London Mathematical Society ISSN 1461–1570

ON CONJUGACY CLASSES OF CONGRUENCE SUBGROUPS OF PSL$(2,\mathbb{R})$.

C. J. CUMMINS

Abstract

Let G be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which is commensurable with $\mathrm{PSL}(2,\mathbb{Z})$. We say that G is a *congruence subgroup* of $\mathrm{PSL}(2,\mathbb{R})$ if G contains a principal congruence subgroup $\overline{\Gamma}(N)$ for some N. An algorithm is given for determining whether two congruence subgroups are conjugate in $\mathrm{PSL}(2,\mathbb{R})$. This algorithm is used to determine the $\mathrm{PSL}(2,\mathbb{R})$ conjugacy classes of congruence subgroups of genus-zero and genus-one. The results are given in a table.

1. *Introduction.*

The principal congruence subgroups of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ are defined as follows:

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\},$$

for $N = 1, 2, 3, \ldots$. The image of $\Gamma(N)$ in $\overline{\Gamma} = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm 1_2\}$ is denoted by $\overline{\Gamma}(N)$. A subgroup of $\overline{\Gamma}$ is called a congruence subgroup if it contains some $\overline{\Gamma}(N)$. The level of a congruence subgroup $\overline{G}$ of $\overline{\Gamma}$ is the smallest N such that $\overline{\Gamma}(N)$ is contained in $\overline{G}$.

It is natural to extend this definition to subgroups of $\mathrm{PSL}(2,\mathbb{R})$. A subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is commensurable with $\overline{\Gamma}$ if $\overline{G} \cap \overline{\Gamma}$ has finite index in both $\overline{G}$ and $\overline{\Gamma}$. In this case, we say that $\overline{G}$ is a congruence subgroup if it contains some $\overline{\Gamma}(N)$.

It was originally conjectured by Rademacher that there are only finitely many congruence subgroups of given genus in $\overline{\Gamma}$. This problem was studied by several authors. Cox and Parry [3, 4] gave effective bounds and computed a list of genus-zero congruence subgroups of $\overline{\Gamma}$. Independently, Thompson [9] showed that, up to conjugation, there are only finitely many congruence subgroups of $\mathrm{PSL}(2,\mathbb{R})$ of fixed genus. Motivated by Thompson's result, and using bounds due to Zograf [10], a list of congruence subgroups of $\mathrm{PSL}(2,\mathbb{R})$ of genus-zero and genus-one was found in [5] and fundamental domains for these groups were found in [6].

The strategy of [5] was, following Thompson, to use a result of Helling which states that every group commensurable with $\overline{\Gamma}$ is conjugate to a subgroup of a certain class of maximal discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. Thus to find all congruence subgroups of genus-zero and genus-one, it suffices to compute all the conjugacy classes of such subgroups inside these "Helling Groups". Further work, however, is required to find the resulting $\mathrm{PSL}(2,\mathbb{R})$ conjugacy classes. In this paper we supply the necessary algorithm and the resulting conjugacy classes are given in Table 1.

Work partly supported by NSERC
Received 12 September 2007, revised 10 November 2008; *published* 21 November 2009.
2000 Mathematics Subject Classification 11F06, 11F22, 20H10, 30F35

In Section 2 background definitions are given. The algorithm for the computation is described in Section 3 and some concluding comments are given in Section 4.

2. *Background*

If $\overline{G}$ is a discrete subgroup of PSL$(2,\mathbb{R})$ which is commensurable with $\overline{\Gamma}$, then $\overline{G}$ acts on the extended upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ by fractional linear transformations and the genus of $\overline{G}$ is defined to be the genus of the corresponding Riemann surface $\mathcal{H}^*/\overline{G}$.

From a computational point of view, it is easier to work with subgroups of Γ and SL$(2,\mathbb{R})$, rather than $\overline{\Gamma}$ and PSL$(2,\mathbb{R})$. There is a 1-1 correspondence between the subgroups of PSL$(2,\mathbb{R})$ and the subgroups of SL$(2,\mathbb{R})$ which contain -1_2, where 1_2 is the identity of SL$(2,\mathbb{R})$. Thus in [5] and in this paper we consider subgroups of SL$(2,\mathbb{R})$ which contain -1_2. If G is subgroup of SL$(2,\mathbb{R})$ and $\overline{G}$ is its image in PSL$(2,\mathbb{R})$, then when we refer to geometric invariants such as the genus or cusp number of G, we mean the corresponding invariants of $\overline{G}$.

We recall the following definition.

Definition 2.1.

$$\Gamma_0(f)^+ = \{e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \,\Big|\, a,b,c,d,e \in \mathbb{Z},\ e \mid\mid f,\ e \mid a,\ e \mid d,\ f \mid c,\ ad-bc=e\},$$

where $e \mid\mid f$ means $e \mid f$ and $\gcd(e, f/e) = 1$.

By the following theorem, the study of groups commensurable with Γ is essentially the study of subgroups of the groups $\Gamma_0(f)^+$, where f is a square-free integer.

Theorem 2.2. *(Helling [7]. See also Conway [2]). If G is a subgroup of $SL(2,\mathbb{R})$ which is commensurable with Γ, then G is conjugate to a subgroup of $\Gamma_0(f)^+$ for some square-free f.*

As noted in the introduction, we many define the notion of a congruence subgroup for subgroups of $\Gamma_0(f)^+$ using the same definition as for subgroups of Γ. However, it turns out to be equivalent, and more convenient, to introduce, following Thompson, the appropriate generalization of $\Gamma(N)$. Recall that

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \pmod N\}.$$

Definition 2.3. $G(n,f) = \Gamma_0(nf) \cap \Gamma(n)$.

Note that $G(n,f)$ is a normal subgroup of $\Gamma_0(f)^+$ and that $G(n,1) = \Gamma(n)$.

Definition 2.4. *Call a subgroup G of $\Gamma_0(f)^+$ a congruence subgroup if $G(n,f) \subseteq G$ for some n.*

The following proposition shows that Definition 2.4 is equivalent, for subgroups of $\Gamma_0(f)^+$, to the standard definition of a congruence subgroup given in the introduction.

https://doi.org/10.1112/S1461157000001510 Published online by Cambridge University Press

PROPOSITION 2.5. *A subgroup G of $SL(2,\mathbb{R})$ contains a subgroup $G(n,f)$ for some n and some f if and only if G contains $\Gamma(m)$ for some m.*

Proof. Suppose G contains $\Gamma(m)$. Then $\Gamma(m) = G(m,1)$ and so G contains $G(m,1)$. Conversely, if G contains $G(n,f)$, then we have $\Gamma(nf) \subseteq \Gamma_0(nf)\cap\Gamma(n)$ and so $\Gamma(nf)$ is contained in G. □

We may also extended the notion of the level of a congruence subgroup.

DEFINITION 2.6. *If G is a congruence subgroup of $\Gamma_0(f)^+$, then let $n = n(G,f)$ be the smallest positive integer such that $G(n,f) \subseteq G$. We call $n(G,f)$ the level of G.*

If $f = 1$, then $\Gamma_0(f)^+ = \Gamma$ and this definition of level coincides with the usual definition, since $G(n,1) = \Gamma(n)$. However, if G is a subgroup of $\Gamma_0(f)^+$ and $f \neq 1$, then it is not necessarily the case that $n = n(G,f)$ is the smallest n such that $\Gamma(n)$ is contained in G. Moreover, if G is a congruence subgroup of both $\Gamma_0(f_1)^+$ and $\Gamma_0(f_2)^+$, with $f_1 \neq f_2$, then $n(G,f_1)$ and $n(G,f_2)$ are not necessarily equal.

3. *The Algorithm*

By Helling's Theorem, every subgroup of SL$(2,\mathbb{R})$ which is commensurable with SL$(2,\mathbb{Z})$ is conjugate to a subgroup of $\Gamma_0(f)^+$ for some square-free, positive integer f. Thus to tabulate all conjugacy classes of congruence subgroups of genus-zero and genus-one, it is sufficient to list the genus-zero and genus-one subgroups of $\Gamma_0(f)^+$ for the (finite) set of values of f such that $\Gamma_0(f)^+$ is genus-zero or genus-one. This was done in [5]. However, in [5] the groups were found up to conjugacy in each $\Gamma_0(f)^+$. If a class occurred for two different values of f, then this was recorded in the tables, but the full SL$(2,\mathbb{R})$ conjugacy classes were not computed.

In this section we give an algorithm to find these classes. Table 1 records the results.

Suppose that K_1 and K_2 are subgroups of $\Gamma_0(f_1)^+$ and $\Gamma_0(f_2)^+$ respectively, and that K_1 and K_2 represent two of the classes in Table 2 of [5]. We want to test K_1 and K_2 for conjugacy in SL$(2,\mathbb{R})$. It would initially seem that we have to consider the case $f_1 \neq f_2$. It turns out, however, that we do not need to consider such cases, since we can find a square-free integer f_{min} such that K_1 and K_2 are both subgroups of $\Gamma_0(f_{min})^+$.

PROPOSITION 3.1. *Suppose K_1 and K_2 are congruence subgroups of $\Gamma_0(f_1)^+$ and $\Gamma_0(f_2)^+$ respectively. If K_1 and K_2 are conjugate in* SL$(2,\mathbb{R})$, *then there is some positive, square-free integer f_{min}, such that K_1 and K_2 are subgroups of $\Gamma_0(f_{min})^+$.*

Proof. First note that if $m^{-1}K_1m = K_1^m = K_2$ for some m in SL$(2,\mathbb{R})$, then m is a multiple of a primitive integer matrix, since K_1 and K_2 are commensurable with SL$(2,\mathbb{Z})$. (This follows easily from the fact that K_1 and K_2 both have $\mathbb{Q}^* = \mathbb{Q}\cup\{\infty\}$ as the set fixed by parabolic elements and so m must map $\mathbb{Q}^*$ to $\mathbb{Q}^*$.) Next, for any element m of SL$(2,\mathbb{R})$ which is equal to λA, where A is a primitive integer matrix, we define the normalized determinant $\langle m\rangle$ to be the determinant of A. It is not difficult to verify that $\langle m\rangle$ is well defined.

If $B \in$ SL$(2,\mathbb{Z})$, then $\langle mB\rangle = \langle m\rangle$. So the set of normalized determinants of the elements of K_1 is finite, since K_1 contains some $\Gamma(N)$ with finite index. Let

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$\{e_1,\ldots,e_k\}$ be the set of normalized determinants of the elements of K_1. These must be square-free, since K_1 is contained in $\Gamma_0(f_1)^+$, so the normalized determinants divide f_1. Then $f_{min} = \mathrm{lcm}(e_1,...,e_k)$ is the smallest square-free integer such that K_1 is contained in $\Gamma_0(f_{min})^+$. The matrices of K_2 have the same normalized determinants as K_1 up to squares, since they are obtained by conjugating by a multiple of a primitive integer matrix. But K_2 is contained in some Helling group and so the normalized determinants of its elements must also be square-free. So the set of its normalized determinants is also $\{e_1,\ldots,e_k\}$. Thus K_2 is also contained in $\Gamma_0(f_{min})^+$. □

So to find all $\mathrm{SL}(2,\mathbb{R})$ conjugates of the congruence subgroups of interest, we can first find all $\mathrm{SL}(2,\mathbb{R})$ conjugates within $\Gamma_0(f)^+$ for each f. Then the full $\mathrm{SL}(2,\mathbb{R})$ classes are obtained using the known intersections of classes for different values of f already computed in Table 2 of [5].

The area of a fundamental domain is invariant under conjugation in $\mathrm{SL}(2,\mathbb{R})$ and so if K_1 and K_2 are subgroups of $\Gamma_0(f)^+$ which are conjugate in $\mathrm{SL}(2,\mathbb{R})$, they have the same index in $\Gamma_0(f)^+$. Also, the number of classes of elliptic fixed points of given order and the cusp number are invariant under $\mathrm{SL}(2,\mathbb{R})$ conjugation. Thus when testing K_1 and K_2 for $\mathrm{SL}(2,\mathbb{R})$ conjugacy these invariants are first tested for equality. Also, by [5] Corollary 4.8, the set of primes dividing the levels of K_1 and K_2 are equal if the groups are conjugate in $\mathrm{SL}(2,\mathbb{R})$.

Given two groups which pass these initial tests, we now want to bound the number of conjugating matrices m which have to be considered. As $\Gamma_0(f)^+$ acts transitively on $\mathbb{Q}\cup\{\infty\}$, by replacing K_2 by a conjugate group in $\Gamma_0(f)^+$ we can arrange for m to have the form $m = \lambda\begin{pmatrix} p & q \\ 0 & r\end{pmatrix}$, with p,q,r integers such that $\gcd(p,q,r) = 1$, $pr > 0$ and $\lambda = (pr)^{-1/2}$. By [5] Proposition 4.7 we have the following constraints: $p \mid \ell_1$, $r \mid (\ell_1/p)\gcd(f,p)$, $0 \leqslant q < p$ and also $\ell_2 \mid \gcd(pr,\ell_1)\ell_1$, where ℓ_i is the level of K_i, $i = 1,2$. This assumes that we fix K_1 and conjugate K_2 in $\Gamma_0(f)^+$. If we also allow conjugations of K_1 in $\Gamma_0(f)^+$, then we can also impose $0 \leqslant q < \gcd(p,r)$, since we can then conjugate K_1 by $\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$. As $K_2^{m^{-1}} = K_1$ we can apply the same arguments to get the conditions: $r \mid \ell_2$ and $p \mid (\ell_2/r)\gcd(f,r)$, but again this assumes we allow a conjugation of K_1 in $\Gamma_0(f)^+$. This produces a finite list of possible conjugating matrices m.

To summarize: given K_1 and K_2, subgroups of $\Gamma_0(f)^+$, we first test the obvious invariants for equality. Then, given the levels of the two groups, we can find a finite list of possible conjugating matrices with the property that, if K_1 and K_2 are conjugate in $\mathrm{SL}(2,\mathbb{R})$, then there is some m in the list and some conjugate K_1' of K_1 in $\Gamma_0(f)^+$ and some conjugate K_2' of K_2 in $\Gamma_0(f)^+$ such that m conjugates K_1' to K_2'. Since there are only finitely many conjugates K_1' and K_2', this leads to a finite number of cases to test and so we have an algorithm for finding all the $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes of congruence subgroups of a given genus.

Since the groups are infinite there is still the problem of giving an algorithm to perform the test for equality. This can be done as follows. It is not difficult to verify that if $m = \lambda\begin{pmatrix} p & q \\ 0 & r\end{pmatrix}$ as above, then $mG(pr\ell_1,f)m^{-1} \subseteq G(\ell_1,f) \subseteq K_1$. Let $\ell_3 = \mathrm{lcm}(pr\ell_1,\ell_2)$, then since $G(\ell_3,f) \subseteq G(pr\ell_1)$, we have $mG(\ell_3,f)m^{-1} \subseteq K_1$.

Moreover, $G(\ell_3, f) \subseteq G(\ell_2, f) \subseteq K_2$. Thus to test if mK_2m^{-1} is equal to K_1, it is sufficient to test if msm^{-1} is in K_1, for all $s \in S$ for some set of generators S of K_2 over $G(\ell_3, f)$. (The fact that $G(\ell_3, f)$ and $G(\ell_2, f)$ are normal subgroups of K_2 and that $K_2/G(\ell_2, f)$ was constructed as a permutation group in [5] simplifies the computation of the cosets). Thus we are finished if we have an algorithm for testing when an element g in $\Gamma_0(f)^+$ is in K_1, but as the group $K_1/G(\ell_1, f)$ was constructed as a subgroup of $\Gamma_0(f)^+/G(\ell_1, f)$ in [5], we can simply test to see if the image of g in $\Gamma_0(f)^+/G(\ell_1, f)$ is in $K_1/G(\ell_1, f)$.

In summary we have the following algorithm.

The Algorithm

Input: A square-free integer f, and two congruence subgroups K_1 and K_2 of $\Gamma_0(f)^+$, of levels ℓ_1 and ℓ_2 respectively.

Output: Return **true** if K_1 and K_2 are conjugate in $\mathrm{SL}(2,\mathbb{R})$ and **false** otherwise.

- $I_1 \leftarrow$ invariants of K_1
- $I_2 \leftarrow$ invariants of K_2
- If $I_1 \neq I_2$ then return **false**
- Else
  - $C_1 \leftarrow$ conjugates of K_1 in $\Gamma_0(f)^+$
  - $C_2 \leftarrow$ conjugates of K_2 in $\Gamma_0(f)^+$
  - $M \leftarrow$ list of possible conjugating matrices
  - For K in C_2
    - For $m = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$ in M
      - $\ell_3 \leftarrow \mathrm{lcm}(pr\ell_1, \ell_2)$.
      - $S \leftarrow$ a set of generators of K over the normal subgroup $G(\ell_3, f)$.
      - $S' \leftarrow mSm^{-1}$
      - If S' is not a subset of $\Gamma_0(f)^+$ then move to the next m
      - For each L in C_1
        - If the image of S' in $\Gamma_0(f)^+/G(\ell_1, f)$ is contained in $L/G(\ell_1, f)$ then return **true**
- If after testing all elements of C_2 we have not returned **true** then return **false**

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4. *Conclusions*

In the previous section an algorithm was given for determining whether two subgroups of $\Gamma_0(f)^+$ are conjugate in $\mathrm{SL}(2,\mathbb{R})$. Table 2 of [5] records when the conjugacy classes of subgroups C_i of $\Gamma_0(f_i)^+$, $i = 1, 2$, are such that $f_1 \neq f_2$ and $C_1 \cap C_2$ is not empty. As explained in the last section, combining these two pieces

https://doi.org/10.1112/S1461157000001510 Published online by Cambridge University Press

of information yields the $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes of congruence subgroups of genus-zero and genus-one.

The algorithm of the previous section was programmed in MAGMA [**1**]. The results are presented in Table 1. Each entry in this table is a list of the $\Gamma_0(f)^+$ conjugacy classes from Table 2 of [**5**] which are subsets of one $\mathrm{SL}(2,\mathbb{R})$ conjugacy class. Cases with a single $\Gamma_0(f)^+$ conjugacy class have been omitted. The notation is as in Table 2 of [**5**]: each class has a label $level(letter)^{genus}_f$, where f is a square-free integer such that the group is contained in $\Gamma_0(f)^+$; *level* is the level of the group with respect to this f (as defined in Definition 2.4); *genus* is its genus; and *letter* is a letter labelling the group amongst all groups of the same level and genus.

Sebbar [**8**] has shown that there are 15 $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes of torsion-free congruence subgroups of genus-zero. As a test of the results of this paper, using the additional data from Table 2 of [**5**], we also find 15 $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes of torsion-free, genus-zero congruence subgroups. Moreover, these give rise to 33 classes of subgroups of the modular group, again in agreement with Sebbar's results.

References

1. Wieb Bosma, John Cannon and Catherine Playoust, 'The Magma algebra system. I. The user language.' *J. Symbolic Comput.* 24 (1997) 235–265. Computational algebra and number theory (London, 1993). 269

2. J. H. Conway, 'Understanding groups like $\Gamma_0(N)$.' 'Groups, difference sets, and the Monster (Columbus, OH, 1993),' (de Gruyter, Berlin, 1996), vol. 4 of *Ohio State Univ. Math. Res. Inst. Publ.* pp. 327–343, pp. 327–343. 265

3. David A. Cox and Walter R. Parry, 'Genera of congruence subgroups in **Q**-quaternion algebras.' *J. Reine Angew. Math.* 351 (1984) 66–112. 264

4. David A. Cox and Walter R. Parry, 'Genera of congruence subgroups in **Q**-quaternion algebras.' Unabridge preprint. 264

5. C. J. Cummins, 'Congruence subgroups of groups commensurable with $\mathrm{PSL}(2,\mathbb{Z})$ of genus 0 and 1.' *Experiment. Math.* 13 (2004) 361–382. 264, 265, 266, 267, 268, 269

6. C. J. Cummins, '*fundamental domains for congruence subgroups of genus 0 and 1, .' submitted for publication . 264*

7. Heinz Helling, *'Bestimmung der Kommensurabilitätsklasse der Hilbertschen Modulgruppe.' Math. Z. 92 (1966) 269–280. 265*

8. Abdellah Sebbar, *'Classification of torsion-free genus zero congruence groups.' Proc. Amer. Math. Soc. 129 (2001) 2517–2527 (electronic). 269*

9. J. G. Thompson, *'A finiteness theorem for subgroups of* $\mathrm{PSL}(2,\mathbf{R})$ *which are commensurable with* $\mathrm{PSL}(2,\mathbf{Z})$*.' 'The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979),' (Amer. Math. Soc., Providence, R.I., 1980), vol. 37 of Proc. Sympos. Pure Math. pp. 533–555, pp. 533–555. 264*

10. P. Zograf, *'A spectral proof of Rademacher's conjecture for congruence subgroups of the modular group.' J. Reine Angew. Math. 414 (1991) 113–116. 264*

https://doi.org/10.1112/S1461157000001510 Published online by Cambridge University Press

Table 1: PSL$(2,\mathbb{R})$ conjugacy classes of congruence subgroups of genus-zero and genus-one.

Classes	*Classes*	*Classes*	*Classes*
$2B_1^0\ 1B_2^0$	$8C_1^0\ 4D_2^0$	$14C_1^0\ 2D_7^0$	$4H_3^0\ 4I_3^0\ 8C_3^0\ 2L_6^0\ 4I_6^0$
$2C_1^0\ 4B_1^0\ 2D_2^0$	$8I_1^0\ 4K_2^0$	$14D_1^0\ 14C_7^0$	$4J_3^0\ 8D_3^0\ 4J_6^0$
$3B_1^0\ 1B_3^0$	$8J_1^0\ 8K_1^0\ 16E_1^0\ 8L_2^0\ 8U_2^0\ 8V_2^0$	$15C_1^0\ 3C_5^0$	$4K_3^0\ 8F_3^0\ 4M_6^0$
$3D_1^0\ 9B_1^0\ 3D_3^0$	$8L_1^0\ 16F_1^0\ 8W_2^0$	$15D_1^0\ 15E_5^0$	$6E_3^0\ 3C_6^0$
$4C_1^0\ 2C_2^0$	$8M_1^0\ 8N_2^0$	$16C_1^0\ 8J_2^0$	$7A_3^0\ 1C_{21}^0$
$4E_1^0\ 8D_1^0\ 2F_2^0\ 4F_2^0$	$8P_1^0\ 8Q_1^0\ 16H_1^0\ 16I_1^0\ 32A_1^0\ 8AB_2^0\ 8Y_2^0\ 8Z_2^0\ 16E_2^0$	$18C_1^0\ 6F_3^0$	$7B_3^0\ 7B_{21}^0$
$4F_1^0\ 8E_1^0\ 4G_2^0$	$8R_1^0\ 8AA_2^0$	$24A_1^0\ 12E_2^0$	$12F_3^0\ 24B_3^0\ 12D_6^0$
$4G_1^0\ 8H_1^0\ 16D_1^0\ 4J_2^0\ 8M_2^0$	$9C_1^0\ 3C_3^0$	$26A_1^0\ 26C_{13}^0$	$12G_3^0\ 12H_3^0\ 12I_3^0\ 24A_3^0\ 12A_6^0\ 12B_6^0\ 12C_6^0$
$5B_1^0\ 1B_5^0$	$9H_1^0\ 9I_1^0\ 27A_1^0\ 9F_3^0\ 9G_3^0$	$28A_1^0\ 28C_7^0$	$12N_3^0\ 6F_6^0$
$5D_1^0\ 5C_5^0$	$9J_1^0\ 9E_3^0$	$30A_1^0\ 30C_5^0$	$12R_3^0\ 12S_3^0\ 24D_3^0\ 24E_3^0\ 12M_6^0\ 12N_6^0\ 12O_6^0$
$5G_1^0\ 25A_1^0\ 5H_5^0$	$10B_1^0\ 10D_5^0$	$36A_1^0\ 12O_3^0$	$21A_3^0\ 21B_3^0\ 21B_{21}^0$
$5H_1^0\ 25B_1^0\ 5I_5^0$	$10C_1^0\ 5D_2^0\ 2C_5^0\ 1E_{10}^0$	$48A_1^0\ 24C_2^0$	$2B_5^0\ 1B_{10}^0$
$6C_1^0\ 2D_3^0$	$10F_1^0\ 5G_2^0\ 10N_5^0\ 5C_{10}^0$	$3B_2^0\ 1C_6^0$	$2D_5^0\ 4B_5^0\ 2G_{10}^0$
$6D_1^0\ 3C_2^0$	$10G_1^0\ 20A_1^0\ 10D_2^0\ 10Q_5^0\ 20E_5^0\ 10G_{10}^0$	$3H_2^0\ 9B_2^0\ 3E_6^0$	$3B_5^0\ 1B_{15}^0$
$6F_1^0\ 3F_2^0\ 2F_3^0\ 1E_6^0$	$12B_1^0\ 12C_3^0$	$5B_2^0\ 1D_{10}^0$	$4C_5^0\ 2D_{10}^0$
$6G_1^0\ 3G_2^0$	$12C_1^0\ 6E_2^0$	$6D_2^0\ 2E_6^0$	$4E_5^0\ 2I_{10}^0$
$6H_1^0\ 12D_1^0\ 6H_2^0$	$12H_1^0\ 6N_2^0$	$6M_2^0\ 2M_6^0$	$6B_5^0\ 2D_{15}^0$
$6I_1^0\ 12E_1^0\ 6L_2^0\ 2H_3^0\ 4G_3^0\ 2K_6^0$	$12I_1^0\ 12J_1^0\ 24B_1^0\ 12G_2^0\ 12H_2^0\ 12P_3^0\ 12Q_3^0\ 24C_3^0\ 12I_6^0\ 12J_6^0$	$9F_2^0\ 9G_2^0\ 9C_6^0$	$8A_5^0\ 4D_{10}^0$
$6J_1^0\ 18D_1^0\ 6G_3^0$	$13A_1^0\ 1B_{13}^0$	$10B_2^0\ 2F_{10}^0$	$10F_5^0\ 10G_5^0\ 5A_{10}^0$
$6K_1^0\ 18F_1^0\ 3I_2^0\ 9E_2^0\ 6H_3^0\ 3G_6^0$	$13B_1^0\ 13D_{13}^0$	$10E_2^0\ 10J_{10}^0$	$10H_5^0\ 20A_5^0\ 10B_{10}^0$
$6L_1^0\ 12G_1^0\ 6O_2^0$	$13C_1^0\ 13E_{13}^0$	$12D_2^0\ 4E_6^0$	$10I_5^0\ 20B_5^0\ 10A_{10}^0$
$7C_1^0\ 1B_7^0$		$12I_2^0\ 12L_6^0$	$10O_5^0\ 20G_5^0\ 10F_{10}^0$
$7G_1^0\ 7B_7^0$		$2C_3^0\ 1D_6^0$	$10P_5^0\ 20F_5^0\ 10E_{10}^0$
		$2E_3^0\ 4B_3^0\ 2H_6^0$	
		$2G_3^0\ 4C_3^0\ 2G_6^0$	
		$4D_3^0\ 2F_6^0$	

Classes		
$15C^0_5$	$15D^0_5$	$5A^0_{15}$
$15G^0_5$	$15H^0_5$	$15A^0_{15}$
$20H^0_5$	$20I^0_5$	$10I^0_{10}$
	$2C^0_7$	$1B^0_{14}$
$2E^0_7$	$4A^0_7$	$2D^0_{14}$
	$6B^0_7$	$3B^0_{14}$
	$14B^0_7$	$7C^0_{14}$
	$2B^0_{11}$	$1B^0_{22}$
	$3A^0_{11}$	$1B^0_{33}$
	$4B^0_{11}$	$2C^0_{22}$
	$2C^0_{15}$	$1E^0_{30}$
	$2E^0_{15}$	$1F^0_{30}$
$2F^0_{15}$	$4A^0_{15}$	$2D^0_{30}$
	$2A^0_{23}$	$1B^0_{46}$
$6C^1_1$	$12B^1_1$	$6C^1_2$
$6D^1_1$	$18C^1_1$	$6D^1_3$
$6E^1_1$	$12K^1_1$	$6M^1_2$
$6F^1_1$ $36C^1_1$ $6K^1_3$	$12T^1_1$ $6Q^1_2$ $12P^1_3$	$18J^1_1$ $18F^1_2$ $6I^1_6$
$7B^1_1$	$49A^1_1$	$7A^1_7$
	$8A^1_1$	$4B^1_2$
$8B^1_1$	$16A^1_1$	$4D^1_2$ $8H^1_2$
	$8C^1_1$	$4E^1_2$
$8D^1_1$	$16C^1_1$	$8T^1_2$
$8F^1_1$ $4F^1_2$	$16E^1_1$ $8X^1_2$	$32A^1_1$ $16I^1_2$

Classes		
$8G^1_1$	$16G^1_1$	$8AC^1_2$ $8V^1_2$
$8H^1_1$	$16I^1_1$	$8AE^1_2$
$8I^1_1$	$16H^1_1$	$8AD^1_2$
$8K^1_1$ $8AH^1_2$	$16M^1_1$ $8AI^1_2$	$32E^1_1$ $16W^1_2$
	$9A^1_1$	$3A^1_3$
$9D^1_1$	$27A^1_1$	$3B^1_3$ $9D^1_3$
	$9E^1_1$	$9B^1_3$
$9J^1_1$	$27C^1_1$	$9H^1_3$ $9I^1_3$
	$10A^1_1$	$2B^1_5$
	$10B^1_1$	$5A^1_2$
	$10D^1_1$	$10C^1_5$
	$10F^1_1$	$5C^1_2$
$10G^1_1$ $2D^1_5$	$20E^1_1$ $4D^1_5$	$10F^1_2$ $2H^1_{10}$
	$10I^1_1$	$5D^1_2$
$10K^1_1$ $10F^1_5$	$20J^1_1$ $20W^1_5$	$10L^1_2$ $10L^1_{10}$
	$11A^1_1$	$1A^1_{11}$
	$11D^1_1$	$11A^1_{11}$
	$12A^1_1$	$4B^1_3$
	$12C^1_1$	$6E^1_2$
$12F^1_1$	$6I^1_2$	$4C^1_3$ $2G^1_6$
	$12I^1_1$	$4G^1_3$
$12J^1_1$	$24E^1_1$	$12H^1_2$

Classes		
	$12L^1_1$	$6L^1_2$
	$12M^1_1$	$6K^1_2$
	$12N^1_1$	$6J^1_2$
$12O^1_1$	$24G^1_1$	$12N^1_2$
$12P^1_1$	$36B^1_1$	$12K^1_3$
$12Q^1_1$ $12V^1_2$	$24I^1_1$ $4H^1_3$ $2H^1_6$	$6P^1_2$ $8J^1_3$ $4U^1_6$
	$12S^1_1$	$12O^1_3$
	$12U^1_1$	$6R^1_2$
$12V^1_1$	$24K^1_1$	$12Y^1_2$
$12W^1_1$ $12AD^1_2$	$24L^1_1$ $12T^1_3$ $12U^1_6$	$12AC^1_2$ $24O^1_3$ $12V^1_6$
	$14C^1_1$	$7E^1_2$
	$14D^1_1$	$7B^1_2$
$14E^1_1$	$7F^1_2$	$2A^1_7$ $1B^1_{14}$
$14L^1_1$	$7N^1_2$	$14C^1_7$ $7B^1_{14}$
	$15B^1_1$	$5B^1_3$
$15C^1_1$	$5C^1_3$	$3B^1_5$ $1B^1_{15}$
	$15E^1_1$	$3C^1_5$
$15G^1_1$	$5F^1_3$	$15E^1_5$ $5A^1_{15}$
	$15H^1_1$	$15F^1_5$
$15I^1_1$	$15D^1_3$	$15I^1_5$ $15A^1_{15}$
	$16B^1_1$	$8G^1_2$

Classes		
	$16D^1_1$	$8N^1_2$
	$16F^1_1$	$8W^1_2$
	$16J^1_1$	$8AB^1_2$
$16K^1_1$	$32C^1_1$	$16Q^1_2$
$16L^1_1$	$32D^1_1$	$16R^1_2$
	$17A^1_1$	$1A^1_{17}$
	$17B^1_1$	$17A^1_{17}$
	$17C^1_1$	$17B^1_{17}$
	$18D^1_1$	$6E^1_3$
	$18E^1_1$	$9D^1_2$
	$18I^1_1$	$9I^1_2$
	$18K^1_1$	$6L^1_3$
	$19A^1_1$	$1A^1_{19}$
	$19B^1_1$	$19A^1_{19}$
$20B^1_1$	$20C^1_1$	$20E^1_5$
	$20D^1_1$	$4C^1_5$
$20F^1_1$	$10E^1_2$	$4E^1_5$ $2G^1_{10}$
	$20I^1_1$	$20V^1_5$
$20K^1_1$ $40B^1_1$ $20AA^1_5$	$20L^1_1$ $20G^1_2$ $40G^1_5$	$40A^1_1$ $20H^1_2$ $20L^1_{10}$
$20M^1_1$	$10K^1_2$	$20Y^1_5$ $10K^1_{10}$
$21A^1_1$	$21B^1_1$	$21A^1_7$
$21C^1_1$	$7A^1_3$	$3C^1_7$ $1B^1_{21}$
$21I^1_1$	$21C^1_3$	$21C^1_7$ $21B^1_{21}$

Classes		
	$24F_1^1$	$12F_2^1$
	$24J_1^1$	$12X_2^1$
	$26A_1^1$	$2C_{13}^1$
	$26B_1^1$	$26F_{13}^1$
	$27B_1^1$	$9C_3^1$
	$30D_1^1$	$6E_5^1$
	$30E_1^1$	$30K_5^1$
	$32B_1^1$	$16G_2^1$
$39A_1^1$	$39B_1^1$	$39A_{13}^1$
$52A_1^1$	$52B_1^1$	$52B_{13}^1$
$5E_2^1$	$25A_2^1$	$5A_{10}^1$
	$6B_2^1$	$2B_6^1$
$6O_2^1$	$18B_2^1$	$6E_6^1$
	$7A_2^1$	$1A_{14}^1$
	$7G_2^1$	$7A_{14}^1$
	$9A_2^1$	$3C_6^1$
	$9B_2^1$	$3D_6^1$
	$9G_2^1$	$9D_6^1$
	$9H_2^1$	$9E_6^1$
$9J_2^1$	$27A_2^1$	$9H_6^1$
	$10B_2^1$	$2B_{10}^1$
	$11A_2^1$	$1A_{22}^1$
	$11B_2^1$	$11A_{22}^1$
	$12E_2^1$	$4K_6^1$
	$12U_2^1$	$4S_6^1$
	$12W_2^1$	$4X_6^1$

Classes		
	$13A_2^1$	$1B_{26}^1$
	$13B_2^1$	$13B_{26}^1$
	$15D_2^1$	$3F_{10}^1$
	$15E_2^1$	$5C_6^1$
	$17A_2^1$	$1C_{34}^1$
	$17B_2^1$	$17C_{34}^1$
$20B_2^1$	$20C_2^1$	$20C_{10}^1$
	$20D_2^1$	$4I_{10}^1$
$20I_2^1$	$20J_2^1$	$20P_{10}^1$
$24J_2^1$	$24L_2^1$	$8M_6^1$
$24K_2^1$	$24M_2^1$	$24E_6^1$
$24N_2^1$	$24O_2^1$	$24H_6^1$
	$24P_2^1$	$8L_6^1$
	$4A_3^1$	$2D_6^1$
	$4D_3^1$	$2E_6^1$
$4E_3^1$	$8C_3^1$	$2F_6^1$ $4M_6^1$
$4F_3^1$	$8D_3^1$	$4N_6^1$
$4I_3^1$	$8K_3^1$ $4V_6^1$	$16E_3^1$ $8O_6^1$
	$5A_3^1$	$1A_{15}^1$
	$6C_3^1$	$3B_6^1$
$6F_3^1$	$12F_3^1$	$6D_6^1$
$6H_3^1$	$12G_3^1$	$6C_6^1$
	$6J_3^1$	$3E_6^1$
	$8E_3^1$	$4J_6^1$
	$8F_3^1$	$4E_6^1$

Classes		
	$8L_3^1$	$4R_6^1$
	$8M_3^1$	$4T_6^1$
$8N_3^1$	$16F_3^1$	$8P_6^1$
$8O_3^1$	$16G_3^1$	$8Q_6^1$
$8P_3^1$	$16H_3^1$	$8V_6^1$
	$10B_3^1$	$5B_6^1$
$12A_3^1$	$24B_3^1$	$12A_6^1$
$12D_3^1$	$24A_3^1$	$12B_6^1$
$12H_3^1$	$24G_3^1$	$12H_6^1$ $12K_6^1$
$12I_3^1$	$24H_3^1$	$12I_6^1$ $12J_6^1$
$12L_3^1$ $24K_3^1$	$24I_3^1$ $12M_6^1$	$24J_3^1$ $12N_6^1$ $12O_6^1$
$12M_3^1$	$24L_3^1$ $12L_6^1$	$48A_3^1$ $24G_6^1$
$12Q_3^1$	$24N_3^1$	$12T_6^1$
$12S_3^1$	$24P_3^1$	$12W_6^1$
	$13A_3^1$	$1A_{39}^1$
	$13B_3^1$	$13F_{39}^1$
	$14A_3^1$	$2D_{21}^1$
	$14B_3^1$	$2E_{21}^1$
	$18J_3^1$	$9G_6^1$
$21A_3^1$	$21B_3^1$	$21A_{21}^1$
$21D_3^1$	$21E_3^1$	$21D_{21}^1$
$42A_3^1$	$42B_3^1$	$42B_{21}^1$
$2C_5^1$	$4A_5^1$	$2E_{10}^1$

Classes		
$3D_5^1$	$9B_5^1$	$3B_{15}^1$
	$4B_5^1$	$2C_{10}^1$
$4F_5^1$	$8B_5^1$	$4K_{10}^1$
$4G_5^1$	$8C_5^1$	$4J_{10}^1$
	$6D_5^1$	$3D_{10}^1$
$6H_5^1$	$3G_{10}^1$	$2F_{15}^1$ $1F_{30}^1$
	$7A_5^1$	$1A_{35}^1$
	$7B_5^1$	$7A_{35}^1$
	$8A_5^1$	$4F_{10}^1$
	$8E_5^1$	$4N_{10}^1$
$10D_5^1$	$20H_5^1$	$10E_{10}^1$
$10E_5^1$	$20I_5^1$	$10F_{10}^1$
	$16A_5^1$	$8F_{10}^1$
$20A_5^1$	$20B_5^1$	$10A_{10}^1$
$20AB_5^1$	$40H_5^1$	$20M_{10}^1$
$20AC_5^1$	$40I_5^1$	$20K_{10}^1$
$20J_5^1$	$20Q_5^1$	$10I_{10}^1$
	$20K_5^1$	$10C_{10}^1$
$20L_5^1$	$20O_5^1$	$10H_{10}^1$
$20M_5^1$	$20N_5^1$	$10J_{10}^1$
	$20P_5^1$	$10D_{10}^1$
$20R_5^1$	$40C_5^1$	$20D_{10}^1$
$20S_5^1$	$40D_5^1$	$20E_{10}^1$
$20X_5^1$	$20Z_5^1$	$10M_{10}^1$
$30C_5^1$	$30D_5^1$	$10A_{15}^1$

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Classes		
$30I_5^1$	$30J_5^1$	$10C_{15}^1$
$40A_5^1$	$40B_5^1$	$20B_{10}^1$
$40E_5^1$	$40F_5^1$	$20J_{10}^1$
	$5A_6^1$	$1C_{30}^1$
	$5D_6^1$	$1D_{30}^1$
	$5I_6^1$	$5C_{30}^1$
	$7A_6^1$	$1C_{42}^1$
	$7B_6^1$	$7A_{42}^1$
$2B_7^1$	$4B_7^1$	$2E_{14}^1$
	$3A_7^1$	$1A_{21}^1$
	$4C_7^1$	$2C_{14}^1$
	$4D_7^1$	$2D_{14}^1$
$4F_7^1$	$8A_7^1$	$2G_{14}^1$ $4E_{14}^1$
$6E_7^1$	$12A_7^1$	$6E_{14}^1$
	$6F_7^1$	$3E_{14}^1$
$14D_7^1$	$28F_7^1$	$14C_{14}^1$
$28A_7^1$ $56A_7^1$	$28B_7^1$ $28A_{14}^1$	$28C_7^1$ $28B_{14}^1$ $28C_{14}^1$
	$3A_{10}^1$	$1B_{30}^1$
$2B_{11}^1$	$4B_{11}^1$	$2B_{22}^1$
$2C_{11}^1$	$4A_{11}^1$	$2C_{22}^1$
$4D_{11}^1$	$8C_{11}^1$	$4C_{22}^1$
	$5A_{11}^1$	$1A_{55}^1$
	$5B_{11}^1$	$5A_{55}^1$
	$6A_{11}^1$	$2D_{33}^1$

Classes		
$33A_{11}^1$	$33B_{11}^1$	$33A_{33}^1$
	$2B_{13}^1$	$1A_{26}^1$
	$3A_{14}^1$	$1B_{42}^1$
	$3D_{14}^1$	$1F_{42}^1$
$2E_{15}^1$	$4C_{15}^1$	$2M_{30}^1$
$2G_{15}^1$	$4B_{15}^1$	$2N_{30}^1$
$2H_{15}^1$	$4G_{15}^1$	$2R_{30}^1$
	$4D_{15}^1$	$2I_{30}^1$
	$4E_{15}^1$	$2H_{30}^1$
	$4H_{15}^1$	$2Q_{30}^1$
	$6D_{15}^1$	$3C_{30}^1$
	$10D_{15}^1$	$5D_{30}^1$
	$2A_{17}^1$	$1B_{34}^1$
	$3B_{17}^1$	$1A_{51}^1$
	$2B_{19}^1$	$1A_{38}^1$
	$2C_{21}^1$	$1A_{42}^1$
	$3B_{22}^1$	$1A_{66}^1$
$2C_{23}^1$	$4A_{23}^1$	$2C_{46}^1$
	$3A_{23}^1$	$1A_{69}^1$
	$2C_{31}^1$	$1A_{62}^1$
	$2C_{33}^1$	$1C_{66}^1$
	$2C_{35}^1$	$1C_{70}^1$
	$3A_{35}^1$	$1C_{105}^1$
	$2E_{39}^1$	$1B_{78}^1$
	$2A_{47}^1$	$1A_{94}^1$
	$2C_{55}^1$	$1A_{110}^1$

https://doi.org/10.1112/S1461157000001510 Published online by Cambridge University Press

C. J. Cummins cummins@mathstat.concordia.ca

Department of Mathematics and Statistics
Concordia University
Montréal
Québec
Canada