On the Topology of Certain Algebraic Threefold Loci

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Introduction.

The study of the topological properties of algebraic surfaces, considered as continua of four real dimensions, has thrown much light on the theory of the birational invariants of such loci[1]. The results obtained for surfaces have been generalised to varieties of higher dimension by Hodge[2], and, particularly, by Lefschetz[3]. Apart from this, little seems to be known about the general topological properties of algebraic loci of three (or more) dimensions, the detailed study of which seems to present considerable difficulty. In particular, apart from the general theorems of Lefschetz, nothing seems to be known about the cycles of three dimensions of an algebraic V_3. The object of the present paper is to study these cycles on certain quite special V_3, in the hope that some insight may be gained into the general theory.

In § 1 we recall a theorem which enables the three-dimensional Betti number of an algebraic V_3 to be calculated in terms of other characters. In § 2 we examine the homology characters of a fairly extensive class of *rational* V_3, and describe the nature of the 3-cycles on them. The subsequent sections are devoted to a study of the general cubic primal in [4]. We show that any 3-cycle of this locus is homologous to a 3-cycle lying on a certain algebraic ruled surface on the variety, and may in fact be obtained by the variation of a straight line. This result has at least a negative interest, for if the locus contained *non-algebraic* 3-cycles we should be able to deduce

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[1] See Lefschetz, **8**, **9**; Zariski, **15**, for an account of the topology of algebraic surfaces.

[2] Hodge, **6**, **7**.

[3] Lefschetz, **8**, Chap. 5.

its irrationality; as it is, the matter is still left in doubt. The work may also be of interest as being an application in detail of the general procedure outlined by Lefschetz.

§ 1. We consider an algebraic locus V_3 of three dimensions, irreducible and free from singularities. We denote by R_i, ($i=1, 2, .., 6$) the i-dimensional Betti number[1]; $R_i = R_{6-i}$, $R_0 = 1$. We consider on V_3 a linear system (∞^3 at least) of algebraic surfaces F without singularities, such that the general curve C of intersection of two surfaces F is irreducible. A pencil of surfaces of the system, whose base-curve is C, will contain a certain number n of surfaces with a double point. It is known[2] that

$$R_3 = n + 2(R_1 + R_2) - 2r_2 - r_1, \qquad (1)$$

where r_2 is the two-dimensional Betti number of F, and r_1 the one-dimensional Betti number of C; if p is the genus of C, $r_1 = 2p$.

§ 2. *The Betti numbers of certain rational varieties.*

Let C_p^ν be an irreducible curve of order ν and genus p, without singularities, lying in ordinary space. Consider the linear system $|\Sigma|$ of surfaces of order m which pass through C_p^ν, m being chosen so large that $|\Sigma|$ is free from fundamental curves or surfaces. Then $|\Sigma|$ maps the prime sections of a rational locus V in higher space, which for m sufficiently large is free from singularities.

The correspondence between V and the space S containing $|\Sigma|$ is (1, 1) except for the points of V mapping the neighbourhoods of points of the base curve C_p^ν of $|\Sigma|$. These points on V lie on a ruled surface R whose genus p is the same as that of C_p^ν, the neighbourhood of any point on C_p^ν corresponding to a line on R.

Since V and its prime sections are regular, $R_1(V) = 0$. Since V is rational it contains a linear system of rational surfaces, corresponding to the planes of S. Any 2-cycle on such a surface is

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[1] For the fundamental theorems of topology used in the sequel we refer to Lefschetz, **10**.

[2] Lefschetz, **8**, 94. (Theorem XI).

algebraic, and hence any 4-cycle of V is algebraic[1]. But a base for algebraic surfaces on V is manifestly formed by the prime section of V and the scroll R. Hence $R_4(V) = 2$, and hence by a known duality theorem[2], $R_2(V) = 2$.

To calculate R_3 we use the formula (1), the system $|F|$ on V being that represented by the planes of S. F is then a rational surface mapped on a plane by curves of order m with ν base-points, and C is a rational curve. Hence $r_2 = \nu + 1$, $r_1 = 0$. A surface F will have a node if, and only if, two of the base points in the plane it represents coincide, since $|\Sigma|$ has no fundamental curves. Hence the number n of such surfaces is equal to the rank of C_p^ν, *i.e.* $2\nu + 2p - 2$. Hence (1) gives

$$\begin{aligned} R_3 &= (2\nu + 2p - 2) + 2(0 + 2) - 2(\nu + 1) - 0 \\ &= 2p. \end{aligned}$$

We shall now show that the $2p$ independent 3-cycles on V are precisely the $2p$ 3-cycles on the scroll R which is transformed into the neighbourhood of C_p^ν. To do this we require a formula connecting $R_3(V)$ with the characters of the complexes R, $V - R$. We have, in fact[3],

$$R_3(V) = r_3(V - R) + R_3(R) - s_3(R) + t_3(R),$$

where $r_3(V - R)$ denotes the number of 3-cycles on $V - R$ which are independent mod. R, $s_3(R)$ is the number of 3-cycles on R which bound on V, and $t_3(R)$ is the number of relative 3-cycles on $V - R$, independent of the $r_3(V - R)$ absolute cycles, whose boundaries are homologous to zero on R. In the present case $t_3(R)$ is zero[4], and since $V - R$ is homeomorphic[5] to $S - C_p^\nu$,

$$r_3(V - R) = r_3(S - C_p^\nu) = 0.$$

Hence, as $R_3(V) = R_3(R) = 2p$, $s_3(R) = 0$, and the $2p$ 3-cycles of R are independent on V.

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[1] See Lefschetz, **8**, 103. The surfaces mapping the planes of S may be used instead of the prime sections of V itself.

[2] Lefschetz, **10**, 140.

[3] Lefschetz, **10**, 150 ; equations (21) and (22).

[4] Lefschetz, **10**, 153 ; Theorem I.

[5] Lefschetz, **11**, 100.

The same method clearly extends to the case when the base-curve of $|\Sigma|$ consists of several totally disconnected curves, the system still being free from fundamental elements. If there are i such curves, of genera $p_1, \ldots, p_i$, it is easily seen that $R_2 = i + 1$; $R_3 = 2\,\Sigma p_i$.

If fundamental curves and surfaces exist for $|\Sigma|$ the problem becomes more complicated, and the V_3 may have singularities. The results obtained above exhibit clearly, however, the *relative* nature of the number R_3 considered as a birational invariant of V_3. In general, we may say that the behaviour of R_3 under birational transformation of V_3 depends on the genera of the fundamental curves of the transformation, and probably on other factors besides.

The special cases considered here include a number of well-known loci, notably most of those with rational or elliptic curve sections.

§ 5. *The Betti numbers and vanishing cycles of the cubic variety in* [4].

We consider the general cubic variety V in space of four dimensions, and denote by $|F|$ the system of prime sections of V. We recall[1] that V is the locus of a double infinity of lines of which six pass through each point of the locus, forming an irreducible system whose Grassmannian locus is a surface ϕ of irregularity 5, and hence having $R_1(\phi) = 10$. Any algebraic surface on V is its complete intersection with another primal[2].

Since the general prime section is a rational surface, any 2-cycle on it is algebraic. Hence every 4-cycle of V is algebraic[3]. Thus, since V only contains complete intersections, any Γ_4 on $V \sim tF$, and $R_2 = R_4 = 1$. A corollary is that any two lines on V are homologous on V, a result otherwise evident, since they correspond to two points of ϕ, and a chain on ϕ joining these defines a continuous deformation of one of the lines into the other on V.

Since V is of class 24 and $R_1 = 0$, it follows from (1) that $R_3 = 10$, since $r_2(F) = 7$ and $r_1(C) = 2$. Thus *there are ten independent 3-cycles on* V.

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[1] See, *e.g.* Baker, 1, vol. VI, 294; Segre, 13, 947. The result is originally due to Fano, 5.

[2] Fano, 4.

[3] See Lefschetz, 8, 103.

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We now consider a pencil $\{F_u\}$ of $|F|$ whose base is an irreducible curve C. For each value of u we have a definite surface F_u of the pencil, and for twenty-four values u_μ of u the corresponding surface F_μ has a node. In the plane of the complex variable u we mark the points u_μ, and a fixed point u_0 corresponding to some fixed surface F_0 of the pencil; and join u_0 to each point u_μ by a set of non-intersecting arcs. If the plane be cut open along these arcs it becomes[1] an open 2-cell E_2. We can now follow out the variation of any locus on F_u as u moves on E_2.

It is proved by Lefschetz that with each critical value u_μ of u there is associated a definite 2-cycle δ_μ of F_0 with the following properties:

(I). If u describes a path on E_2 from u_0 to u_μ then the cycle obtained by the variation of δ_μ vanishes (reduces to a point) when $u = u_\mu$.

(II). If u describes a closed path on E_2 surrounding u_μ and no other critical point, then the initial and final positions Γ_2, Γ_2', of any 2-cycle on F_0 are connected by the homology

$$\Gamma_2' \sim \Gamma_2 + (\Gamma_2 \,.\, \delta_\mu)\, \delta_\mu, \tag{2}$$

where $(\Gamma_2 \,.\, \delta_\mu)$ is the usual Kronecker index. Actually Lefschetz gives the coefficient of δ_μ with a negative sign, but this seems to be a misprint traceable to one in the first equation on p. 92.

In this case we can actually state what the vanishing cycles δ_μ are. As u tends to u_μ, F_u tends to the surface F_μ having a node. On F_μ six lines pass through the node, which arise from the coincidence of two rows of a double-six lying on F_u. Denoting the lines on F_0, in the usual Schläfli notation[2], by a_i, b_i, c_{ij} $(i, j, = 1, \ldots, 6;\ c_{ij}=c_{ji})$, so that a typical double-six is

$$\begin{matrix} a_1, & a_2, & \ldots, & a_6, \\ b_1, & b_2, & \ldots, & b_6, \end{matrix}$$

[1] The procedure is due to Lefschetz, 8, 91. A more detailed account of the construction is given by Zariski, 15, 105, for the case of surfaces; and this account applies to the present case with only a few verbal changes.

[2] For the notation, and general properties of the lines on a cubic surface, we refer to Baker, 1, vol. III, Chap. IV.

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this means that as u tends to u_μ the lines a_i (or rather the lines on F_u obtained by continuous variation from them) tend to coincidence with b_i, so that as u describes a closed path round u_μ enclosing no other singular point the effect on F_0 is to interchange the lines a_i, b_i, and leave the lines c_{ij} unaltered. Thus we can take $a_i - b_i$ to be the cycle δ_μ which vanishes at u_μ, the suffix i being immaterial since the six possible cycles are all homologous on F_0. A base for 2-cycles on F_0 is composed of a plane section C and the lines a_i. If then

$$\Gamma_2 \sim tC + \Sigma\, t_i a_i$$

is a 2-cycle on F_0, then when u describes a path round u_μ, Γ_2 becomes Γ_2', where

$$\begin{aligned}\Gamma_2' &\sim tC + \Sigma\, t_i b_i \\ &\sim \Gamma_2 - (\Sigma\, t_i)\, \delta_\mu,\end{aligned}$$

and since $(C, \delta_\mu) = 0$ and $(a_i, \delta_\mu) = -1$,

$$\Gamma_2' \sim \Gamma_2 + (\Gamma_2 \,.\, \delta_\mu)\, \delta_\mu,$$

in accordance with (2). Also, $(\delta_\mu, \delta_\mu) = -2$, which shows that δ_μ is changed in sign as u turns round u_μ.

§ 4[1]. If now u describes any closed contour on E_2, the lines of F_0 undergo a certain permutation, and the aggregate of all such permutations forms a group G_1 which either coincides with the group G, of order 51840, of all permutations of the lines which preserve their incidence relations, or forms a proper sub-group of G. We shall show that in fact $G_1 \equiv G$.

Any closed path on E_2 starting from u_0 can be deformed into the sum of a finite number of loops each surrounding a single one of the points u_μ. Thus G_1 is generated by the permutations effected when u describes each of these loops. Each of these permutations is of the type described in the previous section, interchanging the two rows of one of the double-sixes on F_0 and leaving the fifteen lines which do not belong to the double-six unaltered.

It is well known[2] that G can be represented as a group of

[1] The results in this section are not used in the rest of the paper, but are inserted for their intrinsic interest.

[2] See Schoute, 12; Coxeter, 2, 165; Todd, 14.

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congruent transformations in Euclidean space of six dimensions, leaving invariant a certain polytope with thirty-six pairs of opposite vertices (in correspondence with the double-sixes on F_0), having also thirty-six primes of twofold symmetry which bisect the diagonals joining pairs of opposite vertices at right angles. The operation of G which interchanges the two rows of a double-six is represented by the reflection in the corresponding prime, and the whole group is simply isomorphic with the group of rotations and reflections of the polytope. Consequently G_1 is generated by reflections in a certain number of the primes of symmetry of the polytope, and so is included in the table of such groups given by Coxeter[1].

We next show that G_1 acts transitively on the lines of F_0; to do which it is sufficient to show that if l_1 and l_2 are any two lines of F_0 there is an operation of G_1 which interchanges l_1 and l_2. Let L_1 and L_2 be the points on the Grassmannian ϕ of the lines on V which correspond to l_1 and l_2. The lines of V which occur on the surfaces of the pencil $\{F_u\}$ all meet the base curve C of the pencil, and conversely any line of V which meets the plane of C meets C and lies on just one of the surfaces $\{F_u\}$. These lines thus belong to a (special) linear complex and are mapped by the points of a curve ω on ϕ (actually a prime section of ϕ), which is irreducible and which passes through L_1 and L_2. This curve ω (or rather the corresponding Riemann surface) is in (27, 1) correspondence with the complex plane on which u is represented, since each F_u contains 27 lines. Thus any arc on ω joining L_1 and L_2 corresponds to an arc in the u-plane, which is a *cycle* since L_1 and L_2 both correspond to the same point u_0. We can choose this arc on ω so that it does not meet any of the arcs whose points correspond to points of the u-plane on the cuts $u_0 u_\mu$. Then the cycle in the u-plane lies on E_2, and, as u describes the cycle, l_1 is carried into l_2. This proves that G_1 is transitive on the lines of F_0.

It is now easily seen that G_1 coincides with G. For if not, it is a proper subgroup of G occurring in the table in Coxeter's paper referred to above. All these groups are easily verified to be subgroups of one or other of the three groups denoted by the symbols

$$[3^{2,1,1}], \quad [3^4] \times [\ ], \quad [3]^3.$$

[1] Coxeter, 3, 480. (Table VI, (iv)).

The first of these is the subgroup of G keeping fixed a definite line of F_0, the second leaves a particular double-six invariant, while the third leaves invariant nine lines of F_0 which belong to a Steiner trihedral. Thus all the proper subgroups of G generated by a partial set of the reflections which generate G are intransitive on the lines of F_0. Hence, since G_1 is transitive, $G_1 \equiv G$.

§ 5. *The homologies connecting the vanishing cycles.*

The vanishing cycles considered in § 3 are all of the type $l_1 - l_2$, where l_1 and l_2 are two non-intersecting lines of F_0. Since F_0 contains 216 pairs of non-intersecting lines there are 216 cycles of this form on F_0, which fall into 36 sets of six homologous cycles corresponding to the 36 double-sixes on F_0. These sets are themselves connected by homologies, since it is easily seen that only six of these cycles can be linearly independent on F_0. For we may take a representative of each of the 36 sets of cycles, for instance

$$a_1 - a_i; \quad a_1 - b_i; \quad a_1 - c_{ij}; \quad b_1 - c_{ij}; \quad a_i - a_j;$$

with the convention $2 \leqq i < j \leqq 6$. The six cycles

$$a_1 - a_3; \quad a_1 - a_4; \quad a_1 - a_5; \quad a_1 - a_6; \quad a_1 - b_1; \quad a_1 - c_{23}$$

are independent on F_0, and the others are expressible in terms of these in virtue of the homologies

$$\left.\begin{aligned}
(a_1-a_2) &\sim (a_1-b_1)+(a_1-a_4)+(a_1-a_5)+(a_1-a_6)-(a_1-a_3)-2(a_1-c_{23})\\
&\sim (a_1-b_1)+(b_1-b_4)+(c_{56}-c_{16})+(c_{63}-c_{13})\\
&\quad -(c_{36}-c_{16})-(a_2-c_{13})-(c_{56}-b_4),\\
(a_i-a_j) &\sim (a_1-a_j)-(a_1-a_i),\\
(a_1-c_{2k}) &\sim (a_1-c_{23})+(a_1-a_3)-(a_1-a_k)\\
&\sim (a_1-c_{23})+(c_{32}-c_{12})-(c_{k2}-c_{12}),\\
(a_1-c_{jk}) &\sim (a_1-c_{j2})+(a_1-a_2)-(a_1-a_k)\\
&\sim (a_1-c_{j2})+(c_{2j}-c_{1j})-(c_{kj}-c_{1j}),\\
(b_1-c_{jk}) &\sim (a_1-c_{jk})-(a_1-b_1).
\end{aligned}\right\} \quad (3)$$

The homologies (3) evidently form a minimal basis for homologies between the vanishing cycles, and the form of these homologies makes it clear that in any such homology we can arrange, by replacing the individual vanishing cycles by others belonging to the same set of six on F_0, that the actual expressions on the two sides of the homology

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sign are algebraically the same when expressed in terms of the lines a_i, b_i, c_{ij}. Consequently we have the following result. *Any homology*

$$\Sigma \lambda_\alpha \delta_\alpha \sim 0 \tag{4}$$

between the vanishing cycles δ_α of F_0 is equivalent to a homology

$$\Sigma \mu_\beta \delta_\beta' \sim 0 \tag{5}$$

in which each of the cycles δ_β' is a vanishing cycle homologous on F_0 to one of the cycles δ_α, and the algebraic sum of the cycles on the left, expressed in terms of the lines of F_0, is zero. [Of course it is not necessarily the case that each of the λ_α cycles δ_α in (4) is replaced by the *same* cycle δ_β'; an example is the first homology (3), where $2(a_1 - c_{23})$ is replaced by $(a_1 - c_{23}) + (c_{56} - b_4)$.]

§ 6. *Determination of the 3-cycles on V.*

It is shown by Lefschetz in his book[1], that for any 3-cycle Γ_3 of V we have the homology $\Gamma_3 \sim \Sigma \lambda_\mu \Delta_\mu + M_3$, where Δ_μ is the 3-chain described by δ_μ as u describes the arc $u_0 u_\mu$, and M_3 is a 3-chain on F_0 whose boundary is $-\Sigma \lambda_\mu \delta_\mu$; and that such a cycle arises from every independent homology of type (4). Now if δ_μ' is one of the other vanishing cycles at u_μ, and Δ_μ' its locus as u describes $u_0 u_\mu$, then $\Delta_\mu' - \Delta_\mu$ is a 3-chain bounded by the 2-cycle $\delta_\mu' - \delta_\mu$. Further, $\delta_\mu' - \delta_\mu$ bounds a 3-chain N_3 in F_0 such that $\Delta_\mu' - \Delta_\mu - N_3$ is a bounding 3-cycle of V. Hence in view of the result of the last paragraph we may suppose that $\Sigma \lambda_\mu \delta_\mu$ is algebraically zero. Then M_3 must be a cycle. But any 3-cycle on $F_0 \sim 0$. Thus every 3-cycle on V is of the form $\Sigma \lambda_\mu \Delta_\mu$ where $\Sigma \lambda_\mu \delta_\mu$ is algebraically zero. Since δ_μ is of the form $l_1 - l_2$, where l_1, l_2 are two lines of F_0 which are interchanged when u describes a closed path round u_μ, Δ_μ is the locus of l_1 as u describes this circuit. Hence *every 3-cycle of V may be regarded as the locus of a line of V as u describes a suitable closed path on E_2; and is therefore reducible to a cycle on the scroll formed by the lines of V which meet C.* The ten independent cycles accord with the fact that the Grassmannian ϕ of the lines of V carries ten independent 1-cycles.

[1] Lefschetz, 8, 93. (Theorem VI).

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