# AUTOMETRIZED BOOLEAN ALGEBRAS I: FUNDAMENTAL DISTANCE-THEORETIC PROPERTIES OF *B*

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**1. Introduction.** There have been several brief studies made [3, 4, 7, 8, 9, 11] of systems in which a "distance function" is defined on the set of pairs of elements of some abstract set to another abstract set. Frequently both of the sets involved are given algebraic structures. One of the more novel of these systems is the naturally metrized group [3, 7] originated by Karl Menger in 1931. This system is analogous to the Euclidean line in that it assigns to each pair, $a$, $b$ of elements of an additively written Abelian group the "absolute value", $(a\text{-}b, b\text{-}a) = (b\text{-}a, a\text{-}b)$, of the "difference" of the elements as "distance". As might be anticipated (since this is precisely the distance function used on the Euclidean line) many of the results valid for the distance function on the Euclidean line are also valid for this "distance function"

In this paper a system will be defined which is also somewhat similar to the Euclidean line in that the "distances" of elements are themselves elements of the same set. A few of the usual notions of metric geometry will then be examined in terms of these "distances". It should be noted that the definitions of metric concepts in this system are analogous to the corresponding definitions for metric spaces; in particular, there is a striking resemblance between the list of properties given in Theorem 1.1 and the usual list of assumptions about the metric of a metric space.

DEFINITION. Let $B$ be a Boolean algebra with meet, join, complement, and inclusion (in the wide sense) denoted by $ab$, $a+b$, $a'$, and $a \subset b$ respectively. (All these notions are discussed in [1]. The notions of metric geometry not defined in this paper are discussed in L. M. Blumenthal's *Distance Geometries*, University of Missouri Studies, vol. XIII (1938).)

Define "distance" of $a$ and $b$ by $d(a, b) = ab' + a'b$. Throughout the paper subsets of $B$ will be denoted by capital English letters and all small English letters will denote elements of $B$. The name, *autometrized Boolean algebra*, seems appropriate for the system thus defined.

*Remark* (i). The "distance function" defined in $B$, $d(a, b) = ab' + a'b$, is also the element corresponding to the ring sum of $a$ and $b$ under the one-to-one correspondence of Boolean algebras and Boolean rings with units [1] so all the results of this paper might, if desired, be re-phrased as results concerning the ring sum in a Boolean ring with a unit.

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*Remark* (ii). Since $d(a, b) = ab' + a'b = (a + b)(a' + b')$,
$$d'(a, b) = ab + a'b' = (a' + b)(a + b').$$

*Remark* (iii). The "distance function" is invariant with respect to complementation; that is, $d(a, b) = d(a', b')$.

*Remark* (iv). The first element[1] of $B$ acts as an "origin" with respect to the "distance function"; that is, $d(0, a) = a$.

THEOREM 1.1. *The "distance function" has the following properties:*

1. *Symmetry:* $d(a, b) = d(b, a)$.
2. *Vanishing:* $d(a, b) = 0$ *if and only if* $a = b$.
3. *Triangle inequality:* $d(a, c) \subset d(a, b) + d(b, c)$.

*Proof.* Part 1 is obvious and it is trivial to verify that $d(a, a) = 0$ for part 2. Suppose now that $d(a, b) = 0$. Then $ab' = a'b = 0$. Taking complements in $ab' = 0$, one finds that $a' + b = 1$. Hence $a' = b'$ and $a = b$.

To prove part 3 take

$$\begin{aligned}(d(a, b) + d(b, c)\, d(a, c) &= (ab' + a'b + bc' + b'c)(ac' + a'c) \\ &= a'bc + a'b'c + ab'c' + abc' = a'c(b + b') + ac'(b' + b) = a'c + ac' \\ &= d(a, c).\end{aligned}$$

## 2. Betweenness.

DEFINITION. An element $b$ *is between* $a$ *and* $c$ provided

$$d(a, c) = d(a, b) + d(b, c).$$

Pitcher and Smiley [10] have laid the foundations for a theory of betweenness in general lattices based on the condition of Glivenko [5, 6]; that is, $b$ *is* G-*between* $a$ *and* $c$ provided

$$\text{(G)} \qquad ab + bc = b = (a + b)(b + c).$$

We shall show that these two varieties of betweenness are equivalent in $B$.

LEMMA 2.1. *The element $b$ is* G-*between $a$ and $c$ if and only if $b'$ is* G-*between $a'$ and $c'$.*

*Proof.* This result follows from complementation in condition (G).

THEOREM 2.1. *The element $b$ is between $a$ and $c$ if and only if $b$ is* G-*between $a$ and $c$.*

*Proof.* Suppose first that $b$ is G-between $a$ and $c$. Then

$$ab + bc = b = (a + b)(b + c)$$

and by Lemma 2.1, $a'b' + b'c' = b' = (a' + b')(b' + c')$.

[1]The symbols 0 and 1 denote the first and last elements of $B$.

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Now $d(a, b) + d(b, c) = a'b + ab' + b'c + bc' = b(a' + c') + b'(a + c)$
$= (ab + bc)(a' + c') + (a'b' + b'c')(a + c)$
$= a'bc + abc' + ab'c' + a'b'c$
$= a'c(b + b') + ac'(b + b') = a'c + ac' = d(a,c).$

Hence $b$ is between $a$ and $c$.

Suppose now that $b$ is between $a$ and $c$. Then

$$a'b + ab' + b'c + bc' = a'c + ac'.$$

Taking meets in this equation with $a$, $a'$, $b$, and $b'$ respectively it is found that

(1) $ab' + ab'c + abc' = ac',$
(2) $a'b + a'b'c + a'bc' = a'c,$
(3) $a'b + bc' = a'bc + abc',$
(4) $ab' + b'c = a'b'c + ab'c'.$

Taking meets in 1 with $b'$, $c$, and $c'$ respectively; in 2 with $b$, $c$, and $c'$ respectively; in 3 with $c'$; and in 4 with $c$ one obtains:

(5) $ab' + ab'c = ab'c',$
(6) $ab'c + ab'c = 0,$
(7) $ab'c' + abc' = ac',$
(8) $a'b + a'bc' = a'bc,$
(9) $a'bc + a'b'c = a'c,$
(10) $a'bc' + a'bc' = 0,$
(11) $a'bc' + bc' = abc',$
(12) $ab'c + b'c = a'b'c.$

From (6) and (10) one obtains:

(13) $ab'c = 0,$
(14) $a'bc' = 0.$

From (13) and (5), (14) and (8), (14) and (11), and (13) and (12), respectively, one finds that:

(15) $ab' = ab'c'$ or $ab' \subset c',$
(16) $a'b = a'bc$ or $a'b \subset c.$
(17) $bc' = abc'$ or $bc' \subset a.$
(18) $b'c = a'b'c$ or $b'c \subset a'.$

Combining (7), (15), and (17), and then combining (9), (16), and (18) one obtains:

(19) $ab' + bc' = ac',$
(20) $a'b + b'c = a'c.$

(Note that we have in effect "factored" $d(a, c) = d(a, b) + d(b, c)$ into equations (19) and (20)).

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Complementation in (19) and (20) yields:

(21) $$(a' + b)\,(b' + c) = a' + c,$$

(22) $$(a + b')\,(b + c') = a + c'.$$

Taking meets between (21) and (22) yields

(23) $$abc + a'b'c' = ac + a'c'.$$

Taking meets in (23) with $ac$ and $a'c'$ respectively we find that:

(24) $$abc = ac \quad \text{or } ac \subset b,$$

(25) $$a'b'c' = a'c' \text{ or } a'c' \subset b'.$$

From (24), $b = b + ac = (a + b)\,(b + c)$. Hence

(26) $$b = (a + b)\,(b + c).$$

Complementation in (25) yields $a + b + c = a + c$ or $b \subset a + c$ so that $b = b(a + c) = ab + bc$. Hence

(27) $$b = ab + bc.$$

Then combining (26) and (27) we find

(G) $$ab + bc = b = (a + b)\,(b + c).$$

Hence $b$ is G-between $a$ and $c$.

*Remark* (i). All of the properties of G-betweenness in general lattices demonstrated by Pitcher and Smiley [10] are valid for betweenness in $B$.

*Remark* (ii). If $B$ is a *metric lattice* [1] then betweenness in $B$ is equivalent to *metric betweenness* (in the wide sense) since Glivenko [5, 6] showed that metric betweenness (in the wide sense) and G-betweenness are equivalent in any metric lattice.

*Remark* (iii). Betweenness in $B$ is equivalent to each of the following:

(G) $$ab + bc = b = (a + b)\,(b + c),$$

(G*) $$ab + bc = b = b + ac,$$

(G**) $$b(a + c) = b = (a + b)\,(b + c).$$

*Proof*. Equivalence of betweenness and (G) was demonstrated in Theorem 2.1. It is obvious that (G), (G*), and (G**) are pairwise equivalent in any *distributive* lattice (in fact, L. M. Blumenthal and the writer have shown that these conditions are pairwise equivalent in any *modular* lattice [2]).

### 3. The group of motions of *B*. Superposability properties.

DEFINITIONS. Let $E$ and $F$ be subsets of $B$. $E$ *is congruent to* $F$, written $E \approx F$, provided there is a single-valued mapping $f\colon E \to F$ such that for $a, b \in E$, $d(f(a), f(b)) = d(a, b)$. It is easily seen that the mapping $f$ is then biuniform and that $F$ is congruent to $E$ by the mapping $f^{-1}\colon F \to E$. The mapping $f$ (or $f^{-1}$) is then called a *congruence* between $E$ and $F$.

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A *motion* of $B$ is a congruence of $B$ with itself. It is clear that these motions form a group of transformations. This group is called *transitive* (simply transitive) provided there is a motion of $B$ sending any given element of $B$ into any other given element. Two subsets, $E$ and $F$, are called *superposable*, written $E \simeq F$, provided there is a motion of $B$ mapping one onto the other. Clearly $E \approx F$ if $E \simeq F$.

$B$ is said to have the *property of n-element superposability* provided that if $E$ and $F$ are each sets of $n$ elements and $E \approx F$ then $E \simeq F$.

$B$ is said to have the *property of free mobility* provided *any* congruence between *any* two subsets of $B$ may be extended to a motion of $B$; that is, provided there is a motion of $B$ which induces the *original* congruence between the two given sets.

THEOREM 3.1. *B has the property of two-element superposability.*

*Proof.* Let $a, b \approx m,n$. Define a mapping of $B$,

$$f(x) = mn(xab + x'a'b') + m'n'(x'ab + xa'b') + mn'(xab' + x'a'b) + m'n(xa'b + x'ab').$$

This mapping is single valued and sends $a$ into $m$ and $b$ into $n$.
For example,

$$\begin{aligned} f(a) &= mn(ab + a'b') + mn'(ab' + a'b) \\ &= mn(mn + m'n') + mn'(mn' + m'n) \\ &= mn + mn' = m. \end{aligned}$$

By laborious expansion and collection of meets one may verify that

$$\begin{aligned} &d(f(x), f(y)) \\ &= f(x)f'(y) + f'(x)f(y) \\ &= (x'y + xy')((ab + a'b')(mn + m'n') + (a'b + ab')(m'n + mn')) \\ &= (x'y + xy')(ab + a'b' + a'b + ab') = x'y + xy' = d(x,y) \end{aligned}$$

so that $f(x)$ is a congruence of $B$ with itself.

*Remark.* The group of motions of $B$ is transitive as a direct consequence of Theorem 3.1.

THEOREM 3.2. *B has the property of free mobility.*

*Proof.* Let $E$ and $F$ be any two congruent subsets of $B$ and let $g\colon E \rightarrow F$ be this congruence. If $E$ contains less than three distinct elements the proof reduces to an application of Theorem 3.1. Otherwise, select $a, b \in E$, $a \neq b$. Then $a,b \approx g(a), g(b)$ and $f(x)$ as defined in the proof of Theorem 3.1 sends $a$ into $g(a)$ and $b$ into $g(b)$. But $f(x)$ is a motion of $B$ so it remains only to verify that $f$ induces $g$ on $E$. Let then $c \in E$. Since $g$ is a congruence, $a, b, c \approx g(a), g(b), g(c)$.
From the six relations

$$\begin{aligned} a'b + ab' &= g'(a)g(b) + g(a)g'(b), & ab + a'b' &= g(a)g(b) + g'(a)g'(b), \\ a'c + ac' &= g'(a)g(c) + g(a)g'(c), & ac + a'c' &= g(a)g(c) + g'(a)g'(c), \\ b'c + bc' &= g'(b)g(c) + g(b)g'(c), & bc + b'c' &= g(b)g(c) + g'(b)g'(c) \end{aligned}$$

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resulting from this congruence, one obtains

$$\begin{aligned} abc + a'b'c' &= g(a)g(b)g(c) + g'(a)g'(b)g'(c), \\ abc' + a'b'c &= g(a)g(b)g'(c) + g'(a)g'(b)g(c), \\ ab'c + a'bc' &= g(a)g'(b)g(c) + g'(a)g(b)g'(c), \\ a'bc + ab'c' &= g'(a)g(b)g(c) + g(a)g'(b)g'(c). \end{aligned}$$

Then

$$\begin{aligned} f(c) &= g(a)g(b)(g(a)g(b)g(c) + g'(a)g'(b)g'(c)) \\ &\quad + g'(a)g'(b)(g(a)g(b)g'(c) + g'(a)g'(b)g(c)) \\ &\quad + g'(a)g(b)(g'(a)g(b)g(c) + g(a)g'(b)g'(c)) \\ &\quad + g(a)g'(b)(g(a)g'(b)g(c) + g'(a)g(b)g'(c)) \\ &= g(a)g(b)g(c) + g'(a)g'(b)g(c) + g(a)g'(b)g(c) + g'(a)g(b)g(c) \\ &= g(c)\ (g(a)g(b) + g'(a)g'(b) + g(a)g'(b) + g'(a)g(b)) \\ &= g(c). \end{aligned}$$

Hence $f$ induces $g$ on $E$, and Theorem 3.2 is proved.

## 4. Other properties of "distance" in *B*.

THEOREM 4.1. *Let $a, b \in B$ and $d(a, b) = c$. Then $d(a, c) = b$ and $d(b, c) = a$.*

*Proof.* $d(a, b) = a'b + ab' = c$ so that $a'b = a'c$. $d'(a, b) = c' = ab + a'b'$ so that $ab = ac'$. Hence $b = ab + a'b = ac' + a'c = d(a, c)$. Similarly one shows that $a = d(b, c)$.

THEOREM 4.2. *Let $c \in B$ and $a \in B$. There is exactly one element $b$ of $B$ so that $d(a, b) = c$. This is sometimes stated by saying that any given element forms a metric basis for $B$.*

*Proof.* Let $b = d(a, c)$. Then by Theorem 4.1, $c = d(a, b)$. Hence there is at least one such element $b$. Suppose that $d(a, x) = c$. Then by Theorem 4.1, $x = d(a, c) = b$ so that the element $b$ is unique.

*Remark.* Isosceles and equilateral triples, common structures in metric spaces, are shown by Theorem 4.2 to be absent from $B$.

## 5. The "metric characterization" of *B*.

DEFINITION. Let $\Sigma$ be an abstract set, to each pair of elements $\alpha, \beta$ of which is attached an element $d(\alpha, \beta)$ of $B$ such that $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$, and $d(\alpha, \beta) = d(\beta, \alpha)$. Such a set will be called a B-*metrized space.*

$B$ is said to have *congruence order $n$* with respect to the class of B-metrized spaces provided any B-metrized space is congruently contained (congruent to a subset of) in $B$ whenever each $n$ elements of the space are congruently contained in $B$. The smallest integer $n$ for which $B$ has congruence order $n$ is called the *best congruence order* of $B$. In the following theorem small greek letters will denote elements of a B-metrized space.

THEOREM 5.1. *Let $B$ be any autometrized Boolean algebra. The best congruence order of $B$ with respect to the class of* B-*metrized spaces is three.*

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*Proof.* We shall first show that $B$ has congruence order three and then that this is the best congruence order of $B$.

Let $\Sigma$ be any B-metrized space with more than two distinct elements (otherwise the proof is trivial), and suppose that each three elements of $\Sigma$ are congruent with three elements of $B$. Select $\alpha, \beta \in \Sigma$, $\alpha \neq \beta$. Then $\alpha, \beta \approx a, b \in B$. Let $\xi \in \Sigma$. Now $\alpha, \beta, \xi \approx a_1, b_1, x_1 \in B$. But $a_1, b_1 \simeq a, b$ by Theorem 3.1. Let $x$ be the image of $x_1$ under this motion. Then $\alpha, \beta, \xi \approx a, b, x$ and we have established a single-valued mapping $x = x(\xi)$ of $\Sigma$ into $B$. It remains to show that "distances" are preserved. Let $\xi, \eta \in \Sigma$ and let $x, y$ be the elements of $B$ corresponding to $\xi, \eta$ under the previously defined mapping. Now $\alpha, \xi, \eta \approx a_2, x_2, y_2 \in B$ by hypothesis. But $a, x \approx \alpha, \xi$. Hence by Theorem 3.1, $a, x \simeq a_2, x_2$. Let $y_3$ be the image of $y_2$ under the motion sending $a_2$ into $a$ and $x_2$ into $x$. Then $a, y_3 \approx a_2, y_2 \approx \alpha, \eta \approx a, y$. Hence by Theorem 4.2, $y_3 = y$ so that $a, x, y \approx a_2, x_2, y_2 \approx \alpha, \xi, \eta$ and $d(x, y) = d(\xi, \eta)$. This establishes that $\Sigma$ is congruently contained in $B$ and that $B$ has congruence order three.

The example consisting of three elements $\alpha, \beta, \gamma$ with $d(\alpha, \beta) = d(\beta, \gamma) = d(\alpha, \gamma) = a$, $d(\alpha, \alpha) = d(\beta, \beta) = d(\gamma, \gamma) = 0$ where $a$ is any element of $B$ different from 0 shows that $B$ cannot have congruence order less than three since any proper subset of this B-metrized space is congruently contained in $B$ but the space itself forms an equilateral triple and so cannot be congruently contained in $B$ which is free from equilateral triples.

## References


[1] Garrett Birkhoff, *Lattice Theory (revised edition)*, Amer. Math. Soc. Colloquium Publications, vol. XXV (1948).

[2] L. M. Blumenthal and David Ellis, *Notes on metric lattices*, Duke Math. J., vol. 16 (1949), 585-590.

[3] David Ellis, *Superposability properties of naturally metrized groups*, Bull. Amer. Math. Soc., vol. 55 (1949), 639-640.

[4] Maurice Frechet, *De l'écart numérique à l'écart abstrait*, Portugaliae Mathematica, vol. 5 (1946), 121-131.

[5] V. Glivenko, *Géométrie des systèmes de choses normées*, Amer. J. Math., vol. 58 (1936), 799-828.

[6] ——— *Contributions à l'étude des systèmes de choses normées*, Amer. J. Math., vol. 59 (1937), 941-956.

[7] Karl Menger, *Beiträge zur Gruppentheorie I. Über einen Abstand im Gruppen*, Math. Z., vol. 33 (1931), 396-418.

[8] ——— *Projective generalizations of metric geometry*, Reports of a Mathematical Colloquium (Notre Dame), Issue 5-6 (1944), 60-75.

[9] ——— *Statistical metrics*, Proc. Nat. Acad. Sci., vol. 28 (1942), 535.

[10] Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc., vol. 52 (1942), 95-114.

[11] G. Baley Price, *A generalization of a metric space with applications to spaces whose elements are sets*, Amer. J. Math., vol. 63 (1941), 546-560.


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