cong \mathcal{O}_{\mathcal{M}'_T} \quad (\text{over } \mathcal{M}_T). \end{aligned} \tag{1}$$

The first two of these isomorphisms are classical. We recall the definition of the third on page 1209. That it is an isomorphism will be an immediate consequence of the construction of $\mathcal{E}ll_T$.

From now on, we let A_T be one of the complex abelian groups $\mathfrak{t}_{\mathbb{C}}$ or $T_{\mathbb{C}}$ or $\mathcal{M}_T$, and we let $\mathcal{F}_T$ be the theory $\mathcal{H}_T$ or $\mathcal{K}_T$ or $\mathcal{E}ll_T$ taking values in sheaves over A_T. Often we will write $+$ for the group operation in A_T and 0 for its unit, with the understanding that these are to be replaced by $\cdot$ and 1 for the multiplicative case $A_T = T_{\mathbb{C}}$.

Let T be a torus, $\lambda \in \Lambda$, let $\mathbb{S}^\lambda$ be the representation sphere (one point compactification) of $\mathbb{C}_\lambda$, and write K_λ for the kernel of $e^{2\pi i\lambda}$ inside T. We may identify the equator of $\mathbb{S}^\lambda$ with T/K_λ. Then the usual Mayer–Vietoris argument gives the following corollary.

Corollary 2.1. *The sheaf $\mathcal{F}_T(\mathbb{S}^\lambda)$ is identified with the kernel of the map*

$$\begin{aligned} \mathcal{O}_{A_T} \oplus \mathcal{O}_{A_T} &\longrightarrow \mathcal{O}_{A_{K_\lambda}} \\ (f,g) &\longmapsto (f-g)|_{A_{K_\lambda}}. \end{aligned}$$

2.2 Stalks

By a point in A_T, we will always mean a closed point. For $a \in A_T$, let

$$T(a) := \bigcap_{a \in A_{T'}} T'$$

be the smallest subgroup of T with $a \in A_{T(a)}$. Let

$$i_a \colon X^{T(a)} \hookrightarrow X$$

be the inclusion of the $T(a)$-fixed points. We will identify the stalk of $\mathcal{F}_T$ at a in two steps.

First, we note that

$$i_a^* \colon \mathcal{F}_T^*(X)_a \xrightarrow{\cong} \mathcal{F}_T^*(X^{T(a)})_a \tag{2}$$

is an isomorphism of T-equivariant cohomology theories. Indeed, it is enough to check this on orbits $X = T/T'$, where it follows from (1). Second, consider the quotient map $p \colon T \to T/T(a)$

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and use the isomorphism[2]

$$\mathcal{F}_T(X^{T(a)}) \cong A_p^*(\mathcal{F}_{T/T'}(X^{T(a)})).$$

Let

$$\begin{aligned} \tau_a\colon A_T &\longrightarrow A_T \\ b &\longmapsto a+b \end{aligned}$$

denote translation by a. Then

$$A_p = A_p \circ \tau_a, \tag{3}$$

and hence

$$\mathcal{F}_T(X^{T(a)})_a \cong \mathcal{F}_T(X^{T(a)})_0.$$

Combining these two steps, we obtain isomorphisms

$$\begin{aligned} \mathcal{H}_T(X)_a &\cong H_T(X^{T(a)}) \otimes_{H_T} \mathcal{O}_{\mathfrak{t}_{\mathbb{C}},0} \\ \mathcal{K}_T(X)_a &\cong K_T(X^{T(a)}) \otimes_{K_T} \mathcal{O}_{T_{\mathbb{C}},1} \end{aligned}$$

and

$$\mathcal{E}ll_T(X)_a \cong \mathcal{E}ll_T(X^{T(a)})_0.$$

Over a sufficiently small neighbourhood U of a (see § 3.3 for details) these isomorphisms extend to an isomorphism of sheaves

$$\mathcal{F}_T(X)|_U \cong (\tau_a)_*(\mathcal{F}_T(X^{T(a)})|_{U-a}). \tag{4}$$

3. The Chern character and the construction of $\mathcal{E}ll_T$

3.1 Completion

In the sheaf-theoretic language, the Atiyah–Segal completion theorem identifies the formal completion of $\mathcal{F}_T$ at $0 \in A_T$ with the Borel equivariant version of $\mathcal{F}$. More precisely, we have the following theorem.

Theorem 3.1 (Completion theorem). *We have an isomorphism of pro-rings*

$$\mathcal{F}_T(X)\hat{_0} \cong \varprojlim_k \mathcal{F}(ET^{(k)} \times_T X),$$

where $ET^{(k)}$ is the k-skeleton of ET.

In the case of K-theory, the right-hand side is $K(ET \times_T X)$, and Theorem 3.1 is [AS69]. For cohomology, the right-hand side is

$$\prod_{n\in\mathbb{Z}} H^{2n}(ET \times_T X; \mathbb{C})$$

(see [Roş03, p. 6]). In § 6, we will see how Theorem 3.1 follows from the formal properties of $\mathcal{F}_T$.

[2] This isomorphism was proved in [AB84] for H, in [Seg68] for K and postulated for $\mathcal{E}ll$ in [GKV95, (1.6.3)]. Again, it will follow immediately from the construction of $\mathcal{E}ll_T$.

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3.2 Roşu's Chern character

Consider the exponential map

$$\exp\colon \mathfrak{t}_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}.$$

This is an analytic map of complex groups, it is not algebraic. For an algebraic sheaf $\mathcal{F}$ on a complex variety, we let $\mathcal{F}^h$ be the analytic sheaf associated to $\mathcal{F}$. The following theorem is a reformulation of the main result in [Roş03].

Theorem 3.2 (Roşu). *Assume*[3] *that $H_T(X)$ is free over H_T. For a small enough analytic open neighbourhood U of $0 \in \mathfrak{t}_{\mathbb{C}}$ there is an isomorphism of analytic sheaves*

$$ch_T\colon \mathcal{K}^h_T(X)|_{\exp(U)} \xrightarrow{\cong} \exp_* \mathcal{H}^h_T(X)|_U,$$

uniquely determined by the commuting diagram

$$\begin{tikzcd}
\mathcal{K}^h_T(X)_1 \arrow[r, "ch_T|_1"] \arrow[d, tail] & \mathcal{H}^h_T(X)_0 \arrow[d, tail] \\
\mathcal{K}_T(X)_{\widehat{1}} \arrow[r, "ch"'] & \mathcal{H}_T(X)_{\widehat{0}}
\end{tikzcd}$$

where the top row is ch_T at the stalk 1, and in the bottom row, ch stands for the (Borel equivariant) classical Chern character.

We will refer to ch_T as *Roşu's Chern character*. Consider now the quotient maps

$$\exp_E\colon \mathbb{C} \longrightarrow E = \mathbb{C}/2\pi i\langle\tau, 1\rangle$$

and

$$y\colon \mathbb{C}^\times \longrightarrow E \cong \mathbb{C}^\times/q^{\mathbb{Z}},$$

where $q = e^{2\pi i\tau}$. These induce the following commuting diagram of complex analytic group homomorphisms.

$$\begin{tikzcd}
\mathfrak{t}_{\mathbb{C}} \arrow[rr, "\exp_{T_{\mathbb{C}}}"] \arrow[dr, "\exp_{\mathcal{M}_T}"'] & & T_{\mathbb{C}} \arrow[dl, "y"] \\
& \mathcal{M}_T &
\end{tikzcd}$$

When it exists, Roşu's Chern isomorphism ch_T will fit into a commuting diagram

$$\begin{tikzcd}
(\exp_{\mathcal{M}_T})_*\mathcal{H}^h_T(X)|_U & & y_*\mathcal{K}^h_T(X)|_{\exp_{T_{\mathbb{C}}}(U)} \arrow[ll, "y_*(ch_T)"'] \\
& \mathcal{E}ll^h_T(X)|_{\exp_{\mathcal{M}_T}(U)} \arrow[ul, "\phi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

of sheaf isomorphisms over a small neighbourhood of 0 in $\mathcal{M}_T$.

[3] This assumption ensures that the restriction maps of $\mathcal{H}^h_T(X)$ and, more importantly, the map $\mathcal{H}^h_T(X)_0 \to \mathcal{H}_T(X)_{\widehat{0}}$ are injective. I do not follow Roşu's argument for arbitrary X in [Roş03, p. 7]. This does not affect the main applications in [Roş03], since for those Knutsen and Roşu do require $H_T(X)$ to be free over H_T.

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3.3 Construction of $\mathcal{E}ll_T$

The construction of $\mathcal{E}ll_T(X)$ was first outlined in [Gro07]. The technical details were filled in in [Roş03], see also [And03] and [Roş01]. This section is a reminder of Grojnowski's construction.

Note first that the properties stated in §§ 2.2 and 3.2 determine $\mathcal{E}ll^h_T(X)$ locally: every $a \in \mathcal{M}_T$ has a small analytic neighbourhood U_a satisfying

$$b \in U_a \Longrightarrow X^{T(b)} \subseteq X^{T(a)}.$$

Choose U_0 small enough such that the logarithm is well defined over it and

$$c \in U_0 \Longrightarrow X^{T_{\log(c)}} = X^{T(c)}.$$

Further, we assume U_a to be small enough to satisfy

$$(U_a - a) \subseteq U_0.$$

Then we are forced into

$$\mathcal{E}ll^h_T(X)|_{U_a} \cong (\tau_a \circ \exp)_* \mathcal{H}^h_T(X^{T(a)})|_{\log(U_a - a)}.$$

We need to understand how these patches are to be glued. Given a non-empty intersection $U := U_a \cap U_b$, we make the additional assumptions[4]

$$a - b \in U_0 \quad \text{and} \quad X^{T(b)} \subseteq X^{T(a)}.$$

LEMMA 3.3. *Let i denote the inclusion of $X^{T(b)}$ in $X^{T(a)}$. After restricting to $\log(U - a)$, the map*

$$i^* \colon \mathcal{H}_T(X^{T(a)}) \xrightarrow{\cong} \mathcal{H}_T(X^{T(b)})$$

becomes an isomorphism of (analytic) sheaves.

Proof. We check the statement on stalks. Let $\gamma \in \log(U - a)$. We claim that we have an equality of simultaneous fixed point sets

$$X^{T(a)} \cap X^{T(\gamma)} = X^{T(b)} \cap X^{T(\gamma)}.$$

The statement then follows from (2) with γ in the role of a. To prove the non-trivial direction of the claim, let $c = \exp(\gamma)$. Then $a + c$ is an element of U_b, and we obtain

$$X^{T(a)} \cap X^{T(\gamma)} = X^{T(a)} \cap X^{T(c)} \subseteq X^{T(a+c)} \subseteq X^{T(b)}.$$

The inclusion in the middle may be checked on orbits $T/T' \subseteq X$, where it follows immediately from the definition of $T(-)$. $\square$

Now let $\gamma := \log(a - b)$. Similarly to the proof of the lemma, one argues that $T(\gamma)$ fixes $X^{T(b)}$. As in (3), we obtain an isomorphism

$$\phi \colon (\tau_\gamma)_* \mathcal{H}_T(X^{T(b)}) \cong \mathcal{H}_T(X^{T(b)}),$$

and hence of the corresponding analytic sheaves. Finally, we have

$$\tau_b \circ \exp \circ \tau_\gamma = \tau_a \circ \exp.$$

The desired glueing isomorphism is the composite

$$(\tau_b \exp)_*(\phi) \circ (\tau_a \exp)_*(i^*).$$

To define the algebraic theory $\mathcal{E}ll_T(-)$, we use Serre's GAGA result.

[4] This is possible by [Roş03, 2.5].

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Theorem 3.4 [Ser56]. *Let X be a projective algebraic variety over $\mathbb{C}$, let X^h be its underlying analytic variety and let $\mathcal{C}oh_{\mathrm{alg}}(X)$ and $\mathcal{C}oh_{\mathrm{an}}(X^h)$ be the categories of coherent algebraic (respectively analytic) sheaves over X. Then the functor*

$$\begin{aligned} \mathcal{C}oh_{\mathrm{alg}}(X) &\longrightarrow \mathcal{C}oh_{\mathrm{an}}(X^h) \\ \mathcal{F} &\longmapsto \mathcal{F}^h \end{aligned}$$

is an equivalence of categories.

3.4 Compact connected Lie groups

Let G be a compact connected Lie group with maximal torus T and Weyl group W. Then $\mathcal{F}_G$ takes values over the scheme

$$A_G = A_T/W \quad \text{and} \quad \mathcal{F}_G(X) = \mathcal{F}_T(X)^W$$

is the sheaf of W-invariant sections. In the elliptic case, these are to be taken as the definition of $\mathcal{M}_G$ and $\mathcal{E}ll_G$.

4. Moment graphs

Moment graphs provide a powerful tool for calculations. Their application to flag varieties and Schubert calculus has a long history, dating back to the work of Kostant, Bernstein–Gelfand–Gelfand, Lascoux–Schützenberger, Kostant–Kumar and others. Let T be a compact torus and X a compact T-manifold. Let

$$i\colon X^T \longrightarrow X$$

be the inclusion of the fixed points in X. For a subtorus T' of T, this factors through the inclusion

$$i_{T'}\colon X^T \longrightarrow X^{T'}.$$

Recall that the equivariant 1-skeleton X_1 of X is defined as the set of all points in X whose orbit is at most one-dimensional. The following theorem was proved for cohomology by Goresky *et al.* [GKM98]. Knutsen and Roşu later generalized it to K-theory and elliptic cohomology [Roş03]. Further generalizations appear in [HHH05].

Theorem 4.1 (Localization theorem). *Assume that $H_T(X)$ is free over H_T and that X_1 consists of a finite number of representation spheres $\mathbb{S}^\lambda$, meeting only at the fixed points. Then the map*

$$i^*\colon \mathcal{F}_T(X) \longrightarrow \mathcal{F}_T(X^T)$$

is injective, and its image is equal to

$$\mathrm{Im}(i^*) = \bigcap_{T'} \mathrm{Im}(i^*_{T'}), \tag{5}$$

where the intersection runs over all subgroups of codimension 1 in T.

The data determining the right-hand side of (5) are recorded in the 'moment graph' of X.

Definition 4.2. In the situation of the theorem, the *moment graph* Γ of X has vertices indexed by the fixed points of X and an oriented edge with label $\lambda \in \Lambda$ from x_1 to x_2 for each $\mathbb{S}^\lambda \subseteq X_1$, containing x_1 as 0 and x_2 as ∞.

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Corollary 4.3 (of Theorem 4.1 and Corollary 2.1). *In the situation of the theorem, $\mathcal{F}_T(X)$ is described by the following equalizer diagram.*

$$\mathcal{F}_T(X) \longrightarrow \bigoplus_v \mathcal{O}_{A_T} \rightrightarrows \bigoplus_{(e,\lambda)} \mathcal{O}_{A_{K_\lambda}}$$

Here $K_\lambda = \ker(e^{2\pi i\lambda})$, the first sum is over the vertices, the second sum is over the edges of the moment graph, and the two arrows are defined in the obvious manner.

This formulation of the theory can be found in the paragraph before (1.3) in [GKM98, p. 27].

Example 4.4 (Partial flag varieties). Let $H \subseteq G$ be compact connected Lie groups of equal rank. Let $T \subseteq H$ be a maximal torus (of both). Let W_G be the Weyl group of G. Then the Weyl group of H can be identified with a subgroup $W_H \subseteq W_G$. Fix a set of positive roots $\mathcal{R}_+$ of G. For $\alpha \in \mathcal{R}_+$, let $s_\alpha \in W_G$ be the corresponding reflection. The following description of the moment graph of G/H can be found in [Tym09, Theorem 3.1]: we have a bijection

$$\begin{aligned} W_G/W_H &\longrightarrow (G/H)^T \\ wW_H &\longmapsto wH. \end{aligned}$$

Writing $[w]$ for the vertex corresponding to the left-coset wW_H, we have an edge labeled α from $[w]$ to $[s_\alpha w]$ whenever $\alpha \in \mathcal{R}_+$ is such that $w^{-1}(\alpha) < 0$ is not a root of H. (For the explicit calculations in [GR1] and [GR2], it turned out to be convenient to relabel the vertices by exchanging w with w^{-1}, but in the paper at hand, we will follow Tymoczko's conventions.)

Often the groups $K_\alpha = \ker(e^{2\pi i\alpha})$, turning up as the stabilizers of one-dimensional orbits in G/H, have an interpretation as fixed points.

Lemma 4.5. *Let G be a compact connected Lie group with maximal torus T and Weyl group W. Let α be a root of G. Then the action of s_α on T leaves the elements of K_α fixed. If $\pi_1(G)$ is torsion free then the inclusion*

$$K_\alpha \subseteq T^{s_\alpha}$$

is an equality.

Proof. The first claim is [BT85, V.(2.9)(iii)]. Recall from [BT85, V.(7.1)] that

$$\pi_1(G) = \Lambda^\vee/\Gamma,$$

where

$$\Lambda^\vee = \ker(\exp) \subseteq \mathfrak{t}$$

and Γ is the sublattice generated by the coroots. Let $x \in \mathfrak{t}$ be such that $\exp(x)$ is fixed under s_α. Then

$$\alpha(x)\check{\alpha} = x - s_\alpha(x)$$

is an element of $\Lambda^\vee$. Since $\Lambda^\vee/\Gamma$ is torsion free, it follows that $\alpha(x)$ is an integer. Hence $e^{2\pi i\alpha(x)} = 1$, and we have proved

$$\exp(x) \in K_\alpha. \qquad \square$$

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4.1 The scheme X_{A_G}

The sheaf $\mathcal{F}_G(X)$ is a sheaf of commutative algebras over A_G. Following [GKV95, (1.7.4)], we let X_{A_G} be the spectrum of $\mathcal{F}_G(X)$. This is a scheme over A_G. The assignment

$$X \longmapsto X_{A_G}$$

is covariantly functorial in X. We have $\mathrm{pt}_{A_G} = A_G$. Writing $\pi\colon X \to \mathrm{pt}$ for the unique map from X to the one point space, the map

$$\pi_{A_G}\colon X_{A_G} \longrightarrow A_G$$

is the structure morphism. In other words, X_{A_G} is determined by the fact that

$$(\pi_{A_G})_*\mathcal{O}_{X_{A_G}} \cong \mathcal{F}_G(X).$$

4.2 Partial flag varieties

Let $H \subseteq G$ be compact, connected Lie groups of equal rank, let $T \subseteq H$ be a maximal torus, W_G the Weyl group of T in G.

Theorem 4.6. *Assume that $\pi_1(G)$ is torsion free. Then we have a W_G-equivariant epimorphism of schemes over A_T*

$$\varphi\colon (G/H)_{A_T} \longrightarrow A_T \times_{A_G} A_H,$$

inducing an isomorphism of schemes over A_G

$$(G/H)_{A_G} \cong A_H.$$

In the case of ordinary cohomology, the assumption that $\pi_1(G)$ is torsion free is not needed.

Proof of Theorem 4.6.. With the notation as in Example 4.4, we write

$$F := W_G/W_H$$

for the set of vertices in the moment graph. Consider the map

$$\begin{aligned} \overline{\varphi}\colon \coprod_{[w]\in F} A_T &\longrightarrow A_T \times_{A_G} A_H \\ ([w], a) &\longmapsto (a, [w^{-1}a]). \end{aligned}$$

Let W_G act on the source of $\overline{\varphi}$ by

$$v \cdot ([w], a) = ([vw], va),$$

and on the target by its usual action on the first factor. Then $\overline{\varphi}$ is W_G-equivariant. By Corollary 4.3 and Example 4.4, we have a coequalizer diagram

$$\begin{array}{c} \displaystyle\coprod_{[w],\alpha} A_{K_\alpha} \rightrightarrows \coprod_{[w]\in F} A_T \longrightarrow (G/H)_{A_T} \\ ([w], \alpha, a) \longmapsto ([w], a) \\ ([w], \alpha, a) \longmapsto ([s_\alpha w], a) \end{array}$$

where the first coproduct runs over the edges of the moment graph. Both maps on the left are W_G-equivariant with respect to the action

$$v \cdot ([w], \alpha, a) = ([vw], v(\alpha), va)$$

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on their source. The universal property of coequalizer yields a W_G-equivariant map

$$\varphi\colon (G/H)_{A_T} \longrightarrow A_T \times_{A_G} A_H,$$

which is easily seen to be an epimorphism. Note that we have identified $\overline{\varphi}$ with A_i, where i is the inclusion of the T-fixed points F in G/H. To obtain the promised map of schemes over A_G, we quotient by the action of W_G. It remains to prove injectivity of φ/W_G. We have

$$\overline{\varphi}([w], a) = \overline{\varphi}([v], b) \iff a = b \quad \text{and} \quad [w^{-1}a] = [v^{-1}a].$$

In the source of φ, we have made the, *a priori* finer, identifications

$$([w], a) \sim ([s_\alpha w], a) :\iff s_\alpha a = a. \tag{6}$$

Here we have used Lemma 4.5, which is why we need the assumption that $\pi_1(G)$ be torsion free. In many cases (6) is sufficient to imply injectivity of φ, but we will see an example where this fails. Assume now that $[w^{-1}a] = [v^{-1}a]$. Then there is an element $u \in W_H$ with

$$w^{-1}a = uv^{-1}a.$$

In $(G/H)_{A_G}$ we have

$$([w], a) \sim ([1], w^{-1}a) = ([1], uv^{-1}a) \sim ([vu^{-1}], a) = ([v], a).$$

Hence φ/W_G is an isomorphism, as claimed. □

We now ask when the map φ of the theorem is injective.

Lemma 4.7. *Let $w \in W_G$ and let T_c^w be a connected component of the subgroup of w-fixed points in T. Then we can write w as a word*

$$w = s_{\alpha_1} \cdots s_{\alpha_l}$$

in (not necessarily simple) reflections such that

$$T_c^w \subseteq T^{s_{\alpha_1}} \cap \cdots \cap T^{s_{\alpha_l}}.$$

Proof. Choose $t \in T_c^w$ with $T_c^w \subseteq \overline{\langle t \rangle}$. Let $Z_G(t)$ be the centralizer of t in G. This is a connected closed subgroup of full rank. Its Weyl group W_Z may be viewed as a reflection subgroup of W_G. All elements of W_Z fix t and hence T_c^w. Since $w \in W_Z$, we can write w as a word in the reflections generating W_Z. □

The following example shows that the s_{α_j} in Lemma 4.7 can not always be chosen independently of the connected component.

Example 4.8. Let $G = G_2$, and consider the element $w \in W$ acting by $(-)^{-1}$ on T. Then $T^w = T[2]$ has four elements. For each non-trivial element of $t \in T^w$ there is a different, unique pair of reflections $s_{\alpha_t}, s_{\beta_t} \in W$ fixing t. For each such pair, $w = s_{\alpha_t} s_{\beta_t}$.

Corollary 4.9 of Lemma 4.7. *In the situation of the lemma, we have*

$$\mathfrak{t}_{\mathbb{C}}^w = \mathfrak{t}_{\mathbb{C}}^{s_{\alpha_1}} \cap \cdots \cap \mathfrak{t}_{\mathbb{C}}^{s_{\alpha_l}}$$

and

$$(T_{\mathbb{C}}^w)_c \subseteq T_{\mathbb{C}}^{s_{\alpha_1}} \cap \cdots \cap T_{\mathbb{C}}^{s_{\alpha_l}}.$$

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Corollary 4.10. *For cohomology and K-theory, the map φ of Theorem 4.6 is an isomorphism. If the centralizers of commuting pairs in G are connected, then φ is also an isomorphism in elliptic cohomology.*

On global sections, φ gives the familiar isomorphisms

$$H_T \otimes_{H_G} H_H \cong H_T(G/H),$$

studied by Borel, Demazure and others, and

$$R(T) \otimes_{R(G)} R(H) \cong K_T(G/H)$$

[McL79].

Remark 4.11. The condition that the centralizers of commuting pairs be connected, should be compared to [Gro07, 3.2].

Example 4.12. Let $G = U(n)$. Then all the groups T^w are, in fact, connected, so that the word in Lemma 4.7 depends only on w, so that the map φ of the theorem is also an isomorphism in the elliptic case.

5. The Ginzburg–Kapranov–Vasserot characteristic class

5.1 Properties of X_{A_G}

We are now ready to discuss some basic properties of the scheme X_{A_G} introduced in § 4.1. In the cases of cohomology and K-theory, these are well known. In the elliptic case, they were conjectured in [GKV95]. While the elliptic case in its full generality remains conjectural, we will prove the special cases relevant to us.

Change of groups. Let $\phi\colon H \to G$ be a map of groups, and let X be a finite G-CW-complex. Then we have the following commuting square.

$$\begin{tikzcd} X_{A_H} \arrow[r] \arrow[d, "\pi_{A_H}"'] & X_{A_G} \arrow[d, "\pi_{A_G}"] \\ A_H \arrow[r, "A_\phi"'] & A_G \end{tikzcd}$$

We write X_{A_ϕ} for the top map. The assignment $(-)_{A_\phi}$ is natural in X.

Induction axiom. Let $K \lhd G$ be a normal subgroup, and let X be a G-space such that the action of K on X is free. Write $p\colon X \to K\backslash X$ and $\phi\colon G \to G/K$ for the quotient maps. Then we have a commuting square, natural in the space X,

$$\begin{tikzcd} X_{A_G} \arrow[r, equal, "\sim"] \arrow[d] & (K\backslash X)_{A_{G/K}} \arrow[d] \\ A_G \arrow[r, "A_\phi"'] & A_{G/K} \end{tikzcd}$$

where the vertical maps are the respective structure maps, and the top map is $(K\backslash X)_{A_\phi} \circ p_{A_G}$.

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Homogeneous spaces. Let $j\colon H \hookrightarrow G$ be the inclusion of a closed subgroup. Let $\iota\colon \mathrm{pt} \to G/H$ denote the inclusion of the point $1H$. Then we have an isomorphism

$$I_H^G\colon (G/H)_{A_G} \cong A_H,$$

fitting into the following commuting diagram.

$$\begin{tikzcd}
 & (G/H)_{A_H} \arrow[dl, "(G/H)_{A_j}"'] & \\
(G/H)_{A_G} \arrow[rr, equal, "\sim", "I_H^G"'] \arrow[dr, "\pi_{A_G}"'] & & A_H \arrow[ul, "\iota_{A_H}"'] \arrow[dl, "A_j"] \\
 & A_G &
\end{tikzcd}$$

Künneth. For a G-space X and an H-space Y, we have a commuting square, natural in all ingredients,

$$\begin{tikzcd}
(X \times Y)_{A_{G\times H}} \arrow[r, equal, "\sim"] \arrow[d] & X_{A_G} \times Y_{A_H} \arrow[d] \\
A_{G\times H} \arrow[r, equal, "\sim"] & A_G \times A_H
\end{tikzcd}$$

whenever the corners are defined. In the special case that $G = H = T$ is a compact torus, we have an isomorphism over A_T

$$(X \times Y)_{A_T} \cong X_{A_T} \times_{A_T} Y_{A_T}.$$

Each of these properties can be reformulated in terms of the sheaves $\mathcal{F}_G(X)$, where the obvious generalization for pairs can be stated (see [GKV95]). The last property that we need is stated most naturally in terms of the reduced theory.

Odd coefficients. Let ϱ be an odd-dimensional orthogonal representation of G. Then $\widetilde{\mathcal{F}}_G$ vanishes on the corresponding representation sphere

$$\widetilde{\mathcal{F}}_G(\mathbb{S}^\varrho) = \{0\}.$$

For elliptic cohomology, the change of groups property, the Künneth property and the vanishing of the odd coefficients follow immediately from the construction of $\mathcal{E}ll_G$ and the corresponding properties of H_G.

Proposition 5.1. *Assuming the change of groups and Künneth properties, the homogeneous spaces property and the induction axiom are equivalent.*

Proof. It is shown in [GKV95, (1.7.5)] that the induction axiom implies the homogeneous spaces property. The other direction is proved by cellular induction: if K acts freely on the orbit G/H then the composite

$$H \hookrightarrow G \to K\backslash G$$

is still injective, and we have

$$(K\backslash G/H)_{A_\phi} \circ p_{A_G} \circ I_H^G = I_H^{K\backslash G}.$$

Hence $(K\backslash G/H)_{A_\phi} \circ p_{A_G}$ is an isomorphism. ◻

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We saw in (1) that the homogeneous spaces property holds if G is a compact torus. It follows that the induction axiom holds for the inclusion $H \subseteq T$ of any closed subgroup of a compact torus. In particular, $G_{A_T} = (T\backslash G)_{A_1}$, and hence

$$G_{A_G} = \mathrm{spec}(\mathbb{C}). \tag{7}$$

Further, we saw in Theorem 4.6 that the homogeneous spaces property holds if $H \subseteq G$ are compact and connected of equal rank and $\pi_1(G)$ is torsion free.

Proposition 5.2. *Let G and K be compact connected Lie groups. Then the induction axiom holds for the inclusion*

$$K \lhd G \times K.$$

Proof. Write T and T' for the maximal tori of G and K. Using Künneth, (1) and (7), we see that the homogeneous spaces property holds for any inclusion of the form

$$j \times 1\colon H \times \{1\} \longrightarrow T_G \times K.$$

The case $G = T_G$ now follows from the proof of Proposition 5.1. In the general case, we have

$$\begin{aligned} X_{A_{G\times K}} &\cong (X_{A_{T\times T'}})/(W_G \times W_K) \\ &\cong (X_{A_{T\times K}})/W_G \\ &\cong ((K\backslash X)_{A_T})/W_G \\ &\cong (K\backslash X)_{A_G}. \end{aligned}$$

This completes the proof. □

5.2 Classifying maps

Let X be a compact G-manifold, and let

$$\xi\colon P \longrightarrow X$$

be a G-equivariant principal K-bundle on X. We make the convention that both groups act from the left and that the actions commute.

Write $G \ltimes X$ for the translation groupoid with objects X, arrows

$$x \xrightarrow{g} gx$$

and composition given by composition in G. Similarly, we have the translation groupoids $(G \times K) \ltimes P$ and $K \ltimes \mathrm{pt}$.

Definition 5.3. The *classifying map* of ξ is the generalized map of Lie groupoids

$$f_\xi\colon G \ltimes X \xleftarrow{\;\simeq\;} (G \times K) \ltimes P \longrightarrow K \ltimes \mathrm{pt}\,.$$

Definition 5.4. The *universal principal K-bundle* is the K-equivariant principal bundle

$$\xi_{\mathrm{univ}}\colon K \ltimes K \longrightarrow K \ltimes \mathrm{pt},$$

where the two left-actions of K are as follows: as an K-equivariant space, K carries the action of K on itself by left-multiplication. It is a principal K-bundle over the one point space via the action $(k_1, k_2) \mapsto k_2 k_1^{-1}$.

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The nomenclature is justified by the following lemma, which follows directly from the definitions.

Lemma 5.5. *We have an isomorphism*

$$f_\xi^*(\xi_{\mathrm{univ}}) \cong \xi.$$

More can be said here: the assignment

$$\xi \longmapsto f_\xi$$

wants to be an equivalence from the category of principal K-bundles over $G \ltimes X$ to the category of generalized maps (i.e., zig-zags like the one in Definition 5.3), and in fact, the former has been used to define the latter, see [HS87] and [Ler10].

Corollary 5.6. *In the situation of Definition 5.3, assume that G is trivial. Then the Borel construction functor, applied to f_ξ, returns the zig-zag*

$$X \xleftarrow{\simeq} EK \times_K P \longrightarrow BK.$$

Choosing a homotopy inverse to the first map, we obtain the more familiar classifying map from X to the classifying space of K.

Proof. This follows, since Borel construction commutes with pull-backs. □

Example 5.7 (Representations). Let $\varrho\colon G \to U(n)$ be a complex representation of G, and let ξ_n be the universal principal $U(n)$-bundle as in Definition 5.4. Consider the action of G on $U(n)$ by left-multiplication with $\varrho(g)$. This makes ξ_n into a G-equivariant principal $U(n)$-bundle over the one point space. The equivalence

$$G \ltimes \mathrm{pt} \xleftarrow{\simeq} (G \times U(n)) \ltimes U(n)$$

of Definition 5.3 has the quasi-inverse

$$g \longmapsto (g, \varrho(g)) \in \mathrm{Stab}(1).$$

Hence the classifying map f_{ξ_n} is equivalent to

$$\varrho\colon G \ltimes \mathrm{pt} \longrightarrow U(n) \ltimes \mathrm{pt}.$$

Example 5.8 (The splitting principle). Assume that K is a compact connected Lie group and $i\colon T \hookrightarrow K$ the inclusion of its maximal torus. Then ξ may be factored as the composite

$$\xi\colon P \xrightarrow{\zeta} T\backslash P \xrightarrow{q} X,$$

where ζ is the quotient map by the T-action. So, ζ is a principal T-bundle, while the fiber of q is the flag variety $T\backslash K$. Over the total space $T\backslash P$ of this flag bundle, the structure group of ξ can be reduced to T. Let

$$\zeta[K]\colon K \times_T P \longrightarrow T\backslash P$$

be the principal K-bundle obtained from ζ by associating the fiber K. Then

$$q^*(\xi) \cong \zeta[K].$$

This fact is known as the *splitting principle*.

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In terms of classifying maps, the splitting principle amounts to the commutativity of the following diagram.

$$\begin{tikzcd}
f_\zeta\colon G \ltimes (T\backslash P) \arrow[d, "q"'] & (G\times T)\ltimes P \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & T\ltimes \mathrm{pt} \arrow[d, "i"] \\
f_\xi\colon G\ltimes X & (G\times K)\ltimes P \arrow[l, "\simeq"'] \arrow[r] & K\ltimes \mathrm{pt}
\end{tikzcd}$$

Definition 5.9 (Compare [GKV95, (1.8)]). Let $\xi\colon P\to X$ be a G-equivariant principal K-bundle with classifying map f_ξ. Then Proposition 5.2 yields a map of schemes

$$c_\xi\colon X_{A_G} = P_{G\times K} \longrightarrow A_K.$$

We will refer to c_ξ as the *Ginzburg–Kapranov–Vasserot characteristic class* of ξ.

6. Thom sheaves

The idea of the Thom sheaf goes all the way back to Grojnowski. We thank the referee for pointing out that the material of this section has substantial overlap with the discussion of Thom sheaves in [AHS04], see also [And03].

Let $\xi\colon V\to X$ be a G-equivariant complex vector bundle. We will write X^ξ for the Thom space of ξ and $z\colon X_+\hookrightarrow X^\xi$ for the zero section. Applying the reduced theory, we obtain a locally free rank one[5] module sheaf $\widetilde{\mathcal{F}}_G(X^\xi)$ over $\mathcal{F}_G(X)$.

Definition 6.1. The *Thom sheaf* of ξ is the line bundle $\mathbb{L}^\xi_G$ over X_{A_G} characterized (up to isomorphism) by

$$\pi_{A_G *}(\mathbb{L}^\xi_G)^{-1}\cong \widetilde{\mathcal{F}}_G(X^\xi).$$

Note that our convention differs from that in [GKV95, 2.1], where the inverse of $\mathbb{L}^\xi_G$ is referred to as the Thom sheaf. The *Euler map* is the map

$$\eta^\xi_G\colon \mathcal{O}_{X_{A_G}} \longrightarrow \mathbb{L}^\xi_G$$

induced by the zero section $z\colon X\to X^\xi$ (compare [GKV95, (2.6)]). If the group G is understood, we drop it from the notation.

6.1 Properties of the Thom sheaf

The following properties of the Thom sheaf are reformulations of well-known facts about Thom classes in cohomology and K-theory. We deduce the elliptic case, whenever the groups involved have been defined.

Naturality. Let $f\colon X\to Y$ be a G-equivariant map, and let ξ be a complex G-vector bundle over Y. Then we have the following commuting diagram of sheaves over X_{A_G}.

$$\begin{tikzcd}
 & \mathcal{O}_{X_{A_G}} \arrow[dl, "f^*_{A_G}\eta^\xi"'] \arrow[dr, "\eta^{f^*\xi}"] & \\
\mathbb{L}^{f^*\xi}_G \arrow[rr, equal, "\sim"] & & f^*_{A_G}\mathbb{L}^\xi_G
\end{tikzcd}$$

[5] For cohomology and K-theory this is a classical result. In the elliptic case it is an immediate consequence of [Gro07, 2.6] and the W-equivariance of the cohomology Thom isomorphism.

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Proof. The map

$$\widetilde{\mathcal{F}}_G(f^\xi)\colon \widetilde{\mathcal{F}}_G(Y^\xi) \longrightarrow \widetilde{\mathcal{F}}_G(X^{f^*\xi})$$

is a map of $\mathcal{F}_G(Y)$-module sheaves. Hence it corresponds to a map

$$(\mathbb{L}^\xi)^{-1} \longrightarrow f_{A_G *}(\mathbb{L}^{f^*\xi})^{-1},$$

whose adjoint

$$f^*_{A_G}(\mathbb{L}^\xi)^{-1} \longrightarrow (\mathbb{L}^{f^*\xi})^{-1}$$

is an isomorphism. In the elliptic case, the last statement follows from [Gro07, 2.6] and the W-equivariance of the Thom isomorphism in cohomology. □

Change of groups. Let $\phi\colon H \to G$ be a map of Lie groups, and let $\xi\colon V \to X$ be a G-equivariant complex vector bundle. Then we have the following commuting diagram of sheaves over X_{A_H}.

$$\begin{tikzcd} & \mathcal{O}_{X_{A_H}} \arrow[dl, "\eta^\xi_H"'] \arrow[dr, "A^*_\phi \eta^\xi_G"] & \\ \mathbb{L}^\xi_H \arrow[rr, equal, "\sim"] & & X^*_{A_\phi}\mathbb{L}^\xi_G \end{tikzcd}$$

Proof. Consider the map of locally free rank one $\mathcal{F}_H(X)$ module sheaves

$$\mathcal{F}_H(X) \otimes_{A^*_\phi \mathcal{F}_G(X)} A^*_\phi \widetilde{\mathcal{F}}_G(X^\xi) \longrightarrow \widetilde{\mathcal{F}}_H(X^\xi).$$

For $[a] \in A_H$, we have an equality of fixed point sets

$$X^{T(a)} = X^{T(\phi(a))}.$$

Hence [Gro07, 2.6] implies that the above map is an isomorphism at the stalk $[a]$. □

Induction. Assume that the induction axiom holds for the inclusion of a normal subgroup $K \lhd G$ and that X is a G-complex on which the action of K is free. Then the change of groups isomorphism $X_{A_G} \cong (K\backslash X)_{A_K}$ is covered by an isomorphism of line bundles identifying $\mathbb{L}^\xi_G$ with $\mathbb{L}^{K\backslash\xi}_{G/K}$ and η^ξ_G with $\eta^{K\backslash\xi}_{G/K}$.

Multiplicativity. Given equivariant complex vector bundles ξ over a G-space X and ζ over an H-space Y, the Künneth isomorphism $(X\times Y)_{A_{G\times H}} \cong X_{A_G} \times Y_{A_H}$ is covered by an isomorphism of line bundles identifying $\mathbb{L}^{\xi\oplus\zeta}_{G\times H}$ with the external tensor product $\mathbb{L}^\xi_G \otimes \mathbb{L}^\zeta_H$ and $\eta^{\xi\oplus\zeta}_{G\times H}$ with $\eta^\xi_G \otimes \eta^\zeta_H$. In the special case where $G = H$, we get the following commuting square of sheaves over A_G.

$$\begin{tikzcd} & \mathcal{O}_{X_{A_G}} \arrow[dl, "\eta^{\xi\oplus\zeta}_G"'] \arrow[dr, "\eta^\xi_G \otimes \eta^\zeta_G"] & \\ \mathbb{L}^{\xi\oplus\zeta}_G \arrow[rr, equal, "\sim"] & & \mathbb{L}^\xi_G \otimes_{\mathcal{O}_{A_G}} \mathbb{L}^\zeta_G \end{tikzcd}$$

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Proof. The induction and (external) multiplicativity properties follow directly from the induction axiom and the Künneth property of $\mathcal{F}_G$ in their formulation for pairs. The internal Künneth for $\mathbb{L}_T^{\xi\oplus\zeta}$ follows from the external Künneth and the change of groups property. $\square$

Universal bundles. The Thom sheaf of the universal complex line bundle ξ_1 over pt (compare Definition 5.4) is the line bundle

$$\mathbb{L}_{U(1)}^{\xi_1} \cong \mathcal{L}(0)$$

of the divisor (0) on $A_{U(1)}$. Recall that $A_{U(1)}$ equals $\mathbb{C}$ or $\mathbb{C}^\times$, in which case we replace 0 by 1, or $\mathbb{C}/\langle\tau, 1\rangle$. The Euler map of ξ_1 is the canonical inclusion

$$\mathcal{O}_{A_{U(1)}} \longrightarrow \mathcal{L}(0).$$

Consider the universal complex n-vector bundle ξ_n over pt. Let T be the maximal torus of $U(n)$. Then

$$\mathbb{L}_T^{\xi_n} \cong \bigotimes_{i=1}^{n} p_i^*\mathcal{L}(0)$$

where

$$p_i\colon A_{U(1)}^n \longrightarrow A_{U(1)}$$

is the projection to the ith factor. This is the line bundle associated to the divisor

$$\sum_{t=i}^{n} \ker(p_i),$$

and the Euler map $\eta_T(\xi)$ is its canonical inclusion of $\mathcal{O}_{A_{U(1)}^n}$ inside it. The $U(n)$-equivariant Thom sheaf and Euler map are obtained by taking the S_n-invariant parts of $\mathbb{L}_T$ and η_T. We will write η_n for the nth universal Euler map.

The following result, which determines all Thom sheaves up to isomorphism, follows immediately from the list of properties above.

Theorem 6.2. *Let $\xi\colon V \to X$ be a G-equivariant vector bundle, and let c_ξ be its Ginzburg–Kapranov–Vasserot characteristic class (c.f. Definition 5.9). Then we have the following commuting square.*

$$\begin{tikzcd}
\mathcal{O}_{X_{A_G}} \arrow[r, equal] \arrow[d, "\eta(\xi)"'] & {c_\xi}^*\mathcal{O}_{A_{U(1)}^n/S_n} \arrow[d, "c_\xi^*(\eta_n)"] \\
\mathbb{L}_G^\xi \arrow[r, equal, "\sim"] & {c_\xi}^*\mathbb{L}^{\xi_n}
\end{tikzcd}$$

Example 6.3. Let $\lambda \neq 0$ be a weight of T, and let $j\colon K_\lambda \hookrightarrow T$ be the kernel of $e^{2\pi i\lambda}$. Consider the T-equivariant line bundle $\xi_\lambda\colon \mathbb{C}_\lambda \to \mathrm{pt}$. By Example 5.7, we have a short exact sequence

$$A_{K_\lambda} \xrightarrow{A_j} A_T \xrightarrow{c_{\xi_\lambda}} A.$$

Hence the Thom sheaf of ξ_λ is

$$\mathbb{L}^{\xi_\lambda} \cong \mathcal{L}(A_{K_\lambda}),$$

the line bundle on A_T associated to the divisor A_j.

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Example 6.4. Let $\varrho\colon G \to U(n)$ be a complex representation, viewed as a G-equivariant complex vector bundle over the one point space. By Example 5.7, we have an isomorphism

$$\mathbb{L}_G^{\varrho} \cong A_{\varrho}^{*}\mathbb{L}_{U(n)}^{\xi_n}$$

of sheaves over A_G.

Example 6.5. If $\varrho\colon G \to U(1)$ is the one-dimensional trivial representation, then A_ϱ factors through the inclusion of zero $i_0\colon A_1 \to A_{U(1)}$. Hence

$$\widetilde{F}_1(\mathbb{S}^2) \cong i_0^{*}\mathcal{I}(0)$$

is identified with the sheaf of invariant differentials on $A_{U(1)}$,

$$\omega := \mathcal{I}(0)/\mathcal{I}(0)^2|_0.$$

Writing ω also for its pull-back to A_G, we obtain

$$\widetilde{F}_G(\mathbb{S}^{2n}) \cong \omega^{\otimes n}.$$

This shows that the periodicity axiom (1.5.5) in [GKV95] is, in part, redundant. We will come back to this in § 6.2.

Example 6.6. Let

$$\chi_\varrho = e^{2\pi i\lambda_1} + \cdots + e^{2\pi i\lambda_n}$$

be the character of ϱ, with $\lambda_k \in \Lambda\backslash\{0\}$ for all k. By Example 6.3, we may identify the G-equivariant Thom sheaf $\mathbb{L}_G^{\varrho}$ over A_T/W with the sheaf of S_n-invariant sections of

$$\bigotimes_{i=0}^{n} \mathcal{L}(A_{K_{\lambda_i}}).$$

Corollary 6.7. *Let $\varrho\colon T \to U(n)$ be a complex representation of T with character*

$$\chi_\varrho = e^{2\pi i\lambda_1} + \cdots + e^{2\pi i\lambda n},$$

$\lambda_i \neq 0$, and write S_ϱ^{2n-1} for its unit sphere inside $\mathbb{C}^n$. Then we have an isomorphism

$$(\mathbb{S}_\varrho^{2n-1})_{A_T} \cong \bigcap_{i=1}^{n} A_{K_{\lambda_i}},$$

where the right-hand side stands for the scheme theoretic intersection over A_T.

Proof. We have a cofiber sequence

$$(\mathbb{S}_\varrho^{2n-1})_+ \longrightarrow \mathbb{S}^0 \longrightarrow \mathbb{S}^\varrho,$$

whose second map is the zero section z in the definition of η_T^{ϱ}. Applying $\widetilde{F}_T$, we obtain the short exact sequence

$$\bigotimes_{i=1}^{n} \mathcal{I}(A_{K_{\lambda_i}}) \longrightarrow \mathcal{O}_{A_T} \longrightarrow \mathcal{F}_T(\mathbb{S}_\varrho^{2n-1}),$$

where the first map is the canonical inclusion. □

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Example 6.8. Consider the representation

$$k\xi_1 := \xi_1 \oplus \cdots \oplus \xi_1$$

of $U(1)$. Then

$$(\mathbb{S}^{2k-1}_{k\xi_1})_{A_{U(1)}} = (0)^{[k]}$$

is the kth infinitesimal neighbourhood of 0 inside $A_{U(1)}$.

We are now in a position to give the promised proof of the completion theorem from the formal properties of X_{A_T}.

Proof of Theorem 3.1. We follow the outline in [GKV95, (1.7.2)]. Writing T as the product of r copies of $U(1)$, we may build ET from the equivariant skeleta

$$ET^{(2k-1)} := \mathbb{S}^{2k-1}_{k\xi_1} \times \cdots \times \mathbb{S}^{2k-1}_{k\xi_1},$$

where the notation is as in the last example. The induction axiom gives the following commuting diagram.

$$\begin{tikzcd}
(ET^{(2k-1)} \times_T X)_{A_1} \arrow[r, equal, "\sim"] \arrow[d] & (ET^{(2k-1)} \times X)_{A_T} \arrow[d] \\
A_1 & A_T \arrow[l]
\end{tikzcd}$$

A combination of the internal and external Künneth properties, together with the last example, yields an isomorphism of schemes over A_T

$$(ET^{(2k-1)} \times X)_{A_T} \cong \left(\prod_{j=1}^{r} (0)^{[k]} \right) \times_{A_T} X_{A_T}.$$

Letting k vary, the right-hand side becomes an ind-scheme over A_1, isomorphic to the formal completion $(X_{A_T})\widehat{_0}$.

6.2 RO(G)-grading and periodicity

We are now ready to define the full theory $\mathcal{E}ll^*_G(-)$, graded by the set of orthogonal representations contained in an indexing universe and their formal differences (see [May96, p. 154] and [Lur]). For any such universe, there is a cofinal system of representations of the form

$$\varrho\colon G \longrightarrow U(n) \longrightarrow O(2n).$$

Hence it suffices to define the groups

$$\widetilde{\mathcal{E}ll}^{\varrho-\sigma}_G(X) := \mathbb{L}^\varrho \otimes_{\mathcal{O}_{\mathcal{M}_G}} \widetilde{\mathcal{E}ll}^0_G(\mathbb{S}^\sigma \wedge X),$$

where ϱ is as above and σ is in our universe. The resulting theory satisfies the axioms of a sheaf-valued RO(G)-graded cohomology theory: $\mathcal{E}ll_G$ is a contravariant functor of X and σ and a covariant functor of ϱ. Each $\mathcal{E}ll^{\varrho-\sigma}_G(-)$ is exact on cofiber sequences and sends wedges to products. There are suspension isomorphisms

$$s^\varrho\colon \widetilde{\mathcal{E}ll}^{*+\varrho}_G(\mathbb{S}^\varrho \wedge X) \xrightarrow{\cong} \widetilde{\mathcal{E}ll}^*_G(X),$$

natural in X and the orthogonal representation ϱ, and satisfying

$$s^{\varrho\oplus\sigma} = s^\sigma \circ s^\varrho.$$

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Remark 6.9. These axioms are immediate from the definitions. Note that exactness is checked on stalks, and we cannot expect sections over an open $\Gamma(U, \mathcal{E}ll^*_G(-))$ to be exact. In other words, a sheaf-valued cohomology theory is *not* the same thing as a sheaf of cohomology theories. The latter has an extensive literature beyond [Lur], for example the work of Hopins *et al.* on topological modular forms, but it will not play a role here.

The theory of Thom sheaves is extended to virtual equivariant complex vector bundles, and we set

$$\mathcal{E}ll^{*+\xi}_G(X) := \widetilde{\mathcal{E}ll}^*_G(X^{-\xi}).$$

Finally, we have periodicity isomorphisms

$$\begin{aligned} \widetilde{\mathcal{E}ll}^*_G(\mathbb{S}^\varrho \wedge X) &\cong \mathbb{L}^{-\varrho} \otimes_{\mathcal{E}ll_G} \widetilde{\mathcal{E}ll}^*_G(X) \\ &\cong \widetilde{\mathcal{E}ll}^{*-\varrho}_G \otimes_{\mathcal{E}ll^*_G} \widetilde{\mathcal{E}ll}^*_G(X) \end{aligned}$$

for complex representations ϱ.

6.3 The Looijenga line bundle and twisted coefficients

I learned a lot of the material in this section from Matthew Ando. Let G be a simple and simply connected compact Lie group, and let ϱ be a representation of G. In [And03], Ando defines equivariant elliptic cohomology with twisted coefficients, where the twist comes from a degree four characteristic class of G. We recall some facts about such classes, see [And03, § 5.2] and [KN97, KNR94]. For a lattice Λ with dual $\Lambda^\vee$, we consider the group of homogeneous polynomials of degree 2 in Λ,

$$S^2_{\mathbb{Z}}(\Lambda) = (\Lambda \otimes \Lambda)/S_2$$

(second symmetric power) and its dual, the group of symmetric bilinear forms on Λ,

$$B(\Lambda, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(S^2_{\mathbb{Z}}(\Lambda), \mathbb{Z}) = (\Lambda^\vee \otimes \Lambda^\vee)^{S_2},$$

(second divided power of $\Lambda^\vee$). The latter is the target of the universal quadratic map out of $\Lambda^\vee$,

$$\begin{aligned} \gamma_2\colon \Lambda^\vee &\longrightarrow (\Lambda^\vee \otimes \Lambda^\vee)^{S_2} \\ x &\longmapsto x \otimes x \end{aligned}$$

(see [And03, 5.2]). Hence its dual is canonically identified with the group of quadratic forms on $\Lambda^\vee$. To be specific, the isomorphism

$$S^2_{\mathbb{Z}}(\Lambda) \longrightarrow Q(\Lambda^\vee, \mathbb{Z}) \tag{8}$$

sends the monomial $\lambda\mu$ to the quadratic form

$$x \longmapsto \lambda(x)\mu(x).$$

Lemma 6.10. *The shuffle product*

$$\begin{aligned} s\colon S^2_{\mathbb{Z}}(\Lambda) &\longrightarrow B(\Lambda^\vee, \mathbb{Z}) \\ \lambda\mu &\longmapsto \lambda \otimes \mu + \mu \otimes \lambda \end{aligned}$$

is injective. Its image consists of the bilinear forms satisfying

$$I(x, x) \in 2\mathbb{Z} \tag{9}$$

for all $x \in \Lambda^\vee$.

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Proof. The isomorphism (8) identifies s with the map

$$Q(\Lambda^\vee, \mathbb{Z}) \longrightarrow B(\Lambda^\vee, \mathbb{Z})$$

sending the quadratic form ϕ to the bilinear form

$$I(x, y) = \phi(x + y) - \phi(x) - \phi(y). \qquad \square$$

If T is a torus with weight lattice Λ, then we have a canonical isomorphism

$$H^*(BT; \mathbb{Z}) = S^*_{\mathbb{Z}}(\Lambda).$$

So, elements of $H^4(BT; \mathbb{Z})$ may be viewed as bilinear forms on $\Lambda^\vee$ satisfying (9). Now let G be a simple and simply connected compact Lie group with highest root $\widetilde{\alpha}$. Let

$$i_{\widetilde{\alpha}} \colon \mathrm{SU}_2(\widetilde{\alpha}) \hookrightarrow G$$

be the inclusion of the copy of SU(2) inside G corresponding to $\widetilde{\alpha}$. Then $i_{\widetilde{\alpha}}$ generates[6]

$$\pi_3(G) \cong \mathbb{Z},$$

and this is the first non-trivial homotopy group of G. As a consequence, one obtains a canonical isomorphism $H^3(G; \mathbb{Z}) = \mathbb{Z}$, and an isomorphism

$$\tau \colon H^3(G; \mathbb{Z}) \xrightarrow{\cong} H^4(BG; \mathbb{Z})$$

(transgression in the Leray–Serre spectral sequence for EG). Further, one has an integral *Chevalley Restriction Theorem* in degree four.

LEMMA 6.11. *For a simple and simply connected compact Lie group G, the restriction map*

$$r \colon H^4(BG; \mathbb{Z}) \longrightarrow H^4(BT; \mathbb{Z})^W$$

is an isomorphism.

Proof. This follows from the proof of Proposition 29.2(b) in [Bor53] combined with the fact that G/T is without torsion [Bot54]. Alternatively, establish that the target of r is infinite cyclic (e.g., by the real Chevalley restriction theorem), and use the commuting square

$$\begin{tikzcd}
H^4(BG; \mathbb{Z}) \arrow[r, "r"] \arrow[d, equal, "i^*_{\widetilde{\alpha}}"'] & H^4(BT; \mathbb{Z})^W \arrow[d, equal, "(\widetilde{\alpha}^\vee)^*"'] \\
H^4(B\,\mathrm{SU}(2); \mathbb{Z}) \arrow[r, "r"] & H^4(B\mathbb{S}^1; \mathbb{Z})^{S_2}
\end{tikzcd}$$

to reduce to the case of SU(2). There, the claim holds by [Bor53, Proposition 29.2(b)]; in fact, r is just the pull-back along the $\mathbb{S}^2$-fibration

$$\mathbb{H}P^\infty \longrightarrow \mathbb{C}P^\infty.$$

Here $\widetilde{\alpha}^\vee$ is the short coroot dual to $\widetilde{\alpha}$, and we have used that the smallest positive definite element $I_{\min}$ of $B(\Lambda^\vee, \mathbb{Z})$ is characterized by $I_{\min}(\widetilde{\alpha}^\vee, \widetilde{\alpha}^\vee) = 2$, see for instance [Loo77]. $\square$

[6] The isomorphism $\pi_3(G) \cong \mathbb{Z}$ is due to Borel for the classical groups and a famous application of Bott–Morse theory in the general case [Bot54]. A detailed exposition of this result can be found in [MT91]. The fact that $i_{\widetilde{\alpha}}$ is a generator is [BS58, Proposition 10.2.(A)].

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The inverse of $r \circ \tau$ sends the invariant bilinear form

$$I \in B(\Lambda^\vee, \mathbb{Z})^W \subset S^2(\mathfrak{t})^W \cong S^2(\mathfrak{g})^{ad},$$

to the Cartan 3-form

$$\omega_I(x, y, z) = I([x, y], z).$$

Indeed, ω_I defines an element of

$$H^3(G; \mathbb{Z}) \subseteq H^3_{dR}(G; \mathbb{R}) = H^3(\mathfrak{g})$$

if and only if I satisfies (9) [PS86, Proposition 4.4.5].

Example 6.12. If the representation ϱ has character

$$e^{2\pi i\lambda_1} + \cdots + e^{2\pi i\lambda_n}$$

then the first Pontryagin class of ϱ is

$$p_1(\varrho) = \sum_{k=1}^{n} \lambda_k^2.$$

This corresponds to the bilinear form

$$2 \sum_{k=1}^{n} \lambda_k \otimes \lambda_k$$

on $\Lambda^\vee$. If G is simply connected, then ϱ admits a spin structure. In that case, one half of this form still satisfies (9). The corresponding cohomology class of BG is called $(p_1/2)(\varrho)$.

Example 6.13. Let ϱ_{ad} be the adjoint representation of a simply connected group G. Then the class $p_1(\varrho_{ad})$ corresponds to the canonical form

$$B = \sum_{\alpha \in \mathcal{R}} \alpha \otimes \alpha,$$

and $(p_1/2)(\varrho_{ad})$ corresponds to

$$\frac{1}{2} B = \sum_{\alpha \in \mathcal{R}_+} \alpha \otimes \alpha$$

(sum over the positive roots).

Definition 6.14 ([Dyn52, § 2] or [KN97, (4.1)]). The *Dynkin index* of a homomorphism

$$\varphi \colon G \longrightarrow H$$

between simple and simply connected compact Lie groups is the degree of the map

$$\mathbb{Z} = \pi_3(G) \xrightarrow{\varphi_*} \pi_3(H) = \mathbb{Z}.$$

Equivalently, the Dynkin index of φ may be defined as the degree of the map

$$\varphi^* \colon H^4(BH; \mathbb{Z}) \longrightarrow H^4(BG; \mathbb{Z})$$

or a ratio of bilinear forms.

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Definition 6.15. Let G be a simple and simply connected compact Lie group. Then the *dual Coxeter number* of G is the integer

$$g := \frac{p_1}{2}(\varrho_{ad}) = \frac{-c_2(\varrho_{ad}^{\mathbb{C}})}{2}$$

in $H^4(BG;\mathbb{Z}) = \mathbb{Z}$.

In other words, g is defined by the identity

$$2gI_{\min} = B,$$

where $I_{\min}$ is the minimal positive definite and W-invariant bilinear form on $\Lambda^\vee$ satisfying (9). The Dynkin index of

$$\varrho_{ad}^{\mathbb{C}} \colon G \longrightarrow \mathrm{SU}(\mathfrak{g}_{\mathbb{C}})$$

equals $2g$ (see [KN97]), and for $\dim(G) \geqslant 5$, the Dynkin index of

$$\varrho_{ad} \colon G \longrightarrow \mathrm{Spin}(\mathfrak{g})$$

equals g. The last statement follows from the fact that $p_1/2$ generates $H^4(B\mathrm{Spin}(n);\mathbb{Z})$ for $n \geqslant 5$, see [McL92, Lemma 2.2].

Example 6.16. The dual Coxeter number of Spin(3) equals 2.

Definition 6.17. Let I be a positive definite symmetric bilinear form on $\Lambda^\vee$, and assume that I satisfies (9). Then the *Looijenga line bundle* associated to I is the invertible sheaf $\mathcal{L}_I$ on $\mathcal{M}_T^h$ with sections

$$\mathcal{L}_I(U) = \{f \in \Gamma\mathcal{O}^h_{y^{-1}U} \mid f(q^x z) = q^{-\frac{1}{2}I(x,x)} z^{-I^\sharp(x)} f(z)\}.$$

Here $x \in \Lambda^\vee$, and q^x stands for the image of τx under $\exp\colon \mathfrak{t}_{\mathbb{C}} \to T_{\mathbb{C}}$, while

$$y\colon T_{\mathbb{C}} \to \mathcal{M}_T$$

is the quotient map. So, if $T = U(1)$, then $q^x = e^{2\pi i \tau x}$.

A generalization of Definition 6.17 can be found in an older, unpublished version of [And03], where Ando credits Hopkins and the referee: let E be an elliptic curve, and consider the following diagram.

$$\begin{tikzcd}
& B(\Lambda^\vee,\mathbb{Z}) \arrow[dr, dashed, "\exists!\,\mathcal{L}"] & \\
\Lambda \arrow[ur, "\gamma_2"] \arrow[rr, "\lambda\mapsto(\lambda\otimes\mathrm{id})^*[\mathcal{L}(0)]"'] & & \mathrm{Pic}(\Lambda^\vee\otimes E)
\end{tikzcd}$$

The theorem of the cube implies that the bottom map is quadratic. Hence the dotted map exists by the universal property of γ_2. Note that this generalized construction only determines $[\mathcal{L}_I]$ up to isomorphism.

Ando has used the constructions described in this section to consider equivariant elliptic cohomology with twisted coefficients, where the twist comes from an element of $H^4(BG;\mathbb{Z})$.

7. Euler classes

Since the elliptic Thom sheaves are in general non-trivial, the notion of Euler class does not have an immediate generalization to elliptic cohomology, and different authors make different choices on this matter, see [And03, 5.3], [GKV95, (2.6)] and [Roş01, p. 10].

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7.1 Thom isomorphisms

Let ξ_1 be the universal complex line bundle of Definition 5.4, and recall that its Thom sheaf is the invertible sheaf

$$\mathbb{L}_{U(1)}^{\xi_1} \cong \mathcal{L}(0)$$

over $A_{U(1)}$. For the additive or multiplicative group this divisor is principal:

$$\begin{aligned}(0) &= \operatorname{div}(x) \quad \text{on } \mathbb{C} \text{ and}\\ (1) &= \operatorname{div}(1-z) \quad \text{on } \mathbb{C}^\times.\end{aligned}$$

The *universal Euler classes* in cohomology and K-theory are the functions

$$e(\xi_1) = \begin{cases} x & \text{on } \mathbb{C} \text{ and}\\ (1-z) & \text{on } \mathbb{C}^\times,\end{cases}$$

and the *universal Thom isomorphisms* are

$$\begin{aligned}\vartheta\colon \mathcal{O}_{A_{U(1)}} &\xrightarrow{\cong} \mathcal{L}(0)\\ f &\longmapsto \frac{f}{e(\xi_1)}\end{aligned}$$

(replace (0) by (1) for K-theory). As a consequence, all Thom sheaves in these theories are trivialized, and the theories possess Chern classes for complex vector bundles.

We will be particularly interested in the case where the complex vector bundle $\xi\colon V \to X$ comes equipped with a spin structure on the underlying real bundle.[7]

The Ginzburg–Kapranov–Vasserot characteristic class of a $U^{[2]}$ bundle ξ factors as

$$\begin{tikzcd} & A_{U^{[2]}(n)} \arrow[dr, "v"] & \\ X_{A_G} \arrow[ur, dashed, "\widetilde{c}_\xi"] \arrow[rr, "c_\xi"'] & & A_{U(n)} \end{tikzcd}$$

where $A_{U^{[2]}(n)}$ is the pull-back in the following cartesian square.

$$\begin{tikzcd} A_{U^{[2]}(n)} \arrow[r, "u"] \arrow[d, "v"'] & A_{U(1)} \arrow[d, "\cdot 2"] \\ A_{U(n)} \arrow[r, "A_{\det}"'] & A_{U(1)} \end{tikzcd}$$

Example 7.1. In the multiplicative case, a point in $A_{U^{[2]}(n)}$ consists of

$$(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n / S_n$$

together with a choice of square root

$$(z_1 \cdots z_n)^{\frac{1}{2}}.$$

---

[7] The existence (and choice) of such a spin structure is equivalent to the existence (and choice) of a square root of the determinant bundle

$$\det(V) = \Lambda^{\mathrm{top}}(V).$$

Manifolds with this kind of structure or, more generally, with a choice of Nth root of $\det(-\tau_M)$, played an important role in the early constructions of elliptic cohomology, based on Hirzebruch's level N-genera, see [Lan88, Fra92, HBJ92]. Their cobordism ring is introduced in [Hoe91], where Höhn uses the terminology *U-manifold with N-structure*. We will refer to U-bundles with 2-structure as *$U^{[2]}$-bundles*.

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Definition 7.2. We write

$$\mathbb{L}^{\xi_n}_{U^{[2]}(n)} := v^*\mathbb{L}^{\xi_n}_{U(n)}$$

for the universal Thom sheaf for n-dimensional complex bundles with spin structure.

In K-theory $\mathbb{L}^{\xi_n}_{U^{[2]}(n)}$ is the target of the *Atiyah–Bott–Shapiro Thom isomorphism*

$$\begin{aligned} \vartheta_{ABS}\colon \mathcal{O}_{A_{U^{[2]}(n)}} &\xrightarrow{\cong} \mathbb{L}^{\xi_n}_{U^{[2]}(n)} \\ f &\longmapsto \frac{f}{e'(\xi_n)}, \end{aligned}$$

with

$$e'(\xi_n) = (z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}}) \cdots (z_n^{\frac{1}{2}} - z_n^{-\frac{1}{2}}).$$

7.2 Theta functions and elliptic Euler classes

On the elliptic curve $E = \mathbb{C}/2\pi i\langle\tau, 1\rangle$ the divisor (0) is no longer principal. In this case, the Thom isomorphisms above are replaced by the theta function formalism.

Definition 7.3. We write

$$\Theta_\varrho = \Gamma\mathcal{L}_{(p_1/2)(\varrho)}$$

for the global sections of the Looijenga line bundle $\mathcal{L}_{(p_1/2)(\varrho)}$ and refer to elements of Θ_ϱ as *Looijenga theta functions (of level $(p_1/2)(\varrho)$).*

Definition 7.4. Let $\varrho\colon G \to U^{[2]}(n)$ be a representation with character

$$e^{2\pi i\lambda_1} + \cdots + e^{2\pi i\lambda_n}.$$

Then the *elliptic Euler class* of ϱ is the function on $T_{\mathbb{C}}$ defined by

$$e_{\mathrm{ell}}(\varrho) := (-1)^n \prod_{i=1}^{n} \sigma(q, z^{\lambda_i}),$$

where

$$\sigma(q, z) = (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \prod_{n\geqslant 1} \frac{(1-q^n z)(1-q^n z^{-1})}{(1-q^n)^2}$$

is the Weierstrass sigma function, and for $z = \exp(x) \in T_{\mathbb{C}}$ and $\lambda \in \Lambda$ we are using the notation

$$z^\lambda := e^{2\pi i\lambda(x)}.$$

In elliptic cohomology, the role of the Thom isomorphism is replaced by the isomorphism of line bundles over $\mathcal{M}_G$

$$\begin{aligned} \vartheta\colon \mathcal{L}^W_{(p_1/2)(\varrho)} &\xrightarrow{\cong} \mathbb{L}^{\varrho}_G \\ f &\longmapsto \frac{f}{e_{\mathrm{ell}}(\varrho)}. \end{aligned}$$

Presumably, these notions generalize to yield a theta function description for the elliptic Thom sheaf of any equivariant $U^{[2]}(n)$-bundle over a nice enough base (for instance, an equivariantly formal space). We do not pursue this here.

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The discussion of this section can be summarized as follows: for simple and simply connected G, we have a commuting diagram

$$\begin{tikzcd}
R(G) \arrow[rr, "\mathbb{L}"] \arrow[d] & & \mathcal{P}ic(\mathcal{M}_G^h) \arrow[d] \\
\mathrm{RO}(G) \arrow[rr, "\widetilde{\mathcal{E}ll}_G(\mathbb{S}^{(-)})"] \arrow[dr, "p_1/2"'] & & \mathcal{P}ic_W(\mathcal{M}_T^h) \\
& H^4(BG;\mathbb{Z}) \arrow[ur, "\mathcal{L}"'] &
\end{tikzcd}$$

where $\mathcal{L}$ is the Ando–Looijenga construction (see Definition 6.17). It follows that, for such G, the $\mathrm{RO}(G)$-graded coefficients are contained in Ando's picture of twisted coefficients. This diagram, together with [BK05, Theorem 2.2], [BK05, § 3] and [KN97] also suggests the following conjecture.

Conjecture 7.5. *If G is simple and simply connected, the Thom sheaf construction*

$$\mathbb{L}\colon R(G) \longrightarrow \mathcal{P}ic(\mathcal{M}_G^h)$$

agrees with the Kumar–Narasimhan–Ramanathan map, the protagonist of [BK05, KN97, KNR94]. In particular, the map $\mathbb{L}$ is surjective, sections of the line bundle $\mathbb{L}(V)$ have an interpretation as conformal blocks, and the W-invariant sections of $\mathcal{L}_{\mathrm{Lo}}^{2g}$ form the dualizing sheaf $\omega_{\mathcal{M}_G}$.

We will come back to this at a different occasion.

7.3 The ring $\widetilde{Th}_*^W$

Let G be a simple and simply connected compact Lie group with weight lattice Λ and Weyl group W. In [Loo77], Looijenga considers two graded rings (see also [And00]).

(1) For a fixed complex elliptic curve $E = \mathbb{C}/\langle\tau, 1\rangle$, the ring of Looijenga theta functions is

$$\Theta_* = \bigoplus_{k\geqslant 0} \Gamma\mathcal{L}_{\mathrm{Lo}}^k,$$

where

$$\mathcal{L}_{\mathrm{Lo}} = \mathcal{L}(I_{\min})$$

is the bundle denoted $\mathcal{L}^{-1}$ in [Loo77]. Elements of the kth summand Θ_k are referred to as elements of level k. The ring Θ_* is denoted $S(E)$ in [Loo77].

(2) The ring of *formal Looijenga theta series* is the graded ring

$$\widetilde{Th}_* \subseteq \mathbb{Z}[\widehat{T}](\!(q)\!),$$

whose elements of level $k \geqslant 0$ are formal Laurent series

$$\theta = \sum_{n\geqslant m} \sum_{\lambda\in\Lambda} c_{\lambda,n}\, e^{\lambda} q^n$$

satisfying, for all $x \in \Lambda^\vee$,

$$x^*\theta = q^{-(k/2)I(x,x)} e^{-kI^\sharp(x)}\theta.$$

Here $I = I_{\min}$ and $x^*\theta$ stands for θ with e^λ replaced by $e^\lambda q^{\lambda(x)}$.

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Looijenga studies the W-invariant sections of Θ_* via a graded map

$$\widetilde{Th}_* \longrightarrow \Theta_*,$$

[Loo77, § 4]. He gives an explicit basis of $\widetilde{Th}_*$ as a free $\mathbb{Z}((q))$-module ([Loo77, (2.5)] see Definition 8.5 below) and proves that $\widetilde{Th}_*^W$ is a polynomial algebra over $\mathbb{Z}((q))$. He then deduces the analogous results for Θ_*, where $\mathbb{Z}((q))$ is replaced by $\mathbb{C}$. One could hope to interpret homogeneous elements of $\widetilde{Th}_*$ as sections of Looijenga bundles $[\mathcal{L}(kI_{\min})]$ for the Tate curve $\mathrm{Tate}(q)$ over $\mathbb{Z}((q))$, but we will not pursue this here.

Instead, let $\mathbb{D}^\times$ be the punctured open disk

$$\mathbb{D}^\times = \{z \in \mathbb{C} \mid 0 < |z| < 1\}.$$

Over $\mathbb{D}^\times$, we have the *analytic Tate curve*

$$C_{\mathrm{Tate}} = (\mathbb{C}^\times \times \mathbb{D}^\times)/(z, q) \sim (zq, q),$$

whose fiber at q is

$$E_q = \mathbb{C}^\times/q^{\mathbb{Z}},$$

and we have the family $\mathfrak{M}_G(C_{\mathrm{Tate}})$ whose fiber over q is the moduli space $\mathfrak{M}_G(E_q)$ of holomorphic principal $G_{\mathbb{C}}$-bundles over E_q. Let $LG_{\mathbb{C}}$ be the group of holomorphic loops

$$\gamma\colon \mathbb{C}^\times \longrightarrow G_{\mathbb{C}}$$

and let $LG_{\mathbb{C}} \rtimes \mathbb{C}^\times_{\mathrm{rot}}$ be its rotation extension. Write $\sim$ for the conjugation relation.

Theorem 7.6 (Looijenga, unpublished). *Assume that $G_{\mathbb{C}}$ is connected. Then there is an embedding*

$$\mathfrak{M}_G(C_{\mathrm{Tate}}) \hookrightarrow (LG_{\mathbb{C}} \rtimes \mathbb{C}^\times_{\mathrm{rot}})/\sim$$

whose image consists of the conjugacy classes $[(\gamma, q)]$ with $q \in \mathbb{D}^\times$.

Sketch of proof (following [BG96, Proposition 1.3]). Fix $q \in \mathbb{D}^\times$. Any holomorphic principal bundle over $\mathbb{C}^\times/q^{\mathbb{Z}}$ becomes trivial over $\mathbb{C}^\times$. Hence it is isomorphic to a quotient

$$P_\gamma = (\mathbb{C}^\times \times G_{\mathbb{C}})/(z, g) \sim (qz, \gamma(z)g),$$

with holomorphic *'multiplier'*

$$\gamma\colon \mathbb{C}^\times \longrightarrow G_{\mathbb{C}}.$$

The bundles P_γ and P_β are holomorphically isomorphic if and only if there exists a holomorphic function $f\colon \mathbb{C}^\times \longrightarrow G_{\mathbb{C}}$ such that, for all $z \in \mathbb{C}^\times$,

$$f(qz)\gamma(z)f(z)^{-1} = \beta(z). \tag{10}$$

□

Corollary 7.7. *If G is simply connected, then the multipliers may be chosen as constant loops with values in the maximal torus $T_{\mathbb{C}}$ of $G_{\mathbb{C}}$, and*

$$\mathfrak{M}_G(E_q) \cong (\Lambda^\vee \otimes E_q)/W$$

is the space we called $\mathcal{M}_G^h$ above.[8]

[8] This should be compared to [FMW98, § 2].

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Proof. If G is simply connected, then $LG_{\mathbb{C}} \rtimes \mathbb{C}^{\times}_{\mathrm{rot}}$ is connected, and its conjugacy classes with $|q| < 1$ are identified with

$$\begin{aligned}\mathfrak{M}_G(C_{\mathrm{Tate}}) &\cong (T_{\mathbb{C}} \times \mathbb{D}^{\times})/W_{\mathrm{aff}}\\ &\cong ((T_{\mathbb{C}} \times \mathbb{D}^{\times})/\check{T})/W.\end{aligned}$$

Here

$$W_{\mathrm{aff}} = \check{T} \rtimes W$$

is the affine Weyl group, and an element γ of

$$\check{T} = \mathrm{Hom}(\mathbb{C}^{\times}, T_{\mathbb{C}}) \subset W_{\mathrm{aff}}$$

acts on the maximal torus $T_{\mathbb{C}} \times \mathbb{C}^{\times}_{\mathrm{rot}}$ of $LG_{\mathbb{C}} \rtimes \mathbb{C}^{\times}_{\mathrm{rot}}$ via

$$(z, q) \longmapsto (\gamma(q)z, q).$$

So, the fiber of $\mathfrak{M}_G(C_{\mathrm{Tate}})$ over q is

$$\begin{aligned}\mathfrak{M}_G(E_q) &\cong (T_{\mathbb{C}}/q^{\Lambda^{\vee}})/W\\ &\cong (\mathfrak{t}_{\mathbb{C}}/(\Lambda^{\vee} + \tau\Lambda^{\vee}))/W\\ &\cong (\Lambda^{\vee} \otimes E_q)/W.\end{aligned}$$

□

For any central extension of $LG_{\mathbb{C}} \rtimes \mathbb{C}^{\times}_{\mathrm{rot}}$ by $\mathbb{C}^{\times}$, the set of conjugacy classes

$$\{[(t, z, q)] \mid q \in \mathbb{D}^{\times}\}$$

maps to $\mathfrak{M}_G(C_{\mathrm{Tate}})$. Let $\widehat{LG_{\mathbb{C}}}$ be the universal central extension of $LG_{\mathbb{C}} \rtimes \mathbb{C}^{\times}_{\mathrm{rot}}$. Then, by the discussion in [And03], the conjugacy classes in $\widehat{LG_{\mathbb{C}}}$ with $|q| < 1$ form the total space of $(\mathcal{L}_{\mathrm{Lo}}^{-1})^{\times}/W$. Here

$$\mathcal{L}_{\mathrm{Lo}} = (\mathbb{C} \times T_{\mathbb{C}} \times \mathbb{D}^{\times})/(t, z, q) \sim (tz^{-I^{\sharp}(x)}q^{-\frac{1}{2}I(x,x)}, zq^{x}, q) \tag{11}$$

is the Looijenga line bundle for the analytic Tate curve. Its fiber over q is the Looijenga bundle for E_q (see Definition 6.17 and [And00, (10.4)]). Caution: the quotient of $\mathcal{L}_{\mathrm{Lo}}^{-1}$ by W may no longer be a line bundle.

It follows that characters of $\widehat{LG_{\mathbb{C}}}$ have an interpretation as W-invariant sections of $\mathcal{L}_{\mathrm{Lo}}$. Similarly, level k characters have an interpretation as W-invariant sections of $\mathcal{L}_{\mathrm{Lo}}^{k}$. In fact, Ando shows in [And00, Corollary 11.5] that the characters of irreducible positive energy representations of level k form a basis of $\widetilde{Th}_k^{W}$, see also [And00, Theorem 11.6] for the precise relationship between $\widetilde{Th}_*^{W}$ and loop group characters.

Remark 7.8. There is a variation of Theorem 7.6 in [EF94, p. 11], where $LG_{\mathbb{C}}$ is replaced with the group of holomorphic loops

$$\gamma\colon U(1) \longrightarrow G_{\mathbb{C}}$$

from the unit circle to $G_{\mathbb{C}}$. In this picture, γ is interpreted as *clutching function*, recording how P_γ is obtained from a trivial bundle by glueing along the boundary of the fundamental domain

$$A_q = \{z \in \mathbb{C} \mid |q| \leqslant |z| \leqslant 1\}$$

of E_q. The loops β and γ yield isomorphic bundles if and only if there exists a holomorphic function

$$f\colon A_q \longrightarrow G_{\mathbb{C}}$$

such that (10) holds for all $z \in U(1)$.

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This construction is similar to Grothendieck and Krichever's description of $\mathfrak{M}_G(E_q)$ as a double coset space of $L^{\mathrm{hol}}G_{\mathbb{C}}$ (see [PS86] and, importantly, [KNR94]). The latter cuts E_q into two pieces, along the boundary of a small disk, while the former cuts E_q along the circle $[0,\tau]$ to obtain A_q. The annulus A_q also plays an important role in [Seg04].

Example 7.9. Let $G = U(1)$, so $G_{\mathbb{C}} = \mathbb{C}^\times$. This is not simply connected, but Theorem 7.6 still holds. Isomorphism classes of holomorphic principal $\mathbb{C}^\times$ bundles over E_τ are in one-to-one correspondence with those of holomorphic line bundles. For $a \in \mathbb{C}$ and $q = e^{2\pi i\tau}$, the line bundle of the divisor (a) on $E_\tau = E_q$ is

$$\begin{aligned}\mathcal{L}((a)) &\cong (\mathbb{C}\times\mathbb{C}^\times)/(t,qz)\sim(-te^{-2\pi ia}z,z)\\ &\cong (\mathbb{C}\times A_q)/(t,qy)\sim(-te^{-2\pi ia}y,y)\end{aligned}$$

for $y \in \partial_{\mathrm{out}}A_q$. (This is not a degree zero bundle.) Hence $\mathcal{L}((a))$ is classified by the multiplier

$$\begin{aligned}\gamma_a\colon \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times\\ z &\longmapsto -e^{-2\pi ia}z.\end{aligned}$$

Since the multiplier of a tensor product may be chosen as the product (in $LG_{\mathbb{C}}$) of their respective multipliers, the above argument yields a multiplier for each divisor on E_τ. In particular, the constant loop

$$z \longmapsto e^{-2\pi ia}$$

is the multiplier of the degree zero line bundle associated to the divisor $(a)-(0)$. In general, the winding number of the multiplier is minus the degree of the line bundle.

We emphasise once again that loop groups are not needed for our derivation of the Kac character formula in § 8.3.

7.4 Push-forwards

Let X and Y be compact, closed smooth manifolds, and let $f\colon X\to Y$ be a complex oriented map in the sense of [Qui71]. That means that we have a factorization

$$\begin{tikzcd} & E \arrow[d, "\xi"] \\ X \arrow[ur, "i"] \arrow[r, "f"'] & Y \end{tikzcd}$$

where ξ is a complex vector bundle and the normal bundle ν of i is equipped with a complex structure.

Definition 7.10. For such a complex oriented map f, one defines the *relative Thom sheaf* as

$$\mathbb{L}(f) = f_{A_G*}\mathbb{L}^{-\nu}\otimes\mathbb{L}^{\xi}.$$

This is a sheaf over Y_{A_G}. The *push-forward* along f is the map

$$f_!\colon \mathbb{L}(f)\longrightarrow \mathcal{O}_{Y_{A_G}}$$

of sheaves over Y_{A_G} that is adjoint to the map

$$f_{A_G*}\mathbb{L}^{-\nu}\longrightarrow\mathbb{L}^{-\xi}$$

induced by the Pontryagin–Thom collapse.

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The following lemma is immediate from the definitions.

Lemma 7.11 (Localization lemma). *Let X be as above with a smooth T-action. Let $i\colon X^T \hookrightarrow X$ be the inclusion of the fixed points and assume that we are given a T-equivariant complex structure on the normal bundle ν of i. Then we have a commuting diagram*

$$\begin{tikzcd} & \mathcal{F}_T^*(X) \arrow[dl, "i^*"'] & \\ \mathcal{F}_T^*(X^T) & & \mathcal{F}_T^{*-\nu}(X^T) \arrow[ll, "z^*"'] \arrow[ul, "i_!"'] \end{tikzcd}$$

where z is the zero section of $(X^T)^\nu$.

Note that the trivial representation does not turn up as a summand inside ν.

Corollary 7.12. *In the situation of the localization theorem (Theorem 4.1), assume that the normal bundle is equipped with a T-equivariant complex structure. Let $\Delta(\nu) \subseteq A_T$ be the closed subset*

$$\Delta(\nu) = \bigcup_{\mathbb{C}_\lambda \subseteq \nu} \ker(A_{e^\lambda}).$$

Then, restricted to $A_T \backslash \Delta(\nu)$, the map i^* becomes an isomorphism with inverse $i_! \circ (z^*)^{-1}$.

8. Character formulas

8.1 Induced representations

Let G be a compact connected Lie group with maximal torus T, and let $B \subseteq G_{\mathbb{C}}$ be a Borel subgroup of its complexification. Such a choice of B is equivalent to a choice of positive roots of G. It endows the flag variety

$$G/T \cong G_{\mathbb{C}}/B$$

with a complex structure such that the tangent space at the coset of 1 is the complex T-representation

$$\mathfrak{g}/\mathfrak{t} \cong_{\mathbb{C}} \bigoplus_{\alpha \in \mathcal{R}_-} \mathbb{C}_\alpha.$$

Similarly, if $H \subseteq G$ is a connected subgroup containing T and P_H the parabolic subgroup corresponding to H, we have a complex structure on the homogenous space

$$G/H \cong G_{\mathbb{C}}/P_H.$$

Definition 8.1. In this situation, we define the map

$$\mathrm{ind}\colon R(H) \longrightarrow R(G)$$

as the following composite.

$$\begin{tikzcd} K_H \arrow[r, equal, "\sim"] \arrow[d, equal, "\vartheta"'] & K_G(G/H) \arrow[d, equal, "\vartheta"] & \\ K_H^{\mathfrak{g}/\mathfrak{h}} \arrow[r, equal, "\sim"] & K_G^\tau(G/H) \arrow[r, "\pi_!"] & K_G \end{tikzcd}$$

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Here, and I apologize for this notation, $\mathfrak{h}$ is the Lie algebra of H, *not* a Cartan subalgebra. Further

$$\tau \cong G \times_H \mathfrak{g}/\mathfrak{h}$$

is the tangent bundle of G/H, and π is the unique map from G/H to the one point space. The push-forward $\pi_!$ is as defined in § 7.4.[9]

The Atiyah–Singer index theorem identifies our definition of ind with the definition of induction found in the representation-theory literature.

Theorem 8.2 (cf. [AS68] or [HBJ92, 5.4]). *Let $\varrho\colon \mathrm{GL} \to (V)$ be a complex representation. Then*

$$\mathrm{ind}([\varrho]) = \sum (-1)^i H^i(G/H, \mathcal{O}(G \times_H V))$$

is the induced representation of ϱ. Here $\mathcal{O}(G \times_H V)$ is the sheaf of holomorphic sections of $G \times_H V$.

8.2 The Weyl character formula

We will now compute the character of these induced representations. As in Theorem 4.6, we let W_H and W_G be the respective Weyl groups, and we let

$$i\colon F \longrightarrow G/H$$

be the inclusion of the T-fixed points $F := (G/H)^T$. Recall that F can be identified with the set W_G/W_H, and that we have

$$i^*\tau \cong_T \coprod_{[w]\in F} (\mathfrak{g}/\mathfrak{h})^w$$

(conjugation by w on the right-hand side). We have a commuting diagram,

$$\begin{tikzcd}
K_H \arrow[r, equal, "\sim"] & K_G(G/H) \arrow[r, equal, "\vartheta"] \arrow[d, "\mathrm{res}"'] & K_G^\tau(G/H) \arrow[r, "\pi_!"] \arrow[d, "\mathrm{res}"'] & K_G \arrow[d, "\mathrm{char}"] \\
& K_T(G/H) \arrow[r, equal, "\vartheta"] \arrow[d, tail, "i^*"'] & K_T^\tau(G/H) \arrow[r, "\pi_!"] \arrow[d, tail, "i^*"'] & K_T \\
& \bigoplus_{[w]\in F} K_T \arrow[r, equal, "\vartheta"] & \bigoplus_{[w]\in F} K_T^{(\mathfrak{g}/\mathfrak{h})^w} & \bigoplus_{[w]\in F} K_T \arrow[l, tail, "z^*"] \arrow[ul, "i_!"'] \arrow[u, "\sum_{[w]\in F}(-)_w"']
\end{tikzcd} \qquad (12)$$

where

$$z\colon F \longrightarrow F^{i^*\tau}$$

is the zero section.

[9] A more common definition of the push-forward $\pi_!$ in K-theory or cohomology is the composite of our $\pi_!$ with ϑ. The reason for our convention is that it generalizes to elliptic cohomology in a canonical way.

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The top row of (12) is the map ind of Definition 8.1. The composite of the vertical arrows on the left sends an H-representation ϱ to $(\chi_\varrho^w)_{[w]\in F}$ (the character of ϱ and its conjugates under W_G). The composite at the bottom is multiplication by the Euler class of $i^*\tau$. On the $[w]$th summand, this is

$$e(i^*\tau)_{[w]} = \prod_{\alpha\in\mathcal{R}} (1 - e^{w(\alpha)}).$$

Here

$$\mathcal{R} := \mathcal{R}_-^G \backslash \mathcal{R}_-^H$$

consists of the negative roots of G that are not roots of H. Using the localization lemma (see Corollary 7.12), we can deduce Weyl's character formula.

Theorem 8.3 (Weyl). *Let ϱ be a representation of H. Then the character of its induced representation equals*

$$\chi_{\mathrm{ind}(\varrho)} = \sum_{[w]\in F} \frac{\chi_\varrho^w}{\prod\limits_{\alpha\in\mathcal{R}} (1 - e^{2\pi i w(\alpha)})}. \tag{13}$$

To be precise, the localization lemma implies the equality (13) in the localized ring

$$R(T)[e(\mathfrak{g}/\mathfrak{h})^{-1}].$$

Since $R(T)$ maps injectively into this localization, and $\chi_{\mathrm{ind}(\varrho)}$ is an element of $R(T)$, it makes sense to interpret (13) as a formula in $R(T)$.

Replacing K-theory by cohomology, we obtain a formula for the composite

$$(Bj)^* \circ (Bk)_!,$$

where j and k are the respective inclusions of T and H in G. Namely, it sends a regular function f on A_H to

$$Bj^*(Bk_!(f)) = \frac{\sum_{[w]\in f} \det(w) f^w}{\prod_{\alpha\in\mathcal{R}} \alpha_{\mathbb{C}}}$$

(compare [AC83]).

8.3 The Kac character formula

We now turn our attention to the elliptic case, making the additional assumption that the partial flag variety G/H carries a $U^{[2]}$-structure. For simplicity of notation, we write

$$\mathrm{Ell}_G^*(X) := \Gamma\mathcal{E}ll_G^*(X)^h$$

for the analytic global sections, noting that the statement holds on the level of sheaves with all sheaves pushed forward to $\mathcal{M}_G$. Let ϱ be a G-representation. The diagram (12) is replaced by

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the following diagram.

$$\begin{tikzcd}
\Theta^{W_H}_{\varrho+\mathfrak{g}/\mathfrak{h}} \arrow[r, equal, "\sim"] & \mathrm{Ell}^{\varrho+\tau}_G(G/H) \arrow[r, "\pi_!"] \arrow[d, "\mathrm{res}"'] & \mathrm{Ell}^{\varrho}_G \arrow[r, equal, "\vartheta"] \arrow[d, "\mathrm{char}"] & \Theta^{W_G}_{\rho} \arrow[d] \\
 & \mathrm{Ell}^{\varrho+\tau}_T(G/H) \arrow[r, "\pi_!"] \arrow[d, "i^*"'] & \mathrm{Ell}^{\varrho}_T \arrow[r, equal, "\vartheta"] & \Theta_{\rho} \\
\bigoplus\limits_{[w]\in F} \Theta_{\varrho+(\mathfrak{g}/\mathfrak{h})^w} \arrow[r, equal, "\vartheta"] & \bigoplus\limits_{[w]\in F} \mathrm{Ell}_T^{\varrho+(\mathfrak{g}/\mathfrak{h})^w} & \bigoplus\limits_{[w]\in F} \mathrm{Ell}_T{}^{\varrho} \arrow[l, "z^*"] \arrow[u, "\sum_{[w]\in F}(-)_w"'] \arrow[ul, "i_!"']
\end{tikzcd} \tag{14}$$

As before, we will write 'ind' for the composite of the arrows in the top row. The Thom sheaf $\mathbb{L}_T^{\mathfrak{g}/\mathfrak{h}}$ is the line bundle over $\mathcal{M}_T$ associated to the divisor

$$\Delta = \sum_{\alpha\in\mathcal{R}} (A_{K_\alpha})$$

on $\mathcal{M}_T$. Here $K_\alpha = \ker(e^{2\pi i\alpha})$. The T-equivariant Euler class of $\mathfrak{g}/\mathfrak{h}$ is the theta function

$$e_{\mathrm{ell}}(\mathfrak{g}/\mathfrak{h}) = \pm \prod_{\alpha\in\mathcal{R}} (z^{\alpha/2} - z^{-\alpha/2}) \prod_{n\geqslant 1} \frac{(1-q^n z^\alpha)(1-q^n z^{-\alpha})}{(1-q^n)^2},$$

where the sign equals $(-1)^{|\mathcal{R}|}$. Set

$$\Phi = \Phi(q) := \prod_{n\geqslant 1} (1-q^n)^2.$$

Theorem 8.4. *Let f be an element of $\Theta^{W_H}_{\varrho+\mathfrak{g}/\mathfrak{h}}$. Then we have*

$$\mathrm{ind}(f) = (-\Phi)^d \frac{\sum_{[w]\in F} \det(w) w(f)}{\prod_{\alpha\in\mathcal{R}} (z^{\alpha/2} - z^{-\alpha/2}) \prod_{n\geqslant 1} (1-q^n z^\alpha)(1-q^n z^{-\alpha})}.$$

Here d is the complex dimension of the partial flag variety G/H. Consider now the special case where G is simple and simply connected, and $H = T$ is the maximal torus. By [Loo77, (3.4)], we have

$$p_1(\mathfrak{g}/\mathfrak{t}) = g \cdot I_{\min},$$

and hence

$$\mathcal{L}_{I_{\mathfrak{g}/\mathfrak{t}}} = \mathcal{L}^g_{\mathrm{Lo}}.$$

Here g is the dual Coxeter number. Assume that we have $(p_1/2)(\varrho) = kI_{\min}$ with $k \in \mathbb{Z}$. Then the top row of (14) becomes a map

$$\mathrm{ind}\colon \Theta_{\mathrm{k+g}} \longrightarrow \Theta_{\mathrm{k}}^{\mathrm{W_G}},$$

where Θ_k are the Looijenga theta functions of level k.

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DEFINITION 8.5 (Looijenga basis). Let $k \in \mathbb{N}$, and let $\lambda \in \Lambda$. The element $\theta_{k,\lambda} \in \Theta_k$ is defined by

$$\theta_{k,\lambda} := \sum_{x \in \Lambda^\vee} q^{(k\phi+\lambda)(x)} e^{2\pi i (kI^\sharp(x)+\lambda)}.$$

Here

$$\phi(x) := \tfrac{1}{2} I_{\min}(x,x)$$

and $I^\sharp\colon \Lambda^\vee \to \Lambda$ is the adjoint of $I_{\min}$.

As λ varies over a set of representatives for $\Lambda/kI^\sharp(\Lambda^\vee)$, the images of the $\theta_{k,\lambda}$ in Θ_k form a basis for Θ_k.

COROLLARY 8.6 (Kac character formula). *In the situation of Theorem 8.4, assume that G is simple and simply connected, and let $H = T$ be the maximal torus. Then we have*

$$\operatorname{ind}(\theta_{k+g,\lambda}) = \frac{(-\Phi)^d \cdot \sum_{w \in W_G} \det(w) \cdot \theta_{k+g,w(\lambda)}}{\prod_{\alpha \in \mathcal{R}_-}(e^{\pi i \alpha} - e^{-\pi i \alpha}) \prod_{n \geqslant 1}(1 - q^n e^{2\pi i \alpha})(1 - q^n e^{-2\pi i \alpha})}.$$

Up to the factor $\pm\Phi(q)^{d+r}$, which is constant in z, this agrees with the Kac character formula for the positive energy representation of the loop group LG of level k and highest weight

$$\lambda + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha.$$

For a presentation of the Kac character formula in this form see [PS86, (14.3.4)] or [And00, 11.4].

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Nora Ganter nganter@unimelb.edu.au

Department of Mathematics and Statistics, The University of Melbourne, Parkville VIC 3010, Australia

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