FURTHER INTEGRALS INVOLVING E-FUNCTIONS

by F. M. RAGAB

PART I

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§ 1. *Introductory.* The formulae to be proved are

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} E(p\,;\;\alpha_r : q\,;\;\rho_s : \lambda^m z)\,d\lambda$$

$$=\pi \operatorname{cosec}(k\pi)(2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{k-\frac{1}{2}}$$

$$\times E\left(p\,;\;\alpha_r : 1-\frac{k}{m}, 1-\frac{k+1}{m}, \ldots, 1-\frac{k+m-1}{m}, \rho_1, \ldots, \rho_q : e^{\pm m\pi i} m^m z\right)$$

$$+ 2^{\frac{1}{2}-\frac{1}{2}m}\pi^{\frac{1}{2}+\frac{1}{2}m} \sum_{\nu=0}^{m-1} \frac{(-1)^{\nu+1} m^{-\frac{1}{2}-\nu} z^{-(k+\nu)/m}}{\sin\left(\frac{k+\nu}{m}\pi\right) \prod_{s=1}^{\nu} \sin\frac{s\pi}{m} \prod_{t=1}^{m-\nu-1} \sin\frac{t\pi}{m}}$$

$$\times E\left(\begin{matrix} p\,;\;\alpha_r + (k+\nu)/m & : e^{\pm m\pi i} m^m z \\ 1+\frac{k+\nu}{m}, 1+\frac{1}{m}, \ldots, 1+\frac{\nu}{m}, 1-\frac{1}{m}, \ldots, 1-\frac{m-\nu-1}{m}, \rho_1+\frac{k+\nu}{m}, \ldots, \rho_q+\frac{k+\nu}{m} & \end{matrix}\right), \quad (1)$$

where m is a positive integer, $p \geqq q+1$, $R(m\alpha_r+k)>0$, $r=1, 2, \ldots, p$, and $|\operatorname{amp} z|<\pi$. For other values of p and q the result holds if the integral is convergent.

Also

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1} E(p\,;\;\alpha_r : q\,;\;\rho_s : z\lambda^m)\,d\lambda$$

$$=\frac{\sin\rho\pi}{\sin\alpha\pi}\Gamma(\rho-\alpha) m^{\alpha-\rho} E\left\{\begin{matrix}\alpha_1, \ldots, \alpha_p, 1-\rho/m, 1-(\rho+1)/m, \ldots, 1-(\rho+m-1)/m : z \\ 1-\alpha/m, 1-(\alpha+1)/m, \ldots, 1-(\alpha+m-1)/m, \rho_1, \ldots, \rho_q\end{matrix}\right\}$$

$$-2^{1-m}\Gamma(\rho-\alpha) m^{\alpha-\rho} \sum_{\nu=0}^{m-1} \frac{\sin(\rho-\alpha)\pi}{\sin\frac{\alpha+\nu}{m}\pi \prod_{s=1}^{\nu}\sin\frac{s\pi}{m} \prod_{t=1}^{m-\nu-1}\sin\frac{t\pi}{m}} z^{-(\alpha+\nu)/m}$$

$$\times E\left\{\begin{matrix}\alpha_1+\frac{\alpha+\nu}{m}, \ldots, \alpha_p+\frac{\alpha+\nu}{m}, 1+\frac{\alpha-\rho+\nu}{m}, 1+\frac{\alpha-\rho+\nu-1}{m}, \ldots, 1+\frac{\alpha-\rho+\nu-m+1}{m} : z \\ 1+\frac{\alpha+\nu}{m}, 1+\frac{1}{m}, \ldots, 1+\frac{\nu}{m}, 1-\frac{1}{m}, \ldots, 1-\frac{m-\nu-1}{m}, \rho_1+\frac{\alpha+\nu}{m}, \ldots, \rho_q+\frac{\alpha+\nu}{m}\end{matrix}\right\} \ldots (2)$$

where m is a positive integer, $p \geqq q+1$, $R(\alpha+m\alpha_r)>0$, $r=1, 2, \ldots, p$, $R(\rho-\alpha)>0$ and $|\operatorname{amp} z|<\pi$. For other values of p and q the formula is valid if the integral converges.

The proofs will be found in § 2. Some double integrals are discussed in § 3.

The following formulae are required in the proof.

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(1) If m is a positive integer and if $R(k)>0$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\lambda^m)\,d\lambda=m^{k-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}E(p+m\,;\ \alpha_r:q\,;\ \rho_s:z/m^m),\ \ldots\ldots(3)$$

where $\alpha_{p+\nu}=(k+\nu-1)/m,\ \nu=1,2,\ldots,m$.

(2) If $p\geqq q+1$,

$$E(p\,;\ \alpha_r:q\,;\ \rho_s:z)=\pi^{p-q-1}\sum_{r=1}^{p}\prod_{t=1}^{q}\sin(\rho_t-\alpha_r)\pi\left\{\prod_{s=1}^{p}{}'\sin(\alpha_s-\alpha_r)\pi\right\}^{-1}z^{\alpha_r}$$

$$\times E\left\{\begin{matrix}\alpha_r,\alpha_r-\rho_1+1,\ldots,\alpha_r-\rho_q+1:\dfrac{(-1)^{p-q-1}}{z}\\ \alpha_r-\alpha_1+1,\ldots*\ldots,\alpha_r-\alpha_p+1\end{matrix}\right\}.\ \ldots\ldots(4)$$

If m is a positive integer,

$$\sin\frac{k\pi}{m}\sin\frac{(k+1)\pi}{m}\ldots\sin\frac{(k+m-1)\pi}{m}=2^{1-m}\sin k\pi.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

(3) If m is a positive integer and if $R(\rho)>R(\alpha)>0$,

$$\int_0^1\lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\lambda^m)\,d\lambda=\Gamma(\rho-\alpha)\,m^{\alpha-\rho}E(p+m\,;\ \alpha_r:q+m\,;\ \rho_s:z),\ \ldots(6)$$

where $\alpha_{p+\nu}=(\alpha+\nu-1)/m$, $\rho_{q+\nu}=(\rho+\nu-1)/m$, $\nu=1,2\ldots,m$.

$$\frac{1}{\Gamma(\rho_{q+1}-\alpha_{p+1})}\int_0^1 t^{-\ {}_{q+1}}(1-t)^{\rho_{q+1}-\alpha_{p+1}-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:zt)\,dt$$

$$=\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}\pi)}E(p+1\,;\ \alpha_r:q+1\,;\ \rho_s:z)+\frac{\sin(\alpha_{p+1}-\rho_{q+1})\pi}{\sin(\rho_{q+1}\pi)}z^{\rho_{q+1}-1}$$

$$\times E(p+1\,;\ \alpha_r-\rho_{q+1}+1:2-\rho_{q+1},\rho_1-\rho_{q+1}+1,\ldots,\rho_q-\rho_{q+1}+1:z),\ \ldots\ldots\ldots(7)$$

where $p\geqq q+1$, $R(\rho_{q+1}-\alpha_{p+1})>0$, $R(\alpha_r-\rho_{q+1})>-1$, $r=1,2,\ldots,p$, (**4**).

§ 2. *Proofs of the Formulae.* Consider the special case of (1) when $p=1$, $q=0$; then the L.H.S. of (1) becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_1::\lambda^m z)\,d\lambda=z^{\alpha_1}\int_0^\infty e^{-\lambda}\lambda^{k+m\alpha_1-1}E\{\alpha_1::1/(\lambda^m z)\}\,d\lambda$$

$$=m^{k+m\alpha_1-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}z^{\alpha_1}E\{m+1\,;\ \alpha_r::1/(zm^m)\},$$

where $\alpha_{\nu+1}=(k+\nu-1)/m+\alpha_1$, $\nu=1,2,\ldots,m$, by (3).

On applying (4) and (5) this becomes (1) with $p=1$, $q=0$. The general case is deduced in the usual way.

When $p=1$, $q=0$, the integral on the left of (2) can be written

$$z^{\alpha_1}\int_0^1\lambda^{\alpha+m\alpha_1-1}(1-\lambda)^{\rho-\alpha-1}E\{\alpha_1::1/(z\lambda^m)\}\,d\lambda$$

$$=z^{\alpha_1}\Gamma(\rho-\alpha)m^{\alpha-\rho}E\left(\begin{matrix}\alpha_1,\alpha_1+\dfrac{\alpha}{m},\alpha_1+\dfrac{\alpha+1}{m},\ldots,\alpha_1+\dfrac{\alpha+m-1}{m}:\dfrac{1}{z}\\ \alpha_1+\dfrac{\rho}{m},\alpha_1+\dfrac{\rho+1}{m},\ldots,\alpha_1+\dfrac{\rho+m-1}{m}\end{matrix}\right),$$

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by (6). On applying (4) and (5) this gives (2) with $p=1$, $q=0$. The general case can then be deduced.

From (1) and (2) many particular cases can be deduced. For instance, if, in (2), $p=q=0$, the value of

$$\int_1^\infty e^{-\lambda^m/z}\lambda^{-\rho}(\lambda-1)^{\rho-\alpha-1}\,d\lambda,$$

where $R(z)>0, R(\rho-\alpha)>0$, is found to be the R.H.S. of (2) with all the linear expressions involving $\alpha_1, \alpha_2, \ldots, \alpha_p, \rho_1, \rho_2, \ldots, \rho_q$ omitted.

§ 3. *Some Double Integrals.* The first of these is

$$\int_0^\infty \int_0^\infty \lambda^{m-1}\mu^{n-1}(1+\lambda+\mu)^{-k}\,E(p\,;\alpha_r:q\,;\ \rho_s:\lambda z/\mu)\,d\lambda\,d\mu$$

$$=\frac{\pi\Gamma(k-m-n)}{\sin m\pi\Gamma(k)}\left\{E\begin{pmatrix}n,\alpha_1,\ldots,\alpha_p:e^{\pm i\pi}z\\1-m,\rho_1,\ldots,\rho_q\end{pmatrix}-z^{-m}E\begin{pmatrix}m+n,\alpha_1+m,\ldots,\alpha_p+m:\ e^{\pm i\pi}z\\m+1,\rho_1+m,\ldots,\rho_q+m\end{pmatrix}\right\},\ \ldots(8)$$

where $p\geqq q+1$, $R(z)>0$, $R(n)>0$, $R(k-m)>0$, $R(m+\alpha_r)>0$, $R(k-n+\alpha_r)>0$, $r=1, 2, \ldots, p$.

Consider the case in which $p=q=0$; then the L.H.S. of (8) is equal to

$$\int_0^\infty \int_0^\infty e^{-\mu/(\lambda z)}\lambda^{m-1}\mu^{n-1}(1+\lambda+\mu)^{-k}\,d\lambda\,d\mu$$

$$=\frac{1}{\Gamma(k)}\int_0^\infty \lambda^{m-1}\,d\lambda\int_0^\infty e^{-\mu/(\lambda z)}\mu^{n-k-1}E\left(k::\frac{\mu}{1+\lambda}\right)d\mu.$$

Here replace μ by $\lambda z\mu$ and get

$$\frac{z^{n-k}}{\Gamma(k)}\int_0^\infty e^{-\mu}\mu^{n-k-1}\,d\mu\int_0^\infty \lambda^{m+n-k-1}E\left(k::\frac{\lambda\mu z}{1+\lambda}\right)d\lambda.$$

The second integral can be written

$$\int_0^1 t^{m+n-k-1}(1-t)^{k-m-n-1}\,E(k::t\mu z)\,dt,$$

where $\lambda=t/(1-t)$, $R(k)>R(m+n)>0$; and, from (7), this is equal to

$$\Gamma(k-m-n)(\mu z)^{k-m-n}E(m+n::\mu z).$$

On applying (1), (8) with $p=q=0$ is obtained. The general case is derived in the usual way.

Next, consider the double integral

$$\int_0^\infty \int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:\lambda^n z/\mu^n)\,d\lambda\,d\mu,\quad \ldots\ldots\ldots\ldots(9)$$

where n is a positive integer, $p\geqq q+1$, $R(l+m+k+n\alpha_r)>0$, $r=1, 2, \ldots, p$, and $|\operatorname{amp} z|<\pi$. On replacing μ by $\lambda\mu$ this can be written

$$\int_0^\infty \lambda^{2l+2m-1}\,d\lambda\int_0^\infty e^{-\lambda^2\mu-1/\mu}\mu^{l+m-k-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\mu^n)\,d\mu,$$

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and, on changing the order of integration and replacing λ by $\lambda/\surd\mu$ in the inner integral it becomes

$$\tfrac{1}{2}\Gamma(l+m)\int_0^\infty e^{-1/\mu}\mu^{-k-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\mu^n)\,d\mu$$

$$=\tfrac{1}{2}\Gamma(l+m)\int_0^\infty e^{-\mu}\mu^{k-1}\,E(p\,;\ \alpha_r:q\,;\ \rho_s:z\mu^n)\,d\mu.$$

This last integral is the integral in (1) with n in place of m.

Now consider the integral

$$\int_0^\infty\int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}\,E(p\,;\ \alpha_r:q\,;\ \rho_s:\lambda^n\mu^n z)\,d\lambda\,d\mu, \quad \ldots\ldots\ldots\ldots\ldots(10)$$

where n is a positive integer, $p\geqq q+1$, $R(l+m+k+n\alpha_r)>0$, $r=1, 2, \ldots, p$, and $|\operatorname{amp} z|<\pi$.

Replace μ by μ/λ and get

$$\int_0^\infty \lambda^{2k-1}\,d\lambda\int_0^\infty e^{-\mu-\lambda^2/\mu}\mu^{l+m-k-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:\mu^n z)\,d\mu.$$

Here change the order of integration, replace λ by $\lambda\surd\mu$, and so get

$$\tfrac{1}{2}\Gamma(k)\int_0^\infty e^{-\mu}\mu^{l+m-1}\,E(p\,;\ \alpha_r:q\,;\ \rho_s:\mu^n z).$$

The last integral is the integral in (1) with n in place of m and $l+m$ in place of k.

Finally, consider the integral

$$\int_0^\infty\int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}E(p\,;\alpha_r:q\,;\ \rho_s:\lambda^{-n}\mu^{-n}z)\,d\lambda\,d\mu, \quad \ldots\ldots\ldots\ldots(11)$$

where n is a positive integer, $p\geqq q+1$, $R(l+m+k)>0$ and $|\operatorname{amp} z|<\pi$.

Here replace μ by μ/λ, and get

$$\int_0^\infty \lambda^{2k-1}\,d\lambda\int_0^\infty e^{-\mu-\lambda^2/\mu}\mu^{l+m-k-1}\,E(p\,;\ \alpha_r:q\,;\ \rho_s:\mu^{-n}z)\,d\mu.$$

On changing the order of integration and replacing λ by $\lambda\surd\mu$ this becomes

$$\tfrac{1}{2}\Gamma(k)\int_0^\infty e^{-\mu}\mu^{l+m-1}E(p\,;\ \alpha_r:q\,;\ \rho_s:\mu^{-n}z)\,d\mu$$

$$=\tfrac{1}{2}\Gamma(k)\,n^{l+m-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}n}\,E(p+n\,;\ \alpha_r:q\,;\ \rho_s:z/n^n),$$

where $\alpha_{p+\nu}=(l+m+\nu-1)/n$, $\nu=1, 2, \ldots, n$, by (3).

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Part II

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§1. *Introductory.* In § 3 two E-function integrals, involving the function $I_n(\mu)$, will be evaluated. Two subsidiary formulae, needed in the proofs, will be established in § 2. The following formulae are also required.

If $R(n) > -\frac{1}{2}$, (1),

$$I_n(\mu) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\left(\frac{\mu}{2}\right)^n \int_{-1}^{1} e^{-\mu\lambda}(1-\lambda^2)^{n-\frac{1}{2}}\,d\lambda. \quad \text{(1)}$$

If m is a positive integer, $R(\alpha)>0$ and $|\operatorname{amp} z| < \pi$, (2),

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha-1}E(p\,;\alpha_r : q\,;\rho_s : z/\lambda^m)\,d\lambda = (2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\alpha-\frac{1}{2}}E(p+m\,;\alpha_r : q\,;\rho_s : z/m^m), \quad \text{(2)}$$

where $\alpha_{p+\nu+1} = (\alpha+\nu)/m$, $\nu = 0, 1, 2, \ldots, m-1$.

If m is a positive integer, $R(\rho) > R(\alpha) > 0$, $|\operatorname{amp} z| < \pi$, (3),

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}E(p\,;\alpha_r : q\,;\rho_s : z/\lambda^m)\,d\lambda = \Gamma(\rho-\alpha)m^{\alpha-\rho}E(p+m\,;\alpha_r : q+m\,;\rho_s : z), \quad \text{(3)}$$

where $\alpha_{p+\nu+1} = (\alpha+\nu)/m$, $\rho_{q+\nu+1} = (\rho+\nu)/m$, $\nu = 0, 1, 2, \ldots, m-1$.

If $p \geqq q+1$, (4),

$$E(p\,;\alpha_r : q\,;\rho_s : z) = \pi^{p-q-1}\sum_{r=1}^{p}\prod_{t=1}^{q}\sin(\rho_t-\alpha_r)\pi\left\{\prod_{s=1}^{p}{}'\sin(\alpha_s-\alpha_r)\pi\right\}^{-1}$$
$$\times z^{\alpha_r}E\left\{\begin{matrix}\alpha_r, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 : e^{\pm(p-q-1)\pi i}.1/z\\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_p+1\end{matrix}\right\}. \quad \text{(4)}$$

§ 2. *Subsidiary Integrals.* The two formulae are as follows.

If $p \geqq q+1$, $R(k+2\alpha_r) > 0$, $r = 1, 2, \ldots, p$, and $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p\,;\alpha_r : q\,;\rho_s : \lambda^2 z)\,d\lambda$$
$$= \frac{\pi\sqrt{\pi}}{\sin(k\pi)}2^k E(p\,;\alpha_r : 1-\tfrac{1}{2}k, \tfrac{1}{2}-\tfrac{1}{2}k, \rho_1, \ldots, \rho_q : 4z)$$
$$-\frac{\pi\sqrt{\pi}}{2\sin(\frac{1}{2}k\pi)}z^{-\frac{1}{2}k}E(p\,;\alpha_r+\tfrac{1}{2}k : 1+\tfrac{1}{2}k, \tfrac{1}{2}, \rho_1+\tfrac{1}{2}k, \ldots, \rho_q+\tfrac{1}{2}k : 4z)$$
$$+\frac{\pi\sqrt{\pi}}{4\cos(\frac{1}{2}k\pi)}z^{-\frac{1}{2}k-\frac{1}{2}}E(p\,;\alpha_r+\tfrac{1}{2}k+\tfrac{1}{2} : \tfrac{3}{2}+\tfrac{1}{2}k, \tfrac{3}{2}, \rho_1+\tfrac{1}{2}k+\tfrac{1}{2}, \ldots, \rho_q+\tfrac{1}{2}k+\tfrac{1}{2} : 4z). \quad \text{(5)}$$

If $p \geqq q+1$, $R(\rho-\alpha) > 0$, $R(\alpha+2\alpha_r) > 0$, $r = 1, 2, \ldots, p$, $|\operatorname{amp} z| < \pi$,

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}E(p\,;\alpha_r : q\,;\rho_s : \lambda^2 z)d\lambda$$
$$= \frac{\Gamma(\rho-\alpha)}{2^{\rho-\alpha+1}}\left[\begin{matrix}2\dfrac{\sin(\rho\pi)}{\sin(\alpha\pi)}E\begin{pmatrix}\alpha_1, \ldots, \alpha_p, 1-\frac{1}{2}\rho, \frac{1}{2}-\frac{1}{2}\rho : z\\ \rho_1, \ldots, \rho_q, 1-\frac{1}{2}\alpha, \frac{1}{2}-\frac{1}{2}\alpha\end{pmatrix}\\ -\dfrac{\sin(\rho-\alpha)\pi}{\sin(\frac{1}{2}\alpha\pi)}z^{-\frac{1}{2}\alpha}E\begin{pmatrix}\alpha_1+\frac{1}{2}\alpha, \ldots, \alpha_p+\frac{1}{2}\alpha, \frac{1}{2}\alpha-\frac{1}{2}\rho+1, \frac{1}{2}\alpha-\frac{1}{2}\rho+\frac{1}{2} : z\\ \rho_1+\frac{1}{2}\alpha, \ldots, \rho_q+\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha, \frac{1}{2}\end{pmatrix}\\ -\dfrac{\sin(\rho-\alpha)\pi}{\cos(\frac{1}{2}\alpha\pi)}z^{-\frac{1}{2}\alpha-\frac{1}{2}}E\begin{pmatrix}\alpha_1+\frac{1}{2}\alpha+\frac{1}{2}, \ldots, \alpha_p+\frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha-\frac{1}{2}\rho+\frac{3}{2}, \frac{1}{2}\alpha-\frac{1}{2}\rho+1 : z\\ \rho_1+\frac{1}{2}\alpha+\frac{1}{2}, \ldots, \rho_q+\frac{1}{2}\alpha+\frac{1}{2}, \frac{3}{2}+\frac{1}{2}\alpha, \frac{3}{2}\end{pmatrix}\end{matrix}\right]. \quad \text{(6)}$$

In proving (5) consider first the case in which $p=1, q=0$; then

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_1::\lambda^2 z)\,d\lambda = z^{\alpha_1}\int_0^\infty e^{-\lambda}\lambda^{k+2\alpha_1-1}E\left(\alpha_1::\frac{1}{\lambda^2 z}\right)d\lambda$$

$$=\pi^{-\frac{1}{2}}2^{k+2\alpha_1-1}z^{\alpha_1}E\{\alpha_1,\ \alpha_1+\tfrac{1}{2}k,\ \alpha_1+\tfrac{1}{2}k+\tfrac{1}{2}::1/(4z)\},\ \text{by (2)};$$

and, on applying (4), the R.H.S. of (5) with $p=1, q=0$, is obtained. The general result is deduced in the usual way.

For (6), proceeding as before and applying (3), it is found that

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}E(\alpha_1::\lambda^2 z)\,d\lambda = z^{\alpha_1}\int_0^1 \lambda^{\alpha+2\alpha_1-1}(1-\lambda)^{\rho-\alpha-1}E\left(\alpha_1::\frac{1}{z\lambda^2}\right)d\lambda$$

$$=\Gamma(\rho-\alpha)2^{\alpha-\rho}z^{\alpha_1}E(\alpha_1,\ \alpha_1+\tfrac{1}{2}\alpha,\ \alpha_1+\tfrac{1}{2}\alpha+\tfrac{1}{2}:\alpha_1+\tfrac{1}{2}\rho,\ \alpha_1+\tfrac{1}{2}\rho+\tfrac{1}{2}:1/z),\ \text{by (3)},$$

and, on applying (4), the R.H.S. of (6) with $p=1, q=0$, is obtained. From this the general result is derived.

§ 3. *The Integral Formulae.* The formulae to be proved are:

$$\int_0^\infty e^{-\mu}\mu^{-m-1}I_n(\mu)E(p\,;\ \alpha_r:q\,;\ \rho_s:z/\mu^2)d\mu$$

$$=-\frac{\sin(n+m)\pi}{2\sqrt{2}\,.\,\pi\cos(m\pi)}$$

$$\times E\begin{pmatrix}\alpha_1,\ \ldots,\ \alpha_p,\ \tfrac{1}{2}n-\tfrac{1}{2}m,\ \tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2},\ \tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m,\ -\tfrac{1}{2}n-\tfrac{1}{2}m:z\\ \rho_1,\ \ldots,\ \rho_q,\ \tfrac{1}{4}-\tfrac{1}{2}m,\ \tfrac{3}{4}-\tfrac{1}{2}m\end{pmatrix}$$

$$-\frac{\cos(n\pi)}{4\sqrt{2}\,.\,\pi\sin(\tfrac{1}{4}+\tfrac{1}{2}m)\pi}z^{-\frac{1}{4}-\frac{1}{2}m}$$

$$\times E\begin{pmatrix}\alpha_1+\tfrac{1}{4}+\tfrac{1}{2}m,\ \ldots,\ \alpha_p+\tfrac{1}{4}+\tfrac{1}{2}m,\ \tfrac{1}{4}+\tfrac{1}{2}n,\ \tfrac{3}{4}+\tfrac{1}{2}n,\ \tfrac{1}{4}-\tfrac{1}{2}n,\ \tfrac{3}{4}-\tfrac{1}{2}n:z\\ \rho_1+\tfrac{1}{4}+\tfrac{1}{2}m,\ \ldots,\ \rho_q+\tfrac{1}{4}+\tfrac{1}{2}m,\ \tfrac{5}{4}+\tfrac{1}{2}m,\ \tfrac{1}{2}\end{pmatrix}$$

$$-\frac{\cos(n\pi)}{4\sqrt{2}\,.\,\pi\cos(\tfrac{1}{4}+\tfrac{1}{2}m)\pi}z^{-\frac{3}{4}-\frac{1}{2}m}$$

$$\times E\begin{pmatrix}\alpha_1+\tfrac{3}{4}+\tfrac{1}{2}m,\ \ldots,\ \alpha_p+\tfrac{3}{4}+\tfrac{1}{2}m,\ \tfrac{3}{4}+\tfrac{1}{2}n,\ \tfrac{5}{4}+\tfrac{1}{2}n,\ \tfrac{3}{4}-\tfrac{1}{2}n,\ \tfrac{5}{4}-\tfrac{1}{2}n:z\\ \rho_1+\tfrac{3}{4}+\tfrac{1}{2}m,\ \ldots,\ \rho_q+\tfrac{3}{4}+\tfrac{1}{2}m,\ \tfrac{7}{4}+\tfrac{1}{2}m,\ \tfrac{3}{2}\end{pmatrix},\ \ldots\ldots\ldots(7)$$

where $R(n-m)>0$, $R(m+2\alpha_r)>-\tfrac{1}{2}$, $r=1, 2, \ldots, p$, $|\operatorname{amp} z|<\pi$; and

$$\int_0^\infty e^{-\mu}\mu^{-m-1}I_n(\mu)E(p\,;\alpha_r:q\,;\rho_s:z\mu^2)\,d\mu$$

$$=\frac{\pi}{\sqrt{2}\sin(n-m)\pi}$$

$$\times E\begin{pmatrix}\alpha_1,\ \ldots,\ \alpha_p,\ \tfrac{1}{4}+\tfrac{1}{2}m,\ \tfrac{3}{4}+\tfrac{1}{2}m & :z\\ \rho_1,\ \ldots,\ \rho_q,\ 1+\tfrac{1}{2}m-\tfrac{1}{2}n,\ \tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{1}{2}n,\ 1+\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{1}{2}n\end{pmatrix}$$

$$+\frac{\pi}{2\sqrt{2}\sin(\tfrac{1}{2}m-\tfrac{1}{2}n)\pi}z^{\frac{1}{2}m-\frac{1}{2}n}$$

$$\times E\begin{pmatrix}\alpha_1-\tfrac{1}{2}m+\tfrac{1}{2}n,\ \ldots,\ \alpha_p-\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{1}{4}+\tfrac{1}{2}n,\ \tfrac{3}{4}+\tfrac{1}{2}n & :z\\ \rho_1-\tfrac{1}{2}m+\tfrac{1}{2}n,\ \ldots,\ \rho_q-\tfrac{1}{2}m+\tfrac{1}{2}n,\ 1-\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{1}{2},\ \tfrac{1}{2}+n,\ 1+n\end{pmatrix}$$

$$+\frac{\pi}{2\sqrt{2}\cos(\tfrac{1}{2}m-\tfrac{1}{2}n)\pi}z^{\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}}$$

$$\times E\begin{pmatrix}\alpha_1-\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2},\ \ldots,\ \alpha_p-\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{3}{4}+\tfrac{1}{2}n,\ \tfrac{5}{4}+\tfrac{1}{2}n & :z\\ \rho_1-\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2},\ \ldots,\ \rho_q-\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{3}{2}-\tfrac{1}{2}m+\tfrac{1}{2}n,\ \tfrac{3}{2},\ 1+n,\ \tfrac{3}{2}+n\end{pmatrix},\ \ldots(8)$$

where $p\geqq q+1$, $R(m)>-\tfrac{1}{2}$, $R(n-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, $|\operatorname{amp} z|<\pi$.

In proving (7) consider the case $p=q=0$; then, from (1), if $R(n)>-\frac{1}{2}$,

$$\text{L.H.S.}=\frac{2^{-n}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-\mu}\mu^{n-m-1}E(::z/\mu^2)\,d\mu\int_{-1}^1 e^{-\mu\lambda}(1-\lambda^2)^{n-\frac{1}{2}}\,d\lambda.$$

Here change the order of integration and replace μ by $\mu/(1+\lambda)$, so getting

$$\frac{2^{-n}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_{-1}^1 (1+\lambda)^{m-\frac{1}{2}}(1-\lambda)^{n-\frac{1}{2}}\,d\lambda\int_0^\infty e^{-\mu}\mu^{n-m-1}E\{::z(1+\lambda)^2/\mu^2\}\,d\mu$$

$$=\frac{2^{-m-1}}{\pi\Gamma(n+\frac{1}{2})}\int_{-1}^1 (1+\lambda)^{m-\frac{1}{2}}(1-\lambda)^{n-\frac{1}{2}}E\left\{\frac{n-m}{2},\,\frac{n-m+1}{2}::\tfrac{1}{4}z(1+\lambda)^2\right\}d\lambda,$$

by (2).

Now put $(1+\lambda)=2\mu$ and the expression becomes

$$\frac{2^{n-1}}{\pi\Gamma(n+\frac{1}{2})}\int_0^1 \mu^{m-\frac{1}{2}}(1-\mu)^{n-\frac{1}{2}}E\left(\frac{n-m}{2},\,\frac{n-m+1}{2}::z\mu^2\right)d\mu.$$

On applying (6), with $\alpha=m+\frac{1}{2}$, $\rho=m+n+1$, so that $\rho-\alpha=n+\frac{1}{2}$, the R.H.S. of (7) with $p=q=0$ is obtained. From this the general case can be deduced.

Finally, for (8), substitute from (1) in the L.H.S. and change the order of integration, so getting, if $R(n)>-\frac{1}{2}$,

$$\frac{2^{-n}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_{-1}^1 (1-\lambda^2)^{n-\frac{1}{2}}\,d\lambda\int_0^\infty e^{-\mu(1+\lambda)}\mu^{n-m-1}E(p\,;\,\alpha_r:q\,;\,\rho_s:z\mu^2)d\mu.$$

Here replace μ by $\mu/(1+\lambda)$, apply (5) and get

$$\frac{2^{-n}}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_{-1}^1 (1+\lambda)^{m-\frac{1}{2}}(1-\lambda)^{n-\frac{1}{2}}$$

$$\times\left[\begin{array}{l}-\dfrac{\pi\sqrt{\pi}}{\sin(m-n)\pi}2^{n-m}\\ \qquad\times E\{p\,;\,\alpha_r:\rho_1,\,\dots,\,\rho_q,\,1+\frac{1}{2}m-\frac{1}{2}n,\,\frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n:4z(1+\lambda)^{-2}\}\\ +\dfrac{\pi\sqrt{\pi}}{2\sin(\frac{1}{2}m-\frac{1}{2}n)\pi}\dfrac{z^{\frac{1}{2}m-\frac{1}{2}n}}{(1+\lambda)^{m-n}}\\ \qquad\times E\left\{\begin{matrix}p\,;\,\alpha_r-\frac{1}{2}m+\frac{1}{2}n & :4z(1+\lambda)^{-2}\\ \rho_1-\frac{1}{2}m+\frac{1}{2}n,\,\dots,\,\rho_q-\frac{1}{2}m+\frac{1}{2}n,\,1-\frac{1}{2}m+\frac{1}{2}n,\,\frac{1}{2} & \end{matrix}\right\}\\ +\dfrac{\pi\sqrt{\pi}}{4\cos(\frac{1}{2}m-\frac{1}{2}n)\pi}\dfrac{z^{\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}}}{(1+\lambda)^{m-n-1}}\\ \qquad\times E\left\{\begin{matrix}p\,;\,\alpha_r-\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2} & :4z(1+\lambda)^{-2}\\ \rho_1-\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2},\,\dots,\,\rho_q-\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2},\,\frac{3}{2}-\frac{1}{2}m+\frac{1}{2}n,\,\frac{3}{2} & \end{matrix}\right\}\end{array}\right]d\lambda.$$

Now replace $1+\lambda$ by 2μ and apply (3), so obtaining the R.H.S. of (8).

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University of Glasgow
and
Faculty of Science
Heliopolis University
Abbassia, Cairo

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