Accretion onto Magnetized Neutron Stars

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1 Introduction
Magnetic fields of about 10¹² G have now been measured in several astrophysical objects (Trümper et al. 1978, Wheaton et al. 1979, Mazets et al. 1981). In the case of pulsating X-ray sources, the presence of a companion star indicates that accretion is probably the energy source for the X-rays. Although no companions of γ-ray burst sources have been identified, several current models of these objects (e.g. Fryxell and Woosley 1982) assume them to be slowly accreting neutron stars.

There are many aspects of accretion onto magnetized neutron stars which have been explored by theorists e.g. the existence of an accretion disc (Lamb et al. 1973), its interaction with the stellar magnetic field (Anzer and Börner 1983) and the formation of a channel or funnel as matter approaches the surface (Arons and Lea 1980). One of the most challenging areas is the investigation of the means by which accreting material is decelerated at the surface of the star, releasing its kinetic energy as radiation. The goal of the present paper is to review this aspect of the problem.

In setting up a theoretical model of the deceleration of an accretion flow several different approaches are possible. Thus, radiation pressure may be assumed to play the dominant role in the dynamics, or, conversely, gas pressure could be taken to be more important. Some of these assumptions may apply only in certain regions of the parameter space encountered e.g. radiation pressure should be important only in high-luminosity sources. Others, such as the assumption of the presence of a shock front, may coexist with rival proposals (in this case deceleration by Coulomb collisions), in the absence of a definitive verdict from observations. In section 2 of this paper the equations which lie behind all approaches to the problem are presented in simplified form, and a rough criterion for division into radiation pressure and gas pressure dominated flows is given.

Although the modifications of physical processes which are brought about by the intensely strong magnetic field are substantial, they will be largely ignored in this review. Where they directly affect the models described, a reference to more detailed treatments is made, but the emphasis here is on the macroscopic aspects of accretion, and, especially, the boundary conditions applied.

The discussion falls naturally into two parts: models in which radiation pressure is assumed to play no role at all are discussed in section 3, together with some models in which radiation pressure turns out to play only a minor part. Section 4, on the other hand, deals with models in which radiation pressure dominates in at least some part of the flow. In some of these gas pressure is everywhere negligible, in others it is important immediately behind a shock front.

2 Equations
In order to render the problem of the radiation hydrodynamics of accretion tractable, several assumptions must be made, in addition to the restriction to stationary solutions with an axis of rotational symmetry. Some of these are common to all treatments, and constitute what might be called the 'rules of the game':

1. The magnetic field of the star is assumed to be so strong that all plasma particles are constrained to move along it. This assumption eliminates the need for an equation of momentum conservation in those directions perpendicular to the field.
2. A geometrical form is prescribed for the magnetic field. This may be dipolar structure or, for the sake of simplicity, radial field lines or a uniform field. The latter possibilities are realistic only in close proximity to the surface.
3. Accreting material is assumed to arrive at only a small part of the surface, usually called the 'polar cap' or 'hot spot'.
4. At some distance above the surface the material velocity is equal to that of free fall, and its density is prescribed (usually constant) on those magnetic field lines which cut the polar cap, and zero elsewhere.
5. The infalling plasma is assumed to be fully ionized and is taken to consist solely of hydrogen.

In addition, approximations are made which apply only to some models of accretion. The diffusion approximation for radiative transport falls into this category, and is employed in all those models in which the effects of radiation pressure on the flow are taken into account. Such an approximation can be justified only in regions in which the mean free path of a photon (for scattering or absorption) is much smaller than any other length scale of interest. The radiation field is then described by two quantities. J—the energy density in radiation as measured in the frame in which matter is locally at rest, and F—the energy flux density vector, also measured in the local rest frame of matter. The transfer equation reduces to two equations, which achieve a relatively simple form when expanded to lowest order in the plasma velocity v/c and when derivatives with respect to time are neglected (Castor 1972)

\[ \nabla F + (v \cdot \nabla) J + 4\pi (v \cdot \nabla) J / 3 = C_1 \tag{2.1} \]

\[ \nabla J / 3 = C_2 \tag{2.2} \]

where \( v \), the plasma velocity, is directed along the magnetic field. \( C_1 \) and \( C_2 \) represent, respectively, energy and momentum exchange between the radiation and the plasma, and contain not only angle integrations over the magnetically modified...
scattering and absorption cross-sections but also integrals over frequency which depend on the spectral properties of the radiation.

Complementary to equations (2.1) and (2.2), one can write for the energy and momentum balance of the plasma:

$$\rho (\mathbf{v} \cdot \nabla) \rho + \mathbf{v} p \rho / (\gamma - 1) = - C_1$$  \hspace{1cm} (2.3)

$$\mathbf{p} (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p \rho - \rho g = - C_2, \hspace{1cm} (2.4)$$

where $\rho$ is the density, $\mathbf{g}$ is the acceleration due to gravity, $p$ is the pressure, and the plasma is assumed to behave as an ideal gas with $\gamma$ as the ratio of specific heats. Conduction of heat, in common with other dissipative processes, has been excluded from equation (2.3). However, equation (2.1) contains a heat flow term in addition to those which are appropriate for a fluid of $\gamma = 4/3$. Thus, whereas equations (2.3) and (2.4) possess a singular point at the sound speed of the gas, $v_s = \sqrt{(\gamma p / \rho)}$, equations (2.1) and (2.2) possess a singular point only if $v \cdot \mathbf{F} = 0$. In describing the deceleration of an accretion flow, the initial velocity is always larger than both $v_s$ and the velocity of sound in the radiation fluid, $v_R = \sqrt{(4J / 9\rho)}$. The final velocity should be lower than both. The transition through the velocity $v_R$ is commonly termed a radiative shock or a collisionless shock, even though there is, strictly speaking, no discontinuity when the diffusion approximation is employed. The transition through $v = v'_R$ is called a collisionless shock if the required dissipation cannot occur through two-body Coulomb collisions between particles. Otherwise, the term collisional shock is used. This choice of terminology is unfortunate in two respects. Firstly, the meaning of 'collisionless shock' is ambiguous, and, secondly, the term 'radiative shock' is used in other branches of astrophysics to mean something quite different. In this paper we will attempt to avoid confusion by calling the smoothed out transition through $v_R$ a 'radiation shock' and the transition through $v'_R$ a 'collisionless' or 'collisional' shock.

Specification of the quantities $C_1$ and $C_2$ which occur in equations (2.1), (2.2), (2.3) and (2.4) is a complex problem. Using the diffusion approximation, Riffert (1983, 1984) has performed the required angle integrations in the non-relativistic approximation, and has formulated the integrations over frequency as a 'Rosseland mean'—the appropriate choice when using this approximation. His results for $C_2$ may be expressed in terms of effective opacities $\kappa_\parallel$ and $\kappa_\perp$ for the directions parallel to and perpendicular to the magnetic field respectively. Thus $C_2$ may be written

$$C_2 = \begin{bmatrix} C_{2\parallel} \\ C_{2\perp} \end{bmatrix} = \begin{bmatrix} - \kappa_\parallel \rho \mathbf{F}_\parallel / \sigma \\ - \kappa_\perp \rho \mathbf{F}_\perp / \sigma \end{bmatrix}.$$  \hspace{1cm} (2.5)

where the suffixes $\parallel$ and $\perp$ denote components parallel and perpendicular to $\mathbf{B}$. For comparison, the expression for $C_2$ in an unmagnetized gas with Thomson scattering as the sole opacity is

$$C_2 = - \kappa_\parallel \rho \mathbf{F} / \sigma$$  \hspace{1cm} (2.6)

The effective opacities are functions only of the ratio of the energy of a cyclotron photon $h\omega = (h\mathbf{B}/mc)$ to the temperature characterizing the radiation spectrum (here assumed to be a Wien spectrum). In full thermodynamic equilibrium this temperature equals the radiation temperature $T_R$ defined by the expression

$$\frac{1}{2} \sigma T^4_R = \mathcal{J}, \hspace{1cm} (2.7)$$

where $\sigma$ is the Stefan-Boltzmann constant. The opacities are displayed in Figure 1.

Other expressions for $C_2$ have been used by Wang and Frank (1981), who employed an approximate representation of the scattering cross-section valid at frequencies much lower than the cyclotron frequency and omitted the effects of line photons. Braun and Yahel (1984) also give prescriptions for $C_2$ derived from simple averaging schemes.

Figure 1 The effective opacities for momentum transfer in a strongly magnetized plasma: $\chi_\parallel$ (parallel to the field—$x_\parallel$ in text) and $\chi_\perp$ (perpendicular to the field—$x_\perp$ in text) in units of the Thomson opacity $\chi_\text{th} (\approx 0.4 \text{ cm}^2 \text{ g}^{-1})$. The abscissa is the ratio of the energy of a photon of the cyclotron frequency $\omega_\text{c}$ to the temperature $T$ characterizing the photon spectrum ($k$ is Boltzmann's constant) (Riffert 1983).

Riffert (1983) has also calculated $C_1$ using the diffusion approximation. However, it is usual to make a further assumption when using this approximation, which eliminates the need for an explicit expression for $C_1$. This is the assumption that gas pressure is everywhere unimportant compared to radiation pressure, resulting in a reduction by one of the number of unknown functions in the system (2.1) to (2.4); equations (2.1) and (2.3) may then be combined to eliminate $C_1$.

The usual justification for this assumption rests on an estimate of the time taken for plasma and radiation to relax into a state of equilibrium by means of Compton scattering. Even allowing only for photons of frequency $\omega < \Omega$, for which energy exchange is less efficient by a factor $7^{1/2}$ compared to the non-magnetic case (Basko and Sunyaev 1975), one arrives at the relatively short relaxation timescale

$$\tau_\text{Compton} \approx \frac{15}{16} \frac{m \sigma (\sigma_\nu v)^{-1}}{\mathcal{J}} = 2.8 \times 10^{-7} [\mathcal{T}(\text{kev})]^{-4} \text{ seconds}$$
where \( \sigma_T \) is the Thomson cross-section. The next step in the argument is to compare gas and radiation pressure using typical densities and, more importantly, typical radiation temperatures defined according to equation (2.7). However, Compton scattering is a process which conserves photon number; the equilibrium distribution of photons is, therefore, a Bose-Einstein distribution with a non-zero chemical potential. In this case the energy density in radiation \( J \) is not related to the characteristic temperature of the equilibrium distribution by equation (2.7). If, for example, there is a dearth of photons, then the equilibrium temperature of the Bose-Einstein distribution, and, consequently, of the plasma, will be significantly higher than the temperature defined by equation (2.7). Nevertheless, the temperature required for gas pressure to be important is so high (~10^{14} K) that it seems the only likely location for this effect is immediately behind a collisionless shock. The thickness of such a layer may then be controlled by the ion-electron relaxation time rather than the plasma-radiation relaxation time.

A knowledge of \( C_1 \) is, however, important for those models in which the effects of radiation pressure can be neglected. Roughly speaking, the luminosity range in question can be estimated as follows: in an optically thin column, radiation may be expected to escape after travelling a distance through the plasma about equal to the column radius \( a \). If, on travelling this distance, the radiation fails to impart sufficient momentum to the plasma to off-set the momentum of free fall, then the effects of radiation pressure may be neglected. One then obtains, using (2.5), a condition on the luminosity \( L \):

\[
L < \frac{\dot{M} G M}{c} \frac{a}{R}, \tag{2.8}
\]

where \( M \) and \( R \) are the stellar mass and radius (Básko and Sunyaev 1976). At this juncture it is convenient to introduce two dimensionless parameters: the free fall velocity at the surface in units of \( c \),

\[
\delta := \left( \frac{2GM}{Ra^2} \right)^{\frac{1}{2}} \tag{2.9}
\]

and a quantity specifying the accretion rate,

\[
\varepsilon := |\sigma / \kappa_\parallel \phi_a| \tag{2.10}
\]

where \( \phi \) is the mass-flux density at the surface. The quantity \( (\delta)^{-1} \) then represents the optical depth across an accretion column in which matter is freely falling at the stellar surface, and, identifying the accreted power \( G M M / R \) with the luminosity, equation (2.8) may be written

\[
\varepsilon > \pi \tag{2.11}
\]

(for a polar cap of area \( \pi a^2 \)). Since \( \delta \equiv 1 / 2 \) for a neutron star, the condition for neglect of radiation pressure is nearly identical with that for optical thinness of an accretion column containing freely falling plasma. Of course, such a discussion ignores several more subtle points. Resonant (line) photons, for example, will almost certainly encounter an optically thick column even at very low accretion rates. Furthermore, the equation of continuity

\[
\nabla \cdot (\rho \mathbf{v}) = 0. \tag{2.16}
\]

Equation (2.15) is understood to apply only in the direction parallel to the magnetic field. Equation (2.14) can be used to eliminate \( F \), and, once the geometry of the accretion column is prescribed, equation (2.16) can be integrated to eliminate \( \rho \).
in terms of the accretion flux at the surface. In one of the geometries used by Basko and Sunyaev (1976), for example, the magnetic field lines are dipolar, and accretion occurs in a thin sheet of thickness $d$. This sheet forms a funnel, rotationally symmetric about the dipole axis, and of radius $a > d$. In terms of distance from the centre of the star $r$, one has, close to the poles,

$$d(r) = d(R) \times (r/R)^{3/2}$$

and

$$a(r) = a(R) \times (r/R)^{3/2}.$$  

Thus, using the approximation

$$\nabla \psi \approx \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \psi r),$$  

valid for $d < < a$ for any scalar function $\psi(r)$, the equation of continuity integrates to

$$\rho \nu r^3 = \text{constant} =: \phi r^3$$

where $R$ is the stellar radius.

In cylindrical geometry,

$$\rho \nu = \text{constant} =: \phi$$

and since this choice displays most of the interesting properties, without some of the complications of the dipole or radial geometries, it has been adopted by several workers in the field.

A further simplification of the basic equations is their reduction to one space dimension. Inoue (1975) and Basko and Sunyaev (1976) achieved this by writing equations ‘averaged’ over the width of the column. This technique has also recently been used by Braun and Yahel (1984). In it, the radiation flux vector is split into its components parallel ($F_\parallel$) and perpendicular ($F_\perp$) to the field lines. In dipole geometry (2.13) then becomes

$$\frac{1}{r^3} \frac{\partial}{\partial r} \left[ r^3 \frac{\partial}{\partial r} (r^3 \nu) + \nu \frac{\partial \nu}{\partial r} + \frac{4J}{3r^2} \frac{\partial}{\partial r} (r^3 \nu) + \nabla \cdot F_\perp \right] = 0.$$  

(2.19)

The only term involving the coordinate direction perpendicular to the field lines is $\nabla \cdot F_\perp$, which represents the leakage of radiation out of the sides of the column. In one dimensional treatments this term must be estimated from equation (2.14), but first the energy sources i.e. the other terms in (2.19) must be allocated, in an ad hoc manner, a dependence on the coordinate perpendicular to the field lines, which, in this case, can be denoted by $x$. Thus, following Basko and Sunyaev (1976) the sources may be assumed either to be spread evenly across the column, or to be concentrated along a plane (for accretion in a thin ‘wall’) or a line (for a filled column). In the case of a thin wall, the accretion column can be represented locally by a plane parallel layer. Assuming that energy is released uniformly across this layer one has

$$\nabla \cdot F_\perp = \text{constant} =: A.$$  

Using (2.14), and applying the boundary condition $cJ/2 = |F_\perp|$ at each edge of the sheet leads to the expression

$$J = 2A \left[ d + 3 \nu (d^2 - x^2) / a \right] / \sigma,$$  

and, hence, on taking the values of $\nu$ and $J$ at the centre of the sheet, equation (2.19) becomes

$$\frac{1}{r^3} \frac{d}{dr} \left[ r^3 \frac{dJ}{dr} + \nu \frac{dJ}{dr} + \frac{4J}{3r^2} \frac{d}{dr} (r^3 \nu) + \frac{8\sigma J}{3\nu d^2} \right] = 0$$

(2.20)

Of course, it is inconsistent to assume that, on the one hand, energy sources are uniform across the column and, on the other, that $J$ varies with $x$ according to equation (2.20). However, one may hope to learn something by solving the more tractable problem resulting from the simplification to one dimension.

In the case of a filled accretion column, a similar calculation leads to the following equation (in spherical geometry)

$$\frac{e \nu}{3\nu^2} \frac{d}{dr} \left[ \nu r^4 \frac{dJ}{dr} + \nu \frac{dJ}{dr} + \frac{4J}{3r^2} \frac{d}{dr} (r^2 \nu) + \frac{4e\nu J}{3R} \right] = 0$$

(2.22)

where $a$ is the radius of the cross-section of the accretion column at $r = R$. The equation of motion is

$$\nu \frac{d\nu}{dr} - \frac{GM}{r^2} = \frac{\nu}{\sigma} J_\parallel = 0$$

(2.23)

with the radial component of the energy flux in radiation, $F_\parallel$, given by

$$F_\parallel = \frac{e \nu r^2 a}{3\nu^2} \frac{d\nu}{dr}$$

(2.24)

and the integrated equation of continuity has been used to eliminate the density.

Having assembled the basic equations used in modelling the accretion process close to the neutron star surface, we are now in a position to discuss the solutions which may be found in the literature. Especially in the case of radiation dominated accretion, it turns out not to be the details of the coefficients used in the equations which distinguishes the models from each other, but rather the boundary conditions which are applied.

3 Models without Radiation Pressure

As discussed above, it is at least plausible that radiation released at the foot of an accretion column should have no effect on the deceleration of the flow, provided that the accretion rate is such that $e > 1$. In this case equations (2.3) and (2.4) govern the flow, with $C_2 = 0$, and $C_1$ calculated as indicated above. However, any solution of these equations which is intended to represent accretion must contain a collisionless or collisional shock, and a typical post-shock temperature is of the order of $10^{12}$ K. Electrons held at such a temperature would produce a copious flux of $\gamma$-rays. Thus, in attempting to model X-ray pulsars one is driven to the assumption that the electron temperature either does not reach this value, or does so only in a very thin transition
layer. The latter assumption underlies treatments by Inoue (1975) and Kirk (1984) in which radiation pressure is assumed to play an important role in the post-shock plasma. The former assumption led Langer and Rappaport (1982) to extend equations (2.3) and (2.4) to allow for a post shock plasma of differing ion and electron temperatures.

In their treatment, each plasma component is represented by a fluid— with $\gamma = 5/3$ for the ions, and $\gamma = 3$ for the electrons, taking into account the assumption that the electrons may move only along the magnetic field lines. To preserve charge neutrality, the number densities of electrons and ions (protons) are required to be equal; a current-free plasma is also demanded by setting the velocities of the two species equal. As a result, three equations replace (2.3) and (2.4), and there is an additional coupling term describing energy exchange between the two species. Assuming that this term does not become singular at the shock front, one can write jump conditions in which each species thermalizes its own kinetic energy. The result is a highly non-isothermal post-shock plasma, with electrons at about $10^9$ K supported by ions at about $10^{12}$ K.

Langer and Rappaport (1982) assume the coupling between the electron and ion fluids to occur via Coulomb collisions, for which they propose a rate modified by the strong magnetic field. The three equations can then be integrated as an initial value problem, using values of the ion and electron temperatures and the plasma velocity obtained by solving the jump conditions for a strong shock. Within a finite distance the post-shock solution cools to zero temperature and zero velocity, and this point is interpreted as the stellar surface. An example of this behaviour is shown in Figure 2.

Instead of using two sets of fluid equations and postulating the existence of a collisionless shock, several authors have preferred to adopt an approach in which the ions in the accretion flow are described as particles; a technique used in the problem of spherically symmetric accretion by Alme and Wilson (1973). The picture is roughly as follows: in free-fall, the plasma has negligible pressure, and its kinetic energy is mainly carried by the ions. On encountering the upper edge of the atmosphere of the neutron star, the electrons interact strongly with atmospheric electrons, probably through a saturated two-stream instability. Within a short distance, accreting electrons and atmospheric electrons are indistinguishable. However, this process has a negligible effect on the accreting ions, which continue to penetrate the atmosphere. The build up of an electrostatic field is prevented by the formation of a small return current in the plasma—the electron gas drifts slowly downwards through the atmospheric ions—and the only process capable of decelerating the fast accreting protons is that of collisions.

As a result of these considerations, four equations arise; two describe accreting protons:

$$\frac{dE}{dr} = Q_1$$
$$\frac{dP}{dr} = Q_2$$

where $E$ is the average energy of the accreting protons and $P$ their average momentum; $Q_1$ and $Q_2$ represent the loss rates of these quantities to the ambient medium. If all accreting protons are represented by just a single particle, $E$ and $P$ are simply related to each other ($E = P^2 / 2m_p$). However, a small spread in the initial velocities of a cloud of accreting particles will be amplified during the deceleration process, so that the average quantities $E$ and $P$ are not simply related. Early calculations of $Q_1$ and $Q_2$ used the single particle approach, and, more importantly, considered Coulomb scattering of protons off cold electrons (Basko and Sunyaev 1975b, Pavlov and Yakovlev 1976). These results were later modified by including the thermal motion of the electrons by Kirk and Galloway (1982), who also abandoned the single particle approach in favour of a full solution of the evolution of the distribution function of accreting protons.

![Figure 2](https://www.cambridge.org/core/terms). Langer and Rappaport (1982), for a magnetic field of $5 \times 10^{12}$ G and an accretion rate of $10^{15}$ g s$^{-1}$ corresponding to $e = 19$ (see eq. 2.10). The scales for the electron temperature $T_e$ and the ion temperature $T_i$ are in Kelvins, for the flow velocity $v$ in cm s$^{-1}$. The stellar surface corresponds to $-3$ on the abscissa; $r$ being the distance from the centre of the star and $r_*$ its radius. The upper abscissa shows the fraction $L(r)/L$ of the total luminosity radiated above the position $r$. 

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The two remaining equations describe the effects of deceleration on the neutron star atmosphere:

\[ C_1 + Q_1 \phi / M_p = 0 \]  

(3.3)

and

\[ \frac{d\phi}{dy} - \rho g = \frac{Q_2 \phi}{M_p}. \]  

(3.4)

Here, \( \phi \) is the (constant) mass flux density and \( M_p \) is the proton mass; (3.3) and (3.4) represent the specialization to one dimension of equations (2.3) and (2.4) applied to a stationary atmosphere.

Equations similar to (3.1)-(3.4) are solved by Wang and Frank (1981), although they include a radiation pressure term in (3.4). Deceleration is approximated by a single, constant scattering cross-section representing the strong interaction \( p - p \) collision, and the equations integrated as an initial value problem by specifying the velocity of accreting particles at the top of the column and the density of the atmosphere at this point. After a finite distance, the particle velocity drops to zero. At this same point the temperature of the background also goes to zero, since equation (3.3) does not provide for any thermal input to the atmosphere other than from accretion.

In a more detailed calculation, Mészáros et al. (1983) and Harding et al. (1984) combine radiation transport schemes designed for inhomogeneous strongly magnetized plasmas with the numerical solutions found by Kirk and Galloway (1982) for the deceleration of test particles in strongly magnetized plasmas. Despite the complexity of the ingredients, the equations solved are still essentially (3.1)-(3.4). As before, the boundary conditions at the top of the atmosphere are the infall velocity and the background density. However, \( C_1 \) is determined by solving the radiation transport problem in the two-stream approximation. The two boundary conditions required for this are free escape of radiation at the top of the atmosphere, and complete reflection at the bottom. As a consequence, the temperature of the background plasma does not tend to zero at the point at which the accreting protons come to rest.

Although cumbersome, this approach does have the advantage of being able to predict the continuum spectra and beaming patterns of X-ray pulsars. The structure of the layers found in this model is shown in Figure 3.

Braun and Yahel (1984) also solve equations (3.1)-(3.4). They use the diffusion approximation for radiation transport, and include the radiation pressure term in (3.4). For \( Q_1 \), the values given by Pavlov and Yakovlev (1976) are used. Unfortunately, these values apply only in the case of a cold plasma, and are substantially in error in the present situation. In addition, the important ram pressure term \( Q_2 \) is omitted. Nevertheless, a solution is obtained in which the particles are completely decelerated within a finite distance; at this point the energy flux in radiation also vanishes.

Table 1 summarizes the four models discussed in this section. Clearly, both the collisionless shock model and the Coulomb deceleration models rest on unproven assertions about the energy and momentum exchange between electrons and ions. In the former case such exchange is assumed not to occur within the shock, whereas, in the latter, the interaction at the upper boundary of the atmosphere must involve a small energy transfer to or from the electrons. The precise assumption in this case appears in the rather arbitrary specification of the density at this point.

Inspection of Figures 2 and 3 reveals that electrons have a much higher temperature in the shock model than in the Coulomb deceleration model. This, coupled with the different beaming patterns associated with column-like (for the shock model) and slab-like (for the Coulomb deceleration model) emission regions should enable comparison with observation (White et al. 1983).

4 Models which include Radiation Pressure

The first solution of equations (2.13)-(2.16) was presented by Davidson (1973), who argued that the gravity term \( g \) could be neglected in high luminosity sources. This has the advantage that the equation of motion (2.15) can be integrated analytically, leaving only one second order partial differential equation, which is solved numerically. The boundary conditions at the stellar surface are those of zero velocity and zero radiation flux. Along the sides, where radiation should escape freely into vacuum, \( J \) is set equal to zero, as it is at the top of the column. Also at the top, the velocity is set to that of free-fall. Although this solution displays some of the features expected of a real accretion column—in particular a pronounced mound of a low
Table 1
Gas Pressure Dominated Accretion Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Type</th>
<th>Boundary Conditions</th>
<th>Microphysics Input/Radiation Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>upper(^1)</td>
<td>lower(^2)</td>
</tr>
<tr>
<td>Wang and Frank (1981)</td>
<td>Coulomb deceleration</td>
<td>(v, J) and (F) fixed, (n = n_{\text{acc}})</td>
<td>(v = 0) [(F = 0)]</td>
</tr>
<tr>
<td>Langer and Rappaport (1982)</td>
<td>collisionless shock</td>
<td>pre-shock: (p = 0), (v = \text{free fall})</td>
<td>(v = 0) [(F = 0)]</td>
</tr>
<tr>
<td>Mézáros (et\ al) (1983)</td>
<td>Coulomb deceleration</td>
<td>(v = \text{free fall}) (\quad n = n_{\text{acc}}) (\quad F = aJ/2)</td>
<td>(v = 0)</td>
</tr>
<tr>
<td>Harding (et\ al) (1984)</td>
<td>Coulomb deceleration</td>
<td>(v = \text{free fall}) (\quad n = n_{\text{acc}}) (\quad F = aJ/2)</td>
<td>(v = 0) [(F = 0)]</td>
</tr>
</tbody>
</table>

\(^1\) \(n\) is the atmospheric density, \(n_{\text{acc}}\) the density of accreting matter, the remaining symbols are defined in section 2.

\(^2\) [ ] denotes 'implied by the other boundary conditions'.

velocity gas lying on the surface—it nevertheless suffers from several inadequacies. The neglect of gravity, for example, means that the structure of the gas mound cannot be described accurately, since the gravity term is important where the gas velocity is below that of sound in a radiation dominated plasma.

Following this paper, two one-dimensional treatments appeared: Inoue (1975) and Basko and Sunyaev (1976) each based on equations (2.22)-(2.24).

Three boundary conditions are required to specify the solution to these equations. At the maximum value of \(r\) to be considered, it is natural to demand as did Davidson (1973) that no deceleration has occurred and, therefore, to require the velocity at this point to equal that of free-fall. On the other hand, the velocity at the base of the column should be such that all kinetic energy has been lost. In a stationary solution the condition \(v(R) = 0\) implies an infinite density. Nevertheless, such solutions exist in flows dominated by gas pressure (e.g. Langer and Rappaport 1982) and it seems natural to look for similar ones for radiation pressure dominated flows. Another boundary condition should derive from the thermal properties of the stellar surface—the energy which flows into or out of the star when the polar cap is held at a given temperature. Ignorance of this functional relationship is, unfortunately, almost complete. The most expedient solution is to require that the energy flux into the star be zero irrespective of the temperature at the base of the column. Although this works well for low luminosities, it turns out to be an impossible demand at high luminosities, as we shall presently see.

At first glance, therefore, it would seem that appropriate boundary conditions are

\[ v(r_1) = \sqrt{2GM/r_1} \]  

(4.1)

at the top of the column \((r = r_1)\), and

\[ v(R) = 0 \]  

(4.2)

and

\[ F_\parallel(R) = 0 \]  

(4.3)

at the base. However, equation (2.23) permits no solution with the property \(v \to 0\) and \(F_\parallel \to 0\) at the lower boundary, unless the gravitational term is neglected completely. In this respect the behaviour of a radiation dominated plasma differs from that of a gas pressure dominated one because, in the former, the presence of a pressure gradient requires a finite energy flux. Physically, equation (4.2) implies an infinite density, so that gas pressure must become important as the base of the column is approached. Equation (2.23) does not describe such a solution.

An alternative approach is to place the lower boundary just above the position at which one expects (4.2) to be satisfied, in a region still dominated by radiation pressure. The condition
of zero energy flux may be extended to include advected energy at this point
\[ \frac{4}{3} \nu \sigma + F_1 = 0 \quad (4.4) \]
and solutions sought in which the velocity is much smaller than the sound speed. Of course, the dropping of condition (4.2) means that another boundary condition must be found, the choice of which depends on which further approximations are applied to equations (2.22-2.24).

Basko and Sunyaev (1976) present solutions to these equations in two approximations. In the first the gravity term is dropped, yielding a one-dimensional version of the problem treated by Davidson (1973). However, they do not apply condition (4.2), but instead demand that the radiation energy density go to zero far above the surface. The behaviour of the solutions is found to depend on the accretion rate. For \( \epsilon > 1 \) (see equation 2.10 for the definition of \( \epsilon \)) the solutions exhibit high velocity at the base of the column indicating that radiation pressure alone cannot decelerate these flows, in agreement with the arguments presented in section 2. For \( \epsilon = 1 \) a solution is obtained in which \( v(R) = 0 \) — a singular case reminiscent of Davidson’s solution. For \( \epsilon < 1 \) deceleration of the flow starts to occur higher above the surface, and tends to be confined to a relatively small range in \( r \), resembling a radiation shock, above an almost static sinking zone. To examine this zone more carefully, the gravity term must be included. On the other hand, inertia is unimportant in a subsonic zone, so that equation (2.23) can again be simplified, this time by dropping the term \( \nu \nu / dr \). The result is two first order equations, because the divergence of the radial flux vanishes identically. In this approximation the upper boundary condition can be chosen to be consistent with the subsonic side of a strong radiation shock i.e.
\[ v = - \frac{\nu \sigma}{r}, \quad (4.5) \]
where \( \nu \sigma : = (2GM/r)^{1/2} \), and
\[ \frac{4}{3} v \sigma = \frac{\phi \sigma_0^2}{r} \left( \frac{v^2}{2} - \frac{\nu \sigma}{2} \right) \]
\[ = \frac{\phi \sigma_0^2}{2r^2} \quad (4.6) \]
(Zel’dovich and Raizer 1967). Both conditions are needed to specify the solution, since the position at which they are to be satisfied is not known a priori. Applying (4.4), the solution is given by
\[ \frac{4}{3} v \sigma = e^{-cr/R} \frac{GM}{R} \left\{ - \frac{e^{-cr/R}}{c^2} - \frac{\phi E_2(\epsilon)}{\epsilon r} E_2(\epsilon r/R) \right\} \quad (4.7) \]
where \( E_2(x) \) is an exponential integral (Abramowitz and Stegun 1972) and
\[ J = A \left( \frac{4}{3} v \sigma e^{-cr/R} \right)^4 \quad (4.8) \]
with \( A \) a constant.

Here the difficulty with the boundary condition (4.4) is apparent. The radius at which the shock front condition (4.6) is fulfilled is finite only when \( \epsilon \) fulfills the restriction
\[ \epsilon < ae^{-e}/\nu E_2(\epsilon). \quad (4.9) \]
For example, for \( a/R = 1/10 \), a solution exists satisfying (4.4) and (4.6) only for accretion rates such that \( \epsilon > 2.2 \). However, a radiation shock and sinking zone cannot form at such a low accretion rate, as discussed above (eq. 2.11). In physical terms, radiation losses from the sides of the column become less efficient as the optical depth across the column builds up. At a critical accretion rate, the column is no longer able to radiate away all the energy which is accreted, and no stationary solution can be found. Of course, this statement rests upon the form of the loss terms used in the one dimensional approximation, and does not necessarily hold for more realistic two-dimensional (axisymmetric) models.

To circumvent this problem, Basko and Sunyaev (1976) proposed a boundary condition at the surface which sets the radiation pressure inside the column equal to the confining pressure of the magnetic field. In this way a significant fraction of the energy of accretion is allowed to flow into the star itself or, more precisely, out of the region in which a solution of the equations is sought. If the magnetic field is sufficiently strong (\( > 2 \times 10^{13} \) G) densities are achieved such that neutrino losses may account for this energy, whereas if the magnetic field is weak, it seems unlikely that a stationary accretion pattern can be maintained. Assuming a stationary solution does exist, the accretion column consists of an extended subsonic section (sinking zone) underneath a radiation shock. The height of the shock increases with accretion rate—in contrast with the behaviour of gas pressure dominated flows—and the total energy

Figure 4 The solution found by Basko and Sunyaev (1976) for the ‘sinking zone’ beneath a radiation shock. The accretion rate (for a filled axisymmetric column of radius \( a \)) corresponds to \( \epsilon = a/R \) (see equation 2.10) where \( R \) is the stellar radius. \( T_{\text{int}} \) is the radiation temperature on the axis of the column defined according to eq. (2.7). \( T_{\text{int}} \) is the temperature which would be required of the side of the column if it were to radiate as a black body, and so represents a lower limit to the actual temperature. \( r \) is the distance from the centre of the star, and the radiation shock is positioned at \( r = r_s \).
loss from the sides of the column tends to a limiting value. Expressions for this limit have been given by Basko and Sunyaev (1976) and depend sensitively on the geometry of the column. In Figure 4, the temperature at the centre of the column is shown as a function of distance from the surface, for a luminosity close to this limit.

Inoue (1975) discussed an approximate solution to the one-dimensional equations, assuming a collisionless shock at the upper boundary of the column as opposed to the radiation shock of Basko and Sunyaev. It was also assumed that matter and radiation come into equilibrium beneath a narrow transition layer immediately below the shock. The remaining boundary conditions are the same as those of Basko and Sunyaev.

Wang and Frank (1982) produced the first accretion model to include gravity in two spatial dimensions. Specifying an axisymmetric accretion column, they numerically integrated equations similar to (2.13) to (2.16) using magnetic field dependent opacities. In addition a ‘flux-limited’ form of the diffusion equation was used. This technique has the advantage of removing obviously unphysical effects which arise when the diffusion equation is applied to an optically thin plasma. It does not, however, remove the problem of finding a suitable boundary condition for the diffusion equation in a region of decreasing plasma density, such as the top of the accretion column.

Two prescriptions have been used to set the boundary conditions to this equation. The first, adopted by Davidson (1973), is to put the energy density in radiation, $J$, equal to zero. If the energy flux across the boundary then turns out to be negligibly small, this condition is a reasonable one. Such would be the case at the top of an accretion column when the freely falling plasma above this point is so optically thick that it prevents any escape of radiation. Basko and Sunyaev (1976) also employed this condition implicitly in using the jump conditions across the radiation shock at the top of their column. However, as the shock moves nearer and nearer to the stellar surface when the accretion rate is increased, the optical depth of the pre-shock plasma above it decreases rapidly and the upper boundary condition becomes inappropriate. The second prescription is based on the Marshak boundary condition (e.g. Pomraning 1973), applied to an interface with vacuum on one side and plasma on the other, and relates $J$ to the component of its gradient normal to the surface:

$$ J = -\frac{2}{5c_p n_w}. $$(4.10)

Application of this condition at the sides of the accretion column where plasma is confined by the magnetic field is relatively uncomplicated. However, at the upper boundary there is no well-defined interface; rather the plasma becomes optically thin at some surface. Braun and Yahel (1984) have used a condition similar to (4.10) in their numerical search for a one-dimensional solution, and Wang and Frank (1982) have applied it at a surface of constant $r$ in their two-dimensional solution. Unfortunately, no attempt has yet been made to apply the condition at the surface at which the plasma becomes optically thin; Wang and Frank placed their upper surface well above the deceleration zone and thus, in optically thick cases, their use of (4.10) is for practical purposes the same as that of setting $J$ to zero.

Several interesting results emerge from Wang and Frank’s work, despite the rather modest numerical means at their disposal. The mound of subsonic plasma first described by Davidson (1973) is obtained at the base of the column for accretion rates such that $\epsilon \approx 1$. At lower accretion rates, radiation pressure is unable to decelerate the flow substantially, in agreement with the conclusions of Basko and Sunyaev (1976). However, at higher accretion rates a structure similar to a radiation shock appears, and moves away from the surface as the accretion rate is increased. Although this is qualitatively the same behaviour as that predicted by Basko and Sunyaev, it is important to note that it is obtained by applying the boundary condition (4.4) at the surface. The implication is that it is not essential that the star absorb any energy at accretion rates such that $\epsilon < 1$, thus refuting the conclusion drawn from the one-dimensional solution. The question of the existence of an upper limit to the luminosity could not, however, be answered by Wang and Frank owing to their limited numerical resources.

Recently, Braun and Yahel (1984) have extended the one-dimensional equations to include gas pressure and the possibility of differing ion and electron temperatures. The boundary condition (4.10) is applied at the top of the column, and a collisionless shock is specified to occur at the same point. Braun and Yahel fail to find an acceptable solution under these conditions and conclude that stationary solutions exist only for deceleration by Coulomb collisions, as described in section 2. However, it is possible that the placement of the collisionless shock at the position at which the boundary condition (4.10) is applied could result in an unphysical constraint on the system; further investigation of this point would be desirable.

![Figure 5 The location of the collisionless shock front in the 2-D accretion model of Kirk (1984). Z and R are cylindrical coordinates; the column is assumed to be axisymmetric. Distances are in units of the column radius, equal in this case to one fifth of the stellar radius. The accretion rate corresponds to $\epsilon = 3$ (see eq. 2.10). The three curves parameterized by $k$ represent three possible choices of the lower boundary condition, $k_{max}$ denoting the choice for which the shock rises highest above the surface.](https://www.cambridge.org/core/fig/5/454-kirk-proc-asa-5-4-1984)
The idea of combining the existence of a collisionless shock with that of an extended sinking zone beneath it in which radiation pressure dominates has been investigated recently in two dimensions (Kirk 1984). In this model it is once more assumed that there exists a thin layer behind the collisionless shock in which plasma and radiation are not in equilibrium with each other. At some undetermined point within this layer the plasma becomes optically thick. Below the layer equations (2.13-2.16) are solved in two dimensions, using cylindrical geometry. Because the plasma becomes optically thick at an undetermined point within a region where the energy density in radiation varies rapidly, it is not possible to apply the Marshak boundary condition. Instead, jump conditions are derived relating pre-shock values of the plasma velocity and radiation flux to values of the velocity and radiation energy density behind the non-equilibrium layer. In this way the extra constraint implied in the procedure used by Braun and Yahel (1984) is avoided. At the bottom of the column, condition (4.4) is applied, and the velocity is required to have a particular functional dependence on position at this boundary, in order that stationary analytic solutions may be obtained.

Results of this investigation are shown in Figure 5 where the position of the collisionless shock front is shown for various

Table 2
Radiation Pressure Dominated Accretion Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Dimension</th>
<th>Analytic/ Numerical</th>
<th>Boundary Conditions</th>
<th>Range of Validity / Salient Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Davidson (1973)</td>
<td>2-D</td>
<td>N</td>
<td>( \mathcal{J} = 0 ) (fixed)</td>
<td>( \mathcal{v} = 0 ), ( \mathcal{F} = 0 ) ( \varepsilon &lt; 1 ) mound formed</td>
</tr>
<tr>
<td>Inoue (1975)</td>
<td>1-D</td>
<td>A</td>
<td>collisionless shock (adiabatic) ( ^2 )</td>
<td>( \mathcal{J}/3 = \text{magnetic pressure} ) ( \varepsilon &lt; 1 ) approximate solution found</td>
</tr>
<tr>
<td>Basko and Sunyaev (1976)</td>
<td>1-D</td>
<td>A</td>
<td>radiation shock (adiabatic) ( ^2 )</td>
<td>( \mathcal{J}/3 = \text{magnetic pressure} ) ( \varepsilon &lt; 1 ) limiting luminosity proposed</td>
</tr>
<tr>
<td>Wang and Frank (1981)</td>
<td>2-D</td>
<td>N</td>
<td>( \mathcal{F} = \sigma J/2 ) ( \mathcal{v} = \text{free fall} ) (fixed)</td>
<td>zero flux of energy ( \varepsilon &lt; 1 ) mound formed</td>
</tr>
<tr>
<td>Braun and Yahel (1984)</td>
<td>1-D</td>
<td>N</td>
<td>collisionless shock ( \mathcal{F} = \sigma J/2 )</td>
<td>zero flux of energy, ( \mathcal{v} = 0 ) gas pressure included, electron temperature ( \neq ) ion temperature, no solution found</td>
</tr>
<tr>
<td>Kirk (1984)</td>
<td>2-D</td>
<td>A</td>
<td>collisionless shock (free)</td>
<td>zero flux of energy, fixed ( \mathcal{v} ) ( \varepsilon &gt; 1 ) mound formed</td>
</tr>
</tbody>
</table>

\( ^1 \) Notation as in sections 2 and 4.
\( ^2 \) 'Adiabatic' means no radiation losses from the shock.
discussed concentrating on the dynamics of the flow near the stellar surface. It is interesting to note that a mound of subsonic gas is obtained even though the solution is valid only for low accretion rates, such that $c > 1$. The run of temperature with height is shown in Figure 6, which may be compared with Basko and Sunyaev's solution (Figure 4).

Table 2 summarizes the six models discussed in this section. It is clear from this table that there are still unresolved areas of understanding. In particular, a numerical study of the problem of the existence of a limiting luminosity remains to be done. Also, further work (probably numerical) on the two-dimensional structure of low luminosity accretion, including the effects of gas pressure, is desirable, as well as the extension to time-dependent models. As yet, comparison of the predictions of these models with observation seems even more distant than in the case of the gas pressure dominated accretion models of section 3.

5 Conclusion
In this paper, models of accretion onto neutron stars have been discussed concentrating on the dynamics of the flow near the stellar surface. Ultimately, these models should reach the level of sophistication which enables them to predict the spectrum of radiation emitted by an accretion flow as well as the beam shape; thus permitting detailed comparison with observation. That such a goal has not yet been achieved reflects the complicated nature of the underlying physical processes—complications largely brought about by the strength of the magnetic field.

It is probably fair to state that those models in which radiation pressure is neglected have been able to incorporate more of the magnetic modifications than have the others. Mészáros et al. (1983) and Harding et al. (1984) come closest to achieving the goal mentioned above by producing predictions of the beam shape and of the spectrum below the cyclotron frequency, although their treatment cannot yield information about the spectrum in the vicinity of the cyclotron line itself. However, it is not completely clear that this model and the others discussed in section 3 are consistent in neglecting radiation pressure (Braun and Yahel 1984). Furthermore, it seems that an important physical process has not yet been taken account of in the calculations of the rate of cooling of the plasma (Kirk et al. 1984). Characteristic properties, such as the increase of the height of the subsonic zone with decreasing luminosity in the model of Langer and Rappaport (1982)—properties which one might hope to trace in the observations—will probably be changed substantially by the inclusion of the new effect.

Radiation pressure provides effects which are intrinsically difficult, quite apart from their magnetic modifications. Indeed, it is only recently that the coupled set of the hydrodynamic equations and the radiative transport equation has been consistently formulated for a strongly magnetized plasma (Riffert 1983). Although the investigations performed without magnetic effects provide useful indications of which sets of boundary conditions may be applicable, they cannot attempt to predict beam shapes or spectra. Nevertheless, there are still basic questions—such as the existence of a limiting luminosity—which could be addressed by fairly simple models, and it is certainly desirable to have a thorough understanding of these before embarking on the inevitably expensive numerical quest for a solution of the full problem.