A RELATION BETWEEN BIHARMONIC GREEN'S FUNCTIONS OF SIMPLY SUPPORTED AND CLAMPED BODIES

JAMES RALSTON AND LEO SARIO

The deflection, under a point load, of a thin elastic plate clamped at the edges is the biharmonic Green's function β with the boundary data $\beta = \partial\beta/\partial n = 0$. If the boundary of the region is reasonably smooth, the construction of β offers no difficulty. In contrast, nothing is known about the existence of β in the general case. The purpose of our study is to give a sufficient condition for the existence of β on a given Riemannian manifold of arbitrary dimension and to construct β. Our results will have, apart from their physical meaning in elasticity, consequences in the biharmonic classification theory of Riemannian manifolds.

To understand the nature of the problem, we recall that the deflection, under a point load, of a simply supported thin elastic plate is the biharmonic Green's function γ which satisfies the boundary conditions $\gamma = \varDelta\gamma = 0$. The function γ is always positive and increases with the region. As a consequence, γ on an arbitrary plate or, more generally, on an arbitrary Riemannian manifold R, can be defined as the directed limit, if it is finite, of the corresponding functions $\gamma_\varOmega$ on regular subregions $\varOmega$ exhausting R (Sario [34]).

Hadamard conjectured in his 1908 prize memoir [14] that the deflection β, under a point load, of a clamped plate as well be always positive. However, this conjecture was disproved by Duffin [8] for an infinite strip, by Loewner [19] and Szegö [52] for certain nonconvex plane regions, and by Garabedian [10] for a sufficiently eccentric ellipse. Interest in the problem was recently revived by the address of Duffin [9]

delivered before the 1974 Annual Meeting of the American Mathematical Society in San Francisco. The possibility of a nonconstant sign of β_Ω on regular subregions Ω of R causes difficulty in the convergence proof as $\Omega \to R$. This is where the problem lies in defining and constructing β.

To gain intuition from a concrete case we first consider the Euclidean N-space E^N. On exhausting balls $\Omega = \{r < \rho\}$, $\rho \to \infty$, the functions $\gamma_\Omega, \beta_\Omega$ with poles at the origin can be explicitly constructed. It turns out that $\gamma_\Omega \to \infty$ and $\beta_\Omega \to \infty$ at every point if $N \leq 4$, whereas both $\gamma = \lim_\Omega \gamma_\Omega$ and $\beta = \lim_\Omega \beta_\Omega$ exist on E^N if $N > 4$. This simultaneous existence of γ and β raises the question: is there some relationship between the existence of γ and β in the general case (and, in particular, on a plate of arbitrary shape)? That the answer is in the affirmative is our main result: *on an arbitrarily given Riemannian manifold, if γ exists, so does β*. The interesting problem whether or not the converse is true is not a topic of the present study.

A necessary and sufficient condition is known for the existence of γ: the harmonic measure ω of the ideal boundary of the manifold, with values 1 on a compact set, has to be in L^2 (Sario [34]). This test now carries over as a sufficient condition for the existence of β. As an example, the above result on the existence of β on E^N for $N > 4$ follows immediately, and is independent of the pole and exhaustion. Similarly, β is seen to exist on the Poincaré N-ball $B^N = \{r < 1\}$ with the metric $ds = (1 - r^2)^\alpha |dx|$, α a constant, if $\alpha > -3/2$ for $N = 2$, $\alpha \in (-3, 1)$ for $N = 3$, and $\alpha < (N - 2)^{-1}$ for $N > 3$.

A comprehensive bibliography, which includes recent work in the field, is attached.

1. To construct the biharmonic Green's function γ on the Euclidean N-space E^N, we exhaust E^N by N-balls $\Omega = \{r < \rho\}$, $\rho \to \infty$. We recall that the biharmonic fundamental singularity σ at $r = 0$ is $r^2 \log r$ for $N = 2$, r for $N = 3$, $\log r$ for $N = 4$, and r^{-N+4} for $N > 4$, the singularity manifesting itself in that $\sigma \notin C^4$ at the origin. We normalize σ to vanish for $r = \rho$ and then subtract a multiple of the regular biharmonic function $r^2 - \rho^2$ such that the Laplace-Beltrami operator $\Delta = d\delta + \delta d$ acting on γ_Ω,

$$\Delta \gamma_\Omega = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial r} \left(\sqrt{g} \, \frac{\partial}{\partial r} \gamma_\Omega \right),$$

gives zero values for $r = \rho$. The resulting γ_Ω on $\bar{\Omega}$ with pole 0 is for $N = 2, 3, 4, >4$, respectively,

$$\gamma_\Omega = \begin{cases} r^2 \log \dfrac{r}{\rho} - (r^2 - \rho^2) \,, \\[2ex] -r + \rho + \dfrac{1}{3}\rho^{-1}(r^2 - \rho^2) \,, \\[2ex] -\log \dfrac{r}{\rho} + \dfrac{1}{4}\rho^{-2}(r^2 - \rho^2) \,, \\[2ex] r^{-N+4} - \rho^{-N+4} + N^{-1}(N-4)\rho^{-N+2}(r^2 - \rho^2) \,. \end{cases}$$

We define γ on E^N as $\gamma = \lim_{\rho \to \infty} \gamma_\Omega$ and conclude that γ with pole 0 does not exist on E^N for $N = 2, 3, 4$, whereas for $N > 4$ it is

$$\gamma = r^{-N+4} \,.$$

For β_Ω on Ω the construction is the same except that now a multiple of $r^2 - \rho^2$ is to be subtracted which makes $\partial\beta_\Omega/\partial n = 0$ for $r = \rho$. We obtain for $N = 2, 3, 4, >4$,

$$\beta_\Omega = \begin{cases} r^2 \log \dfrac{r}{\rho} - \dfrac{1}{2}(r^2 - \rho^2) \,, \\[2ex] -r + \rho + \dfrac{1}{2}\rho^{-1}(r^2 - \rho^2) \,, \\[2ex] -\log \dfrac{r}{\rho} + \dfrac{1}{2}\rho^{-1}(r^2 - \rho^2) \,, \\[2ex] r^{-N+4} - \rho^{N+4} + \dfrac{1}{2}(N-4)\rho^{-N+2}(r^2 - \rho^2) \,. \end{cases}$$

As $\rho \to \infty$, we again deduce that $\beta = \lim_{\rho \to \infty} \beta_\Omega$ does not exist on E^N for $N \leq 4$, whereas for $N > 4$ it is

$$\beta = r^{-N+4} \,.$$

That for the singularity $y \in E^N$, the existence of $\gamma = \gamma(x, y)$ and $\beta = \beta(x, y)$ for $N > 4$ is independent of (x, y) and the exhaustion will be seen in No. 9.

The biharmonic Green's functions γ and β exist on E^N for precisely the same dimensions: $N > 4$.

This observation suggests an inquiry into a possible relationship between the existence of γ and β on every Riemannian manifold.

2. Given an arbitrary Riemannian manifold R, let Ω be a regular subregion of R. Take a point $y \in \Omega$ and a small geodesic ball B_y about y, $\alpha_y = \partial \beta_y$, $\alpha_y \cap \partial \Omega = \emptyset$, and orient both α_y and $\partial \Omega$ positively with respect to B_y and Ω, respectively. Let $\beta_\Omega = \beta_\Omega(x, y)$, $\gamma_\Omega = \gamma_\Omega(x, y)$ be the biharmonic Green's functions on $\overline{\Omega}$, with the biharmonic fundamental singularity at y normalized by $\int_{\alpha_y} *d\Delta\beta_\Omega = \int_{\alpha_y} *d\Delta\gamma_\Omega = -1$, and with boundary data

$$\beta_\Omega = \frac{\partial \beta_\Omega}{\partial n} = \gamma_\Omega = \Delta \gamma_\Omega = 0 \quad \text{on } \partial \Omega \ .$$

For $f \in C_0^\infty(\Omega)$, set

$$(\beta_\Omega f)(x) = \int_\Omega \beta_\Omega(x, y) * f(y) \ , \qquad (\gamma_\Omega f)(x) = \int_\Omega \gamma_\Omega(x, y) * f(y) \ .$$

By general results on elliptic boundary value problems (e.g., Hörmander [16, Ch. X]), there are unique functions $u_\beta, u_\gamma \in C^\infty(\overline{\Omega})$ satisfying

$$\Delta^2 u_\beta = \Delta^2 u_\gamma = f \quad \text{on } \Omega \ ,$$

$$u_\beta = \frac{\partial u_\beta}{\partial n} = u_\gamma = \Delta u_\gamma = 0 \quad \text{on } \partial \Omega \ .$$

For every u_β, u_γ with these properties,

$$u_\beta = \beta_\Omega f \ , \qquad u_\gamma = \gamma_\Omega f \ .$$

In fact, by Stokes' formula

$$\int_{\partial\Omega - \alpha_y} u_\beta * d\Delta\beta_\Omega - \Delta\beta_\Omega * du_\beta - \beta_\Omega * d\Delta u_\beta + \Delta u_\beta * d\beta_\Omega$$

$$= -\int_{\Omega - B_y} u_\beta * \Delta^2\beta_\Omega - \Delta\beta_\Omega * \Delta u_\beta - \beta_\Omega * f + \Delta u_\beta * \Delta\beta_\Omega \ .$$

As the geodesic radius of α_y tends to 0, $\int_{-\alpha_y} \to u_\beta(y)$, and we obtain $u_\beta = \beta_\Omega f$. Similarly, the equality

$$\int_{\partial\Omega - \alpha_y} u_\gamma * d\Delta\gamma_\Omega - \Delta\gamma_\Omega * du_\gamma - \gamma_\Omega * d\Delta u_\gamma + \Delta u_\gamma * d\gamma_\Omega$$

$$= -\int_{\Omega - B_y} u_\gamma * \Delta^2\gamma_\Omega - \Delta\gamma_\Omega * \Delta u_\gamma - \gamma_\Omega * f + \Delta u_\gamma * \Delta\gamma_\Omega$$

gives $u_\gamma = \gamma_\Omega f$.

3. Let Ω, Ω' with $\Omega \subset \Omega'$ be regular subregions of R, and set $(f, g) = \int_\Omega f * g$.

LEMMA 1. *For* $f \in C_0^\infty(\Omega)$,

$$(f, \beta_\Omega f) \leq (f, \beta_{\Omega'} f) \ .$$

Proof. On Ω, consider the Sobolev space

$$H_{2,\beta}(\Omega) = \left\{ u \,|\, D^\alpha u \in L^2(\Omega),\ |\alpha| \leq 2,\ u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and the functional on $H_{2,\beta}(\Omega)$,

$$J(u) = \int_\Omega \frac{1}{2} \Delta u * \Delta u - u * f \ .$$

For $u \in H_{2,\beta}$ one has the elliptic estimate for $|\alpha| \leq 2$,

$$(1) \qquad\qquad \|D^\alpha u\|^\Omega \leq C_\alpha \|\Delta u\|^\Omega \ ,$$

where $\|\cdot\|^\Omega$ stands for the L^2 norm over Ω. Thus $J(u)$ is bounded below. Let $\{u_n\}$ be a sequence such that $J(u_n) \to \inf_{H_{2,\beta}} J(u)$. Then $\{\Delta u_n\}$ is a bounded sequence in $L^2(\Omega)$. Using (1) and the Rellich compactness theorem, we may choose a subsequence $\{u_{n_k}\}$ converging to $\bar{u}$ in $L^2(\Omega)$ and converging weakly to $\bar{u}$ in $H_2(\Omega)$. Since $H_{2,\beta}$ is a closed subspace of $H_2(\Omega)$, it follows that $\bar{u} \in H_{2,\beta}$. Now $J(u_n) \to \inf_{H_{2,\beta}} J(u)$ implies $\lim_{k\to\infty} \|\Delta u_{n_k}\| = \|\Delta\bar{u}\|$. Hence, $\{\Delta u_{n_k}\}$ converges to $\Delta\bar{u}$ in $L^2(\Omega)$ and

$$\inf_{H_{2,\beta}} J(u) = J(\bar{u}) \ .$$

Thus $\bar{u}$ satisfies

$$0 = \int_\Omega \Delta\bar{u} * \Delta\varphi - \varphi * f$$

for all $\varphi \in H_{2,\beta}$. By direct computation, u_β also satisfies this equation. A fortiori,

$$0 = \int_\Omega \Delta(\bar{u} - u_\beta) * \Delta\varphi$$

for all $\varphi \in H_{2,\beta}$. In particular, this holds for $\varphi = \bar{u} - u_\beta$ and thus (1) implies $\bar{u} = u_\beta$. As a consequence,

$$\inf_{u \in H_{2,\beta}} J(u) = J(u_\beta) = \int_\Omega \frac{1}{2} \Delta u_\beta * \Delta u_\beta - u_\beta * \Delta^2 u_\beta \ .$$

An integration by parts gives

$$J(u_\beta) = -\frac{1}{2} \int_\Omega \Delta u_\beta * \Delta u_\beta .$$

Every $u \in H_{2,\beta}(\Omega)$ can be extended to a function $u \in H_{2,\beta}(\Omega')$ by setting $u = 0$ on $\Omega' - \Omega$. Therefore,

$$\min_{H_{2,\beta}(\Omega)} J(u) \geq \min_{H_{2,\beta}(\Omega')} J(u) ,$$

that is,

$$\int_\Omega \Delta(\beta_\Omega f) * \Delta(\beta_\Omega f) \leq \int_{\Omega'} \Delta(\beta_{\Omega'} f) * \Delta(\beta_{\Omega'} f) ,$$

which gives

$$\int_\Omega \Delta^2(\beta_\Omega f) * \beta_\Omega f \leq \int_\Omega \Delta^2(\beta_{\Omega'} f) * \beta_{\Omega'} f .$$

The Lemma follows.

4. We now compare β_Ω and γ_Ω on the same regular subregion $\Omega \subset R$.

LEMMA 2. *For $f \in C_0^\infty(\Omega)$,*

$$(f, \beta_\Omega f) \leq (f, \gamma_\Omega f) .$$

Proof. For the Sobolev spaces

$$H_{2,\gamma}(\Omega) = \{u \,|\, D^\alpha u \in L^2(\Omega), \, |\alpha| \leq 2, \, u \,|\, \partial\Omega = 0\} ,$$
$$H_2(\Omega) = \{u \,|\, D^\alpha u \in L^2(\Omega), \, |\alpha| \leq 2\} ,$$

we have $H_{2,\beta} \subset H_{2,\gamma} \subset H_2$, and therefore,

$$\min_{H_{2,\beta}(\Omega)} J(u) \geq \min_{H_{2,\gamma}(\Omega)} J(u) .$$

An argument completely analogous to that in No. 3, with $H_{2,\beta}$ and u_β replaced by $H_{2,\gamma}$ and u_γ, gives

$$J(u_\gamma) = \inf_{H_{2,\gamma}(\Omega)} J(u) .$$

Then we have

$$-\frac{1}{2} \int_\Omega \Delta u_\beta * \Delta u_\beta = J(u_\beta) \geq J(u_\gamma) = -\frac{1}{2} \int_\Omega \Delta u_\gamma * \Delta u_\gamma .$$

As a consequence,

$$\int_\Omega \Delta u_\beta * \Delta u_\beta \leq \int_\Omega \Delta u_\gamma * \Delta u_\gamma \,.$$

Here

$$\int_\Omega \Delta u_\gamma * \Delta u_\gamma = -\int_{\partial \Omega} \Delta u_\gamma * du_\gamma + \int_\Omega d\Delta u_\gamma * du_\gamma$$

$$= \int_{\partial \Omega} u_\gamma * d\Delta u_\gamma + \int_\Omega u_\gamma * \Delta^2 u_\gamma = \int_\Omega u_\gamma * f \,,$$

and we infer that

$$\int_\Omega u_\beta * f \leq \int_\Omega u_\gamma * f \,,$$

hence the Lemma.

5. Thus far Ω has been a fixed regular subregion of R. We now let Ω exhaust R, and indicate inner products taken over Ω by the subscript Ω. We know that either $\lim_{\Omega \to R} \gamma_\Omega(x, y) = \infty$ for every (x, y), or else the biharmonic Green's function $\gamma(x, y) = \lim_{\Omega \to R} \gamma_\Omega(x, y)$ on R exists for every (x, y), with the convergence uniform on compact sets of R (Sario [34]). Let $L_0^2(R)$ be the space of L^2 functions on R with compact supports.

LEMMA 3. *If γ exists on R, then as $\Omega \to R$, $(g, \beta_\Omega f)^\Omega$ converges for all $f, g \in L_0^2(R)$.*

Proof. Since C_0^∞ is dense in L^2, Lemmas 1 and 2 remain valid for $f \in L_0^2(R)$, supp $f \subset \Omega$. If γ exists, the uniform convergence $\gamma_\Omega \to \gamma$ on compact sets entails the existence of a constant M_f such that

$$(f, \gamma_\Omega f)^\Omega < M_f$$

for $f \in C_0^\infty(R)$ and hence for $f \in L_0^2(R)$. By Lemmas 1 and 2 for $f \in L_0^2(\Omega)$, $(f, \beta_\Omega f)^\Omega$ converges. For $f, g \in L_0^2(R)$,

$$2(g, \beta_\Omega f)^\Omega = (f + g, \beta_\Omega(f + g))^\Omega - (f, \beta_\Omega f)^\Omega - (g, \beta_\Omega g)^\Omega$$

and therefore $(g, \beta_\Omega f)^\Omega$ converges.

6. To prove the convergence of β_Ω in every compact set K of an R which carries γ, we may assume that $y \notin K$. In fact, if $\rho(x, y)$ is the geodesic distance between x and y, we may choose, for sufficiently small

constants $\rho_1, \rho_2 > 0$,

$$K = \{x \,|\, \rho_1 \le \rho(x, y) \le \rho_2\} \,.$$

The uniform convergence of β_Ω on K will imply that of the harmonic function $\varDelta \beta_\Omega$ on K and hence on $K_0 = \{x \,|\, 0 \le \rho(x, y) \le \rho_2\}$. A fortiori, the potential part in the Riesz decomposition of β_Ω converges uniformly on K_0 and so does, by the convergence of β_Ω on K, the harmonic part on K and consequently on K_0.

Having chosen an arbitrary but then fixed compact set K of R, $y \notin K$, we take a compact set K_y, $y \in K_y$, $K \cap K_y = \emptyset$, and a function $\varphi \in C_0^\infty(R)$, $\varphi \ge 0$, $\varphi \,|\, K_y = 1$, $K_\varphi = \operatorname{supp} \varphi$ a regular region, $K_\varphi \cap K = \emptyset$, and let Ω henceforth contain $K_\varphi \cup K$. We denote by β_φ the biharmonic Green's function β on K_φ and set

$$\tilde{\beta}_\Omega = \beta_\Omega - \varphi \beta_\varphi \,, \qquad \varDelta_x^2 \tilde{\beta}_\Omega = f \,,$$

where f is independent of Ω. Since $\varDelta^2(\tilde{\beta}_\Omega - \beta_\Omega f) = 0$ and $\tilde{\beta}_\Omega - \beta_\Omega f = \partial(\tilde{\beta}_\Omega - \beta_\Omega f)/\partial n = 0$ on $\partial \Omega$, we have

$$\tilde{\beta}_\Omega = \beta_\Omega f \,.$$

Denote the L^2 norm by $\| \cdot \|$ and choose compact sets K_1, K_2 with $K \subset \operatorname{int} K_1 \subset K_1 \subset \operatorname{int} K_2 \subset K_2$ and $K_2 \cap K_\varphi = \emptyset$, $K_2 \subset \Omega$.

LEMMA 4. *The L^2 norm $\|\tilde{\beta}_\Omega\|^{K_2}$ is bounded in Ω.*

Proof. For the functions $g \in L_0^2(R)$, the restrictions $g \,|\, K_2$ form a Banach space B, on which $\tilde{\beta}_\Omega$ gives a bounded linear functional $(g, \tilde{\beta}_\Omega)^{K_2}$,

$$|(g, \tilde{\beta}_\Omega)|^{K_2} \le k_\Omega \| g \|^{K_2} \,, \qquad k_\Omega = \|\tilde{\beta}_\Omega\|^{K_2} \,.$$

By the Uniform Boundedness Principle, either there exists a constant k independent of Ω with

$$|(g, \tilde{\beta}_\Omega)|^{K_2} \le k \| g \|^{K_2}$$

for all $g \in B$, or else

$$|(g, \tilde{\beta}_\Omega)|^{K_2} \to \infty$$

for some g (in fact for g in a dense set in B) as $\Omega \to R$. But we know from Lemma 3 that $(g, \tilde{\beta}_\Omega) = (g, \beta_\Omega f)$ converges for every g. Therefore, the first alternative occurs. In particular, for $g = \tilde{\beta}_\Omega \,|\, K_2$,

$$(\|\tilde{\beta}_\Omega\|^{K_2})^2 \le k \|\tilde{\beta}_\Omega\|^{K_2} \,,$$

hence the Lemma.

7. We are ready to state our main result:

THEOREM. *On an arbitrarily given Riemannian N-manifold R, N ≥ 2, if the biharmonic Green's function of the simply supported body,*

$$\gamma(x, y) = \lim_{\Omega \to R} \gamma_\Omega(x, y)$$

exists for some, and hence every, (x, y), then so does the biharmonic Green's function of the clamped body,

$$\beta(x, y) = \lim_{\Omega \to R} \beta_\Omega(x, y) ,$$

and the convergence is uniform on compact sets of R.

Proof. By virtue of the interior regularity estimate (cf. Agmon [1, Sec. 5]) in Sobolev norms,

$$\|\tilde{\beta}_\Omega\|_{s+4}^{K_1} \leq C(\|f\|_s^{K_2} + \|\tilde{\beta}_\Omega\|^{K_2}) ,$$

Lemma 4 implies that

$$\|D^\alpha \tilde{\beta}_\Omega\|^{K_1} \text{ is bounded in } \Omega$$

for every $m \geq 0$, $|\alpha| \leq m$. Therefore, by the Rellich compactness theorem (cf. [1, Sec. 3]), there exists a sequence $\{\tilde{\beta}_n = \tilde{\beta}_{\Omega_n}\}$ such that, for $|\alpha| \leq m$,

$$\|D^\alpha(\tilde{\beta}_{n+p} - \tilde{\beta}_n)\|^{K_1} \to 0$$

as $n \to \infty$, $p \geq 0$. This in turn implies by Sobolev's Inequality (cf. [1, Sec. 3]) that, for $|\alpha| \leq m$,

$$|D^\alpha(\tilde{\beta}_{n+p} - \tilde{\beta}_n)| \to 0 .$$

In particular, for $\alpha = 0$ we have by $\tilde{\beta}_n|K = \beta_n|K$ the desired uniform convergence of β_n on K. The uniqueness of the limiting function β on R follows from

$$(g, \beta_n g) \nearrow \sup_{\{\Omega\}} (g, \beta_\Omega g) ,$$

which implies

$$(g, \beta g) = \sup_{\{\Omega\}} (g, \beta_\Omega g)$$

for all $g \in L_0^2(R)$ with supp $g \subset K$.

This completes the proof of the Theorem.

8. The criteria known for the existence of γ are now available, as sufficient conditions, for the existence of β. In particular, for a fixed regular subregion Ω_0 of $R, \bar{\Omega}_0 \subset \Omega$, let ω_Ω be the harmonic measure on $\bar{\Omega} - \Omega_0$ with $\omega_\Omega | \partial \Omega_0 = 1$, $\omega_\Omega | \partial \Omega = 0$. The limiting function $\omega = \lim_{\Omega \to R} \omega_\Omega$ is the harmonic measure of the ideal boundary of R on $R - \Omega_0$. It is known (e.g. Sario-Nakai [37]) that $\omega \equiv 1$ if and only if R belongs to the class $\mathcal{O}_G$ of parabolic Riemannian manifolds, characterized by the nonexistence of the harmonic Green's function $g(x, y) = \lim_{\Omega \to R} g_\Omega(x, y)$. Moreover, γ exists on R if and only if $R \notin \mathcal{O}_G$ and $\omega \in L^2(R - \Omega_0)$ (Sario [34]). The Theorem yields:

COROLLARY. *If $R \notin \mathcal{O}_G$ and $\omega \in L^2(R - \Omega_0)$, then β exists on R.*

For the sake of completeness we give a (slightly simplified) proof. Choose $x, y \in R$ and then Ω_0 with $x, y \in \Omega_0$. For $\Omega \supset \bar{\Omega}_0$ we have by virtue of $\gamma_\Omega = \varDelta \gamma_\Omega = 0$ on $\partial \Omega$,

$$\gamma_\Omega(x, y) = \int_\Omega g_\Omega(x, z) * g_\Omega(z, y) = \int_\Omega g_\Omega(z, x) * g_\Omega(z, y) \ .$$

If $R \notin \mathcal{O}_G$, then $g_\Omega(z, x) \to g(z, x)$, $g_\Omega(z, y) \to g(z, y)$ on R, and $\omega_\Omega \to \omega \neq 1$ on $R - \bar{\Omega}_0$. There exists a compact set C and constants $0 < m < M < \infty$ such that both $g_\Omega(z, x)$ and $g_\Omega(z, y)$ are between m and M for all $z \in \partial \Omega_0$ and all $\Omega \supset C$. Therefore they are between $m\omega_\Omega$ and $M\omega_\Omega$ on $\partial \Omega_0 \cup \partial \Omega$, hence on $\bar{\Omega} - \Omega_0$, and we have

$$m^2 \int_{\Omega - \Omega_0} \omega_\Omega * \omega_\Omega < \int_{\Omega - \Omega_0} g_\Omega(x, z) * g_\Omega(z, y) < M^2 \int_{\Omega - \Omega_0} \omega_\Omega * \omega_\Omega \ .$$

If $\omega \in L^2(R - \Omega_0)$, the integral in the middle converges and so does $\gamma_\Omega(x, y)$. Conversely, if $\gamma_\Omega(x, y)$ converges, so does the integral in the middle, and $\omega \in L^2(\Omega - \Omega_0)$. The Theorem then gives the Corollary.

9. As an immediate application of the Corollary, we have the result of No. 1 that E^N carries β if $N > 4$. Moreover, we now see that this fact is independent of the pole y and the exhaustion $\{\Omega\}$ of E^N. Indeed, for $N > 2$, $\Omega_0 = \{r < 1\}$, $\rho > 1$, $\Omega = \{r < \rho\}$, $\rho \to \infty$,

$$\omega_\Omega = \frac{r^{-N+2} - \rho^{-N+2}}{1 - \rho^{-N+2}} \to r^{-N+2} \ .$$

For $N > 4$, we have $\|\omega\| < \infty$, hence β exists on E^N.

For further illustration of the Corollary, consider the Poincaré N-ball

$$B_\alpha^N = \{r < 1 \,|\, ds = (1 - r^2)^\alpha \,|dx|\} \,,$$

α a constant. A function $h(r)$ on B_α^N is harmonic if and only if

$$\Delta h(r) = -\frac{1}{r^{N-1}(1 - r^2)^{N\alpha}}\{r^{N-1}(1 - r^2)^{(N-2)\alpha}h'(r)\}' = 0 \,,$$

that is,

$$h(r) = c \int_a^r r^{-N+1}(1 - r^2)^{-(N-2)\alpha}dr \,.$$

This is bounded, hence $B_\alpha^N \notin \mathcal{O}_G$, for every α if $N = 2$, for $\alpha < (N - 2)^{-1}$ if $N > 2$. The harmonic measure is

$$\omega(r) \sim \begin{cases} c \log r & \text{for } N = 2 \,, \\ c(1 - r)^{-(N-2)\alpha+1} & \text{for } N > 2 \,. \end{cases}$$

For $0 < \rho_0 < \rho < 1$,

$$(\|\omega\|^{\{\rho_0 < r < \rho\}})^2 \sim \begin{cases} c(1 - \rho)^{2\alpha+3} & \text{for } N = 2 \,, \\ c(1 - \rho)^{-(N-4)\alpha+3} & \text{for } N > 2 \,, \end{cases}$$

and

$$B_\alpha^N \text{ carries } \beta \text{ if} \begin{cases} \alpha > -\dfrac{3}{2} & \text{for } N = 2 \,, \\ -3 < \alpha < 1 & \text{for } N = 3 \,, \\ \alpha < \dfrac{1}{N - 2} & \text{for } N > 3 \,. \end{cases}$$

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University of California, Los Angeles