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IITAKA $C_{n,m}$ CONJECTURE FOR 3-FOLDS OVER FINITE FIELDS

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Abstract. We prove Iitaka $C_{n,m}$ conjecture for 3-folds over the algebraic closure of finite fields. Along the way we prove some results on the birational geometry of log surfaces over nonclosed fields and apply these to existence of relative good minimal models of 3-folds.

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§1. Introduction

Iitaka conjecture

Let X be a normal projective variety over a field k, L a Cartier divisor on X, and $N(L)$ the set of all positive integers m such that the linear system $|mL| \neq \emptyset$. For an integer $m \in N(L)$, let $\Phi_{|mL|}$ be the rational map defined by $|mL|$. The Kodaira dimension $\kappa(L)$ is defined as

$$\kappa(L) = \begin{cases} -\infty, & \text{if } N(L) = \emptyset \\ \max\{\dim \Phi_{|mL|}(X) \mid m \in N(L)\} & \text{if } N(L) \neq \emptyset \end{cases}$$

If L is a $\mathbb{Q}$-Cartier divisor, $\kappa(L) := \kappa(mL)$ for any natural number m so that mL is Cartier. This does not depend on the choice of m.

Throughout this paper, a contraction is a projective morphism $f\colon X \to Z$ between varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$. The following conjecture due to Iitaka (in characteristic zero) is of fundamental importance in the classification theory of algebraic varieties.

CONJECTURE 1.1. ($C_{n,m}$) *Let $f\colon X \to Z$ be a contraction between smooth projective varieties of dimension n, m respectively, over an algebraically closed field k. Assume the generic fiber F is smooth. Then*

$$\kappa(K_X) \geqslant \kappa(K_F) + \kappa(K_Z).$$

One can formulate a more general problem when F is not smooth either by assuming it is geometrically integral with a resolution or by considering F as a variety over the function field of Z (for example, see Corollary 1.3).

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Over the field of complex numbers, the conjecture has been studied by Kawamata [22, 23, 25], Kollár [29], Viehweg [43, 44, 44], Birkar [3], Chen and Hacon [12], Cao and Păun [8], and so forth. We refer to [13] for a collection of results over $\mathbb{C}$. In positive characteristic, Chen and Zhang proved the conjecture for fibrations of relative dimension one [13], and Patakfalvi proved it when Z is of general type and the generic geometric fiber satisfies certain properties [36, Theorem 1.1] (see also [37, Corollary 4.6]).

In this paper, we prove:

THEOREM 1.2. *Conjecture $C_{n,m}$ holds when $n=3$, $k=\bar{\mathbb{F}}_p$, and $p>5$.*

Note that the smoothness of X_η implies that f is separable [31, Section 3.2.2]. The case $C_{3,2}$ follows from [13], so the main result here is $C_{3,1}$. Our main tools are the log minimal model program for 3-folds developed recently by Hacon, Xu, and Birkar [5, 16, 47], birational geometry of log surfaces over nonclosed fields (see below), and the semi-positivity results of Patakfalvi [37]. The reason for the restriction $k=\bar{\mathbb{F}}_p$ is that it is often easier to prove semi-ampleness of divisors over finite fields; for example, if $K_X\sim_{\mathbb{Q}} f^*D$ for some $D\equiv 0$ on Z, then $D\sim_{\mathbb{Q}} 0$ is automatic over $k=\bar{\mathbb{F}}_p$ but the same conclusion would perhaps require substantial effort over other fields; this is a major issue also in characteristic zero [22].

Since resolution theory holds in dimension three in positive characteristics, we get

COROLLARY 1.3. *Let $f\colon X\to Z$ be a contraction, from a smooth projective three- dimensional variety to a smooth projective curve over $\bar{\mathbb{F}}_p, p>5$. Let $\tilde{F}$ be a smooth model of the generic geometric fiber of f. Then*

$$\kappa(K_X)\geqslant\kappa(K_{\tilde{F}})+\kappa(K_Z).$$

Log surfaces over nonclosed fields

Let $X\to Z$ be a contraction between normal varieties and let F be its generic fiber. As is well known, in char $p>0$, F may not be smooth even if X and Z are smooth. Actually F may even be geometrically nonreduced. This creates difficulties because proofs in birational geometry are often based on induction and in this case we cannot simply apply induction and lift information from F to X. On the other hand, F has nice properties if we think of it as a variety over the function field of Z without passing to the algebraic closure of this function field. For example, if X is smooth, then F

is regular. In particular, relevant to this paper is the case in which X is a 3-fold and Z is a curve. So it is natural for us to consider surfaces over a not necessarily algebraically closed field k.

It is easy to define pairs, singularities, minimal models, and so forth over an arbitrary field. See Sections 2.4 and 2.5 for more details.

THEOREM 1.4. *Let (X, B) be a projective dlt pair of dimension two over a field k where B is a $\mathbb{Q}$-boundary. Then we can run an LMMP on $K_X + B$ which ends with a log minimal model or a Mori fiber space.*

THEOREM 1.5. *Let (X, B) be a projective klt pair of dimension two over a field k where B is a $\mathbb{Q}$-boundary. Assume $K_X + B$ is nef and that $\kappa(K_X + B) \geqslant 0$. Then $K_X + B$ is semi-ample.*

These results were proved by Tanaka [41] not long ago. Actually he proves more general statements; in particular, he proves 1.5 without the assumption $\kappa(K_X + B) \geqslant 0$. We give a self-contained proof of the above theorems. Our proof of 1.4 is perhaps the same as that in [41] which closely follows Keel's techniques [26]. However, our proof of 1.5 seems to be different from his. He relies on another paper [40] but our proof is short and direct which follows Mumford's ideas [33] and uses a result of Totaro [42]. In fact we worked out these proofs before [41] appeared.

Relative good minimal models of 3-folds

As mentioned earlier our motivation for considering surfaces over non-closed fields is to treat 3-folds over curves.

THEOREM 1.6. *Let (X, B) be a projective klt pair of dimension three where B is a $\mathbb{Q}$-boundary, and $f\colon X \to Z$ be a contraction onto a curve, over $\bar{\mathbb{F}}_p$ with $p > 5$. Let F be the generic fiber of f. If $\kappa((K_X + B)|_F) \geqslant 0$, then (X, B) has a good log minimal model over Z.*

Actually the proof of the theorem works over any algebraically closed field of char $p > 5$ except when $\kappa((K_X + B)|_F) = 1$. In this case we make use of the fact that any nef and big divisor on a surface over $\bar{\mathbb{F}}_p$ is semi-ample.

As far as Theorem 1.2 is concerned we only need special cases of 1.6 which in turn only needs special cases of 1.5. We only need the case when $B = 0$ and F is smooth, or when $\kappa((K_X + B)|_F) = 0$ but F admits a contraction onto an elliptic curve. See Remark 3.17 for some more detailed explanations.

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§2. Preliminaries

We follow Kollár [27] to define canonical sheaves and divisors, adjunction, pairs, singularities, and so forth which we discuss below.

2.1 Relative canonical sheaves

Let $f\colon X \to Z$ be a morphism of schemes where Z is regular and excellent, and X is pure dimensional and of finite type over Z. Let S be a closed subscheme of X and $U := X \setminus S$. Assume that

- codimension of S in X is at least 2; and
- U is a locally closed local complete intersection in some $\mathbb{P}^n_Z$.

Let $\mathcal{I}$ be the ideal sheaf of the closure of U in $\mathbb{P}^n_Z$ and let $j\colon U \to X$ be the inclusion map of U in X. Now define the relative canonical sheaf as

$$\omega_{X/Z} = j_*((\omega_{\mathbb{P}^n_Z/Z} \otimes (\det \mathcal{I}/\mathcal{I}^2)^\vee)|_U).$$

Note that $\mathcal{I}/\mathcal{I}^2$ is locally free on U. Moreover, $\omega_{\mathbb{P}^n_Z/Z}$ is as usual defined to be $\mathcal{O}_{\mathbb{P}^n_Z}(-n-1)$.

2.2 Relative canonical sheaves and divisors of normal schemes

Let $f\colon X \to Z$ be a quasi-projective morphism of schemes where Z is regular and excellent, and X is integral and normal. The set of regular points of X is an open subset U of X by definition of excellent schemes (cf. [7, p. 382]). Let S be any closed subscheme of X containing the singular points and such that the codimension of S in X is at least 2. Such an S exists because X is normal. If we embed U as a locally closed subscheme into some $\mathbb{P}^n_Z$, then U is a locally closed local complete intersection because U is regular (cf. [7, Proposition 2.2.4]). Therefore, we can define the relative canonical sheaf $\omega_{X/Z}$ as in the previous subsection. Under this situation, this sheaf is of the form $\mathcal{O}_X(K_{X/Z})$ for some divisor $K_{X/Z}$ which we refer to as the canonical divisor of X over Z (when Z is the spectrum of a field, we usually drop Z and just write ω_X and K_X if the ground field is obvious from the context).

If $Y \to Z$ is another quasi-projective morphism from a normal integral scheme Y with $K_{Y/Z}$ being $\mathbb{Q}$-Cartier, and if we are given a Z-morphism $h\colon X \to Y$, then we let $K_{X/Y} = K_{X/Z} - h^*K_{Y/Z}$, which is compatible with the definition of above paragraph if Y is regular.

Now assume that Z is integral and let F be the generic fiber of $X \to Z$. Let V and T be the inverse images of U and S under the morphism $F \to X$.

Then F is normal, V is regular, and the codimension of T in F is at least 2. We consider F with its natural scheme structure over K, the function field of Z. By the definition of canonical sheaves, ω_V is the pullback of $\omega_{U/Z}$. Therefore, K_V is the pullback of $K_{U/Z}$. Moreover, if $K_X + B$ is $\mathbb{Q}$-Cartier for some $\mathbb{Q}$-divisor B, then we can write $K_F + B_F$ for the pullback of $K_X + B$ to F where B_F is canonically determined by B: more precisely, B_F is the closure of the pullback of $B|_U$ to V.

2.3 Intersection theory

For a short introduction to intersection theory on a proper scheme X over a field k, see [28, Section 1.5]. Note that intersection numbers depend on the ground field k. For a detailed treatment of intersection theory on regular surfaces, see [31, Chapter 9]. Although [31] does not seem to treat the Riemann–Roch formula, it holds on regular projective surfaces. More precisely, if X is a regular surface projective over a field k and if L is a Cartier divisor, then

$$\mathcal{X}(L) = \tfrac{1}{2}L \cdot (L - K_X) + \mathcal{X}(\mathcal{O}_X)$$

where K_X means the relative canonical divisor of X over k, and $\mathcal{X}(N) := h^0(N) - h^1(N) + h^2(N)$ for any divisor (or sheaf) N which also depends on the ground field k. The formula can be proved as in the case of smooth surfaces over algebraically closed fields. The main point is that it can be reduced to Riemann–Roch on curves which holds in a quite general setting (cf. [31, Section 7.3]). See [41, Section 1.3] for a complete proof.

2.4 Pairs and singularities

Let k be a field. A pair (X, B) over k consists of a normal quasi-projective variety X over k and an $\mathbb{Q}$-Weil divisor B with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We usually refer to B as a $\mathbb{Q}$-boundary. See [27, Definitions 1.5 and 2.8] for definitions in more general settings.

For any projective birational morphism $f: W \to X$ from a normal variety W, we can write $K_W + B_W = f^*(K_X + B)$ for some unique $\mathbb{Q}$-Weil divisor B_W. For a prime divisor D on W we define the log discrepancy $a(D, X, B)$ to be $1 - b$ where b is the coefficient of D in B_W. We say (X, B) is lc (resp. klt) if $a(D, X, B) \geqslant 0$ (resp. $a(D, X, B) > 0$) for any D on any such W. On the other hand, we say (X, B) is dlt if there is a closed subset $Z \subset X$ of codimension at least two such that $a(D, X, B) > 0$ for any D whose image

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in X is inside Z and such that outside Z we have: X is regular and $\operatorname{Supp} B$ has simple normal crossing singularities.

We say f is a log resolution of (X, B) if W is regular and $\operatorname{Supp} B_W$ has simple normal crossing singularities. Log resolutions exist when $\dim X = 2$ [38] or if k is algebraically closed and $\dim X \leqslant 3$; in these situations one can check whether (X, B) is lc or klt by looking at one log resolution. Moreover, if $\dim X = 2$, then a minimal resolution of X exists.

2.5 Minimal models and Mori fiber spaces

Let (X, B) be an lc pair and (Y, B_Y) be a $\mathbb{Q}$-factorial dlt pair, over a field k, equipped with projective morphisms $X \to Z$ and $Y \to Z$ and a birational map $\phi\colon X \dashrightarrow Y$ commuting with these morphisms such that $\phi_* B = B_Y$ and such that ϕ^{-1} does not contract divisors. Assume in addition that

$$a(D, X, B) \leqslant a(D, Y, B_Y)$$

for any prime divisor D on birational models of X with strict inequality if D is on X and exceptional/Y. We say (Y, B_Y) is a log minimal model of (X, B) over Z if $K_Y + B_Y$ is nef/Z. We say (Y, B_Y) is a Mori fiber space of (X, B) over Z if there is a $K_Y + B_Y$-negative extremal contraction $Y \to T/Z$ with $\dim Y > \dim T$.

2.6 Minimal models of 3-folds

For 3-folds we have the following result.

THEOREM 2.7. *Let (X, B) be a projective klt pair of dimension three and $X \to Z$ a contraction to projective variety, over an algebraically closed field k of char $p > 5$.*

(1) *If $K_X + B$ is pseudo-effective over Z, then (X, B) has a log minimal model over Z.*
(2) *If $K_X + B$ is not pseudo-effective over Z, then (X, B) has a Mori fiber space over Z.*
(3) *If $K_X + B$ is nef over Z, and $K_X + B$ or B is big over Z, then $K_X + B$ is semi-ample over Z.*

Part (1) is proved in [16] for canonical singularities, and in [5] in general. Part (2) is proved in [11] for terminal singularities, and in [6] in general. Part (3) is proved in various forms in [5, 6, 47].

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2.8 Adjunction

Let X be a normal projective variety over a field k. Let $B \geqslant 0$ be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let S be a component of $\lfloor B \rfloor$. Then we can write the pullback of $K_X + B$ to the normalization S^ν as $K_{S^\nu} + B_{S^\nu}$ where the different $B_{S^\nu} \geqslant 0$ is canonically determined. If (X, B) is lc outside a codimension $\geqslant 3$ subset of X, then B_{S^ν} is a boundary. See [27, Proposition 4.5] for more details.

2.9 Varieties over $\bar{\mathbb{F}}_p$

Varieties over finite fields enjoy some special properties which we exploit. For example, any numerically trivial divisor on a projective variety over $\bar{\mathbb{F}}_p$ is torsion [26]. Another example is this:

THEOREM 2.10. [39, Theorem 0.1 and 0.2] *Let X be a normal projective surface over $\bar{\mathbb{F}}_p$. Let Δ be an effective $\mathbb{Q}$-divisor on X. If $K_X + \Delta$ is pseudo-effective, then there exists a minimal model (X', Δ') of (X, Δ), and $K_{X'} + \Delta'$ is semi-ample.*

2.11 Semi-positivity of direct images of pluricanonical sheaves

The following result is extracted from [37, 1.5, 1.6, 1.7 and the paragraph below 1.7]. It holds in a more general form but this is all we need in this paper.

THEOREM 2.12. [37] *Let $f\colon X \to Z$ be a surjective morphism from a normal projective variety to a smooth projective curve over an algebraically closed field k. Assume K_X is $\mathbb{Q}$-Cartier and that general fibers are strongly F-regular.*

- *If K_X is nef over Z and K_X is semi-ample on the generic fiber of f, then $K_{X/Z}$ is nef.*
- *If K_X is ample over Z, then $f_*\mathcal{O}_X(mK_{X/Z})$ is a nef vector bundle for any sufficiently divisible natural number m.*

We apply the theorem only when X is a 3-fold and general fibers have canonical singularities.

2.13 Varieties with elliptic fibrations

For fibrations whose general fibers are elliptic curves, we can use a weak canonical bundle formula which allows us to do induction.

THEOREM 2.14. *Let $f\colon X \to Z$ be a contraction between smooth projective varieties over an algebraically closed field k such that the geometric generic fiber is a smooth elliptic curve. Then $\kappa(K_{X/Z}) \geqslant 0$.*

Proof. This follows from [13, 3.2]. ∎

2.15 Nef divisors with Kodaira dimension one

LEMMA 2.16. *Let X be a normal surface projective over a field k. Let L be a nef $\mathbb{Q}$-divisor with $\kappa(L) = 1$. Then L is semi-ample.*

Proof. Let $X \dashrightarrow Z$ be the rational map defined by the linear system $|mL|$ for some sufficiently divisible $m > 0$. Then $\dim Z = 1$. We can replace X with the normalization of the graph of $X \dashrightarrow Z$; hence, assume $X \dashrightarrow Z$ is a morphism. We can in addition assume $L \geqslant H \geqslant 0$ where H is the pullback of some ample $\mathbb{Q}$-divisor on Z. Since L is not big, its support does not intersect the generic fiber of $X \to Z$.

Let F be a fiber of $X \to Z$ which has a common component with L. Let a be the smallest rational number such that $L - aF \leqslant 0$ near F. Then $L - aF$ has no common component with F; otherwise there would be two components C, D of F such that C intersects D, C is not a component of $L - aF$ but D is a component of $L - aF$ which implies $(L - aF) \cdot C < 0$, a contradiction. These arguments show that L is the pullback of some $\mathbb{Q}$-divisor on Z which is necessarily ample, hence L is semi-ample. ∎

2.17 Generically trivial divisors

We recall a result of Kawamata adapted to char $p > 0$.

LEMMA 2.18. [6, Lemma 5.6] *Let $f\colon X \to Z$ be a contraction between normal projective varieties over an algebraically closed field k and L a nef/Z $\mathbb{Q}$-divisor on X such that $L|_F \sim_{\mathbb{Q}} 0$ where F is the generic fiber of f. Assume* $\dim Z \leqslant 3$ *if k has char $p > 0$. Then there exist a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\psi} & Z \end{array}$$

with ϕ, ψ projective birational, and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Z' such that $\phi^* L \sim_{\mathbb{Q}} f'^* D$.

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2.19 Easy additivity of Kodaira dimensions

The following result is well-known to experts [15, Propostion 1].

LEMMA 2.20. *Let $f\colon X \to Z$ be a contraction between normal varieties projective over a field k. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and H a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z. Then*

$$\kappa(D + f^*H) \geqslant \kappa(D|_F) + \dim Z$$

where F is the generic fiber of f.

Proof. Since D is effective, it is enough to prove the statement with H replaced by any positive multiple and D replaced by $D + lf^*H$ for some $l > 0$. If $V \to X$ is a morphism, we denote the pullback of D to V by D_V (similar notation for other divisors). Let m be a sufficiently divisible natural number and let $d = \dim_K H^0(mD_F) - 1$ where K is the function field of Z. Let S be the normalization of the image of $\phi_{mD_F}\colon F \dashrightarrow \mathbb{P}^d_K$ whose dimension is equal to $\kappa(D_F)$. Moreover, ϕ_{mD_F} induces a (not unique) map $\psi\colon X \dashrightarrow \mathbb{P}^d_Z$ over Z which restricts to ϕ_{mD_F}. Let T be the normalization of the image of ψ. Let Y be the normalization of the graph of $X \dashrightarrow \mathbb{P}^d_Z$ and G the generic fiber of $Y \to Z$. We have induced morphisms $Y \to T$, $G \to S$, and $G \to F$.

Let A on $\mathbb{P}^d_Z$ be the pullback of a hyperplane via the projection $\mathbb{P}^d_Z \to \mathbb{P}^d_k$. Perhaps after changing D up to $\mathbb{Q}$-linear equivalence we can assume $mD_G \geqslant A_G$. Thus replacing D with $D + lf^*H$ for some l we can assume $mD_Y \geqslant A_Y$. Therefore, we may replace X with T and replace D with A_T. But then the statement is trivial in this case because we can assume $A + f^*H$ is ample. □

2.21 Covering Theorem

THEOREM 2.22. [21, Theorem 10.5] *Let $f\colon X \to Y$ be a proper surjective morphism between smooth complete varieties. If D is a Cartier divisor on Y and E an effective f-exceptional divisor on X, then*

$$\kappa(f^*D + E) = \kappa(D).$$

Here by f-exceptional we mean: for any prime divisor P on Y, there is a prime divisor Q on X mapping onto P such that Q is not a component of E.

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§3. Log surfaces over nonclosed fields

In this section, k will denote a field which is not necessarily algebraically closed. Shafarevich [38] studied the minimal model theory of regular surfaces over nonclosed fields and Dedekind rings (see also [31]), and Manin [32] and Iskovskikh [20] treated the special case of rational surfaces. None of them seems to have discussed the abundance problem. If k is perfect (e.g., when char $k=0$) or if the surface is smooth over k, then one can often reduce problems to the algebraically closed case by passing to the algebraic closure. But our main point here is that we can actually prove many things by working over k rather than the algebraic closure when char $k>0$.

3.1 Curves with negative canonical divisor

As a preparation we collect some results about curves.

Lemma 3.2. *Let X be a local complete intersection integral projective curve over a field k, and let $l=H^0(\mathcal{O}_X)$. Assume that* $\deg_k K_X<0$. *Then*

(i) $\mathrm{Pic}^0(X)=0$;
(ii) *X is a conic over l, and* $\deg_l K_X=-2$;
(iii) *if X is normal and* char $k>2$, *then* $X_{\bar{l}}\cong\mathbb{P}^1_{\bar{l}}$.

Proof. By assumption $\deg_k K_X<0$, we get $h^1(\mathcal{O}_X)=h^0(K_X)=0$ which implies $p_a(X)\leqslant 0$. Then (i) and (ii) follow from [31, Chapter 9, Proposition 3.16], and (iii) is [11, Lemma 6.5]. □

3.3 Reduced boundary of dlt pairs

Lemma 3.4. *Assume (X,B) is a $\mathbb{Q}$-factorial dlt pair of dimension two over a field k. Then every irreducible component of $\lfloor B\rfloor$ is regular.*

Proof. Let S be a component of $\lfloor B\rfloor$ and let $x\in S$ be a closed point. As (X,S) is plt, S is regular at x by [27, 3.35]. □

Proposition 3.5. *Let (X,B) be a $\mathbb{Q}$-factorial dlt pair of dimension two projective over a field k where B is a $\mathbb{Q}$-boundary. Assume S is a component of $\lfloor B\rfloor$ and A is an ample $\mathbb{Q}$-divisor such that*

- $(K_X+B)\cdot S<0$;
- $(K_X+B+A)\cdot S=0$; *and*
- K_X+B+A *is nef and big.*

Then there is a birational morphism $\sigma : X \to Y$ with Y normal and projective, whose exceptional locus is equal to S, and the resulting pair $(Y, B_Y := \sigma_* B)$ is $\mathbb{Q}$-factorial dlt. Moreover, $(K_X + B) \cdot S \geqslant -2$.

Proof. By perturbing the coefficients of B (i.e., by replacing B with $B - P$ and replacing A with $A + P$ for some appropriate P) we can assume $S = \lfloor B \rfloor$. Since $K_X + B + A$ is nef and big, we can write $K_X + B + A \sim_{\mathbb{Q}} H + D$ where H is ample and $D \geqslant 0$. Since $(H + D) \cdot S = 0$, S is a component of D and $S^2 < 0$. Let $\epsilon > 0$ be a small rational number such that $A' := A + \epsilon S$ is ample. Then S is the only curve on X such that $(K_X + B + A') \cdot S < 0$. Let t be the smallest real number such that $L := K_X + B + A' + tA$ is nef. We want to show L is semi-ample and that $L \cdot S = 0$. If char $k = 0$, the last sentence and the other claims of the proposition can be reduced to the algebraically closed case by passing to the algebraic closure. So we assume char $k > 0$.

By definition L is nef and big but not ample; then there is a curve C with $L \cdot C = 0$ by Nakai–Moishezon criterion [14, Theorem 1.21], which implies t is a rational number. Actually $C = S$ by construction.

By Lemma 3.4, S is regular. Then by adjunction (2.8) we can write $K_S + B_S = (K_X + B)|_S$ where $B_S \geqslant 0$. Since $\deg_k(K_S + B_S) < 0$, we have $\deg_k K_S < 0$. This implies $\mathrm{Pic}^0(S) = 0$ by Lemma 3.2. Therefore, $L|_S \sim_{\mathbb{Q}} 0$ which implies that L is semi-ample [26], so it defines a birational contraction $\sigma : X \to Y$ contracting exactly S so that L_Y is ample where L_Y is the pushdown of L.

The dlt property of (Y, B_Y) is obvious once we show Y is $\mathbb{Q}$-factorial where B_Y is the pushdown of B. Let R_Y be a prime divisor on Y and R its birational transform on X. There is $s \geqslant 0$ such that $(R + sS) \cdot S = 0$. Since L is the pullback of an ample divisor on Y, the divisor $M := mL + R + sS$ is nef and big on X, and $\mathbb{E}(M) = S$ for any $m \gg 0$. Moreover, $M|_S \sim_{\mathbb{Q}} 0$, so by [26, Theorem 0.2], M is semi-ample, thus it is the pullback of some ample divisor M_Y on Y. But then $R_Y = M_Y - mL_Y$ is $\mathbb{Q}$-Cartier. This shows Y is $\mathbb{Q}$-factorial. Finally

$$(K_X + B) \cdot S = \deg_k(K_S + B_S) \geqslant \deg_k K_S = -2. \qquad \square$$

3.6 Base point freeness

PROPOSITION 3.7. *Let (X, B) be a klt pair of dimension two projective over a field k where B is a $\mathbb{Q}$-boundary. Assume L is a nef and big $\mathbb{Q}$-divisor so that $L - (K_X + B)$ is nef. Then L is semi-ample.*

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Proof. If char $k=0$, we can pass to the algebraic closure of k in which case the theorem is well-known. So we assume char $k>0$.

Since L is nef and big, by [26, Theorem 1.9], there exist a birational morphism $X \to V$ to a proper algebraic space V and a reduced divisor D on X such that, the exceptional locus is equal to D, and that $L \equiv 0/V$.

Let $\phi\colon W \to X$ be a log resolution of $(X, B+D)$. Let Δ_W be the sum of the birational transform of B_V plus the reduced exceptional divisor of $W \to V$ where B_V is the pushdown of B on V. Let R_W be an ample divisor on W and let L_W be the pullback of L. Also let $G = L-(K_X+B)$ and G_W be its pullback. Fix $m \gg 0$ and let t be the smallest number such that

$$N_W := K_W + \Delta_W + G_W + tR_W$$

is nef. Note that by construction, $K_W+\Delta_W+G_W = L_W+E_W$ where $E_W \geqslant 0$ and its support is equal to the exceptional locus of $W \to V$. Moreover, N_W is nef and big but not ample, so by [10], there is a curve S with $N_W \cdot S = 0$. Since $(K_W+\Delta_W+G_W)\cdot S<0$, $E_W \cdot S<0$, hence S is a component of E_W which is contracted over V, so it is a component of Δ_W. In addition, t is a rational number and $(K_W+\Delta_W)\cdot S<0$. Therefore, by Proposition 3.5, S can be contracted by a birational morphism $W \to W'$ with an induced morphism $W' \to V$. Continuing this process gives an LMMP on $K_W+\Delta_W$ over V. It terminates with some model Y on which $K_Y+\Delta_Y$ is nef$/V$.

Since $K_Y+\Delta_Y \equiv E_Y/V$, E_Y is nef over V and since E_Y is exceptional over V, we deduce $E_Y=0$ by the negativity lemma (which holds over arbitrary fields). Therefore, $Y=V$ because E_Y contains all the exceptional curves of $Y \to V$. Thus V is projective and $\mathbb{Q}$-factorial. Now L_V is ample and it pulls back to L, hence L is semi-ample. □

PROPOSITION 3.8. *Let (X,B) be a klt pair of dimension two projective over a field k where B is a $\mathbb{Q}$-boundary. Assume L is a nef $\mathbb{Q}$-divisor so that $L-(K_X+B)$ is nef and big, and L is not numerically trivial. Then L is semi-ample.*

Proof. By Proposition 3.7, we can assume L is not big. Moreover, replacing X with its minimal resolution we can assume X is regular. Let $G := L-(K_X+B)$. By the Riemann–Roch theorem for regular surfaces (see 2.3), for any sufficiently divisible natural number m we have

$$\mathcal{X}(mL) = \tfrac{1}{2}mL\cdot(mL-K_X) + \mathcal{X}(\mathcal{O}_X).$$

Since G is big and L is not numerically trivial, $L \cdot G > 0$, and since

$$mL - K_X \sim_{\mathbb{Q}} (m-1)L + B + G,$$

$L \cdot (mL - K_X) > 0$, hence $\mathcal{X}(mL)$ is large when m is large. This implies $h^0(mL) \geqslant 2$ for such m because $h^2(mL) = h^0(K_X - mL) = 0$. Therefore, L is semi-ample by Lemma 2.16. ∎

3.9 Running the LMMP

Proof of Theorem 1.4. The proof is broken into several steps.

Step 1. Assume $K_X + B$ is pseudo-effective but not nef. First suppose X is $\mathbb{Q}$-factorial (we see in the next step that this is automatically satisfied). Let H be an ample divisor on X and let t be the smallest number such that $L = K_X + B + tH$ is nef. Obviously L is nef and big. Moreover, t is rational which can be seen as in the proof of Proposition 3.5. Although we can apply Proposition 3.7 to deduce that L is semi-ample and defines a contraction but we want to modify the situation so that the contraction contracts only one curve. Pick a curve C such that $L \cdot C = 0$. Let $\Delta = (1-\delta)B + \epsilon C$ for certain small rational numbers $\epsilon, \delta > 0$ so that (X, Δ) is klt, $(K_X + \Delta) \cdot C < 0$, and $\delta B + tH$ is ample. Now let t' be the smallest number such that $L' := K_X + \Delta + \delta B + t'H$ is nef. Then $t < t'$ because $C^2 < 0$, so L' is nef and big, and $\delta B + t'H$ is ample. Note that C is the only curve satisfying $L' \cdot C = 0$.

Now by Proposition 3.7, L' is semi-ample and it defines a nontrivial birational contraction $X \to Y$ contracting C with $(K_X + B) \cdot C < 0$. Let R_Y be a prime divisor on Y and R its birational transform on X. Let r be the number such that $(R + rC) \cdot C = 0$. If $m > 0$ is sufficiently large, then $L'' := mL' + R + rC$ is nef and big. Moreover, applying 3.7 to $L' + L''$ shows that L'' is the pullback of an ample divisor on Y, hence R_Y is $\mathbb{Q}$-Cartier. Therefore, Y is $\mathbb{Q}$-factorial and (Y, B_Y) is dlt. Now replace (X, B) with (Y, B_Y) and repeat the argument.

Step 2. In this step we show that the dlt property of (X, B) implies X is $\mathbb{Q}$-factorial. Since the pair is dlt, there is a log resolution $\phi\colon W \to X$ such that the log discrepancy $a(D, X, B) > 0$ for every curve D contracted by ϕ. Fix $m \gg 0$. By Bertini Theorem (similar argument as in [47, Proposition 2.3]), there exists a divisor $H^m \sim_{\mathbb{Q}} mH$ on X such that $(X, B + H^m)$ is dlt. Let Γ_W on W be the sum of the birational transform of $B + H^m$ and the reduced exceptional divisor of $W \to X$. Then $K_W + \Gamma_W = \phi^*(K_X + B + H^m) + E_W$ where $E_W \geqslant 0$ is contracted over X. Then applying a similar procedure as above, we can run an LMMP for $K_W + \Gamma_W$. Since $m \gg 0$, by projection

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formula the curves C contracted by the LMMP intersect ϕ^*H trivially, and by Proposition 3.5, such curves are contracted by ϕ. In other words, the LMMP is over X. The LMMP contracts E_W so it ends with X which means X is $\mathbb{Q}$-factorial. This and the previous step together prove the theorem when $K_X + B$ is pseudo-effective.

Step 3. From now on we assume $K_X + B$ is not pseudo-effective. If there is a curve C such that $(K_X + B) \cdot C < 0$ and such that there is a birational morphism $X \to Y$ with exceptional divisor equal to C, then we replace (X, B) with (Y, B_Y). So we can assume there is no such C.

Pick an ample divisor A and let t be the smallest number such that $K_X + B + tA$ is pseudo-effective. By the last paragraph $K_X + B + tA$ is nef: otherwise, we can run an LMMP for $K_X + B + tA$ which is also an LMMP for $K_X + B$ contracting some curve C, a contradiction.

If $\rho(X) = 1$, then we already have a Mori fiber space. So assume $\rho(X) > 1$. Then there is another ample divisor H such that A is not numerically equivalent to hH for any number h. Let s be the smallest number such that $K_X + B + sH$ is pseudo-effective. Arguing as above, $K_X + B + sH$ is nef. By our choice of A and H both $K_X + B + tA$ and $K_X + B + sH$ cannot be numerically trivial at the same time. We may assume $K_X + B + tA$ is not numerically trivial.

Step 4. In this step we assume t is a rational number. By Proposition 3.8, $K_X + B + tA$ is semi-ample defining a contraction $f\colon X \to Z$ onto Z of dimension one. Assume there is a fiber F of f which is not irreducible. Let C be a component of F. Then $C^2 < 0$. We can find a $\mathbb{Q}$-boundary Δ such that (X, Δ) is klt, $K_X + \Delta$ is pseudo-effective, and $(K_X + \Delta) \cdot C < 0$. So we can contract C. But since $(K_X + B) \cdot C < 0$, this contradicts the first paragraph of Step 3. Therefore, we can assume all the fibers of f are irreducible. But this means f is extremal and so f is a Mori fiber space.

Step 5. Finally we show t is indeed a rational number. Assume not. We derive a contradiction. Let $L = K_X + B + tA$. We can assume A is an effective Q-divisor. For each sufficiently divisible natural number m, let a_m be the number so that $\lfloor mL \rfloor = mL - a_m A$. Since t is not rational, there is an infinite set Π of such m so that the a_m form a strictly decreasing sequence with $\lim_{m\in\Pi} a_m = 0$. On the other hand, for each $m \in \Pi$, let a'_m be the number so that

$$\begin{aligned}(mL - a'_m A) &\cdot ((m-1)L + B + (t - a_m)A)\\ &= (mL - a'_m A) \cdot (\lfloor mL \rfloor - K_X) = 0.\end{aligned}$$

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Since $\lim_{m\in\Pi} a_m = 0$ and $L^2 = 0$, we can see

$$\lim_{m\in\Pi} a'_m = \lim_{m\in\Pi} \frac{mL\cdot((m-1)L+B+(t-a_m)A)}{A\cdot((m-1)L+B+(t-a_m)A)} = \frac{L\cdot(B+tA)}{A\cdot L} > 0.$$

Thus we can assume

$$(mL - a_mA)\cdot(\lfloor mL\rfloor - K_X) > \tau(m-1)A\cdot L$$

for some $\tau > 0$ independent of m. Therefore,

$$\mathcal{X}(\lfloor mL\rfloor) = \tfrac{1}{2}(mL - a_mA)\cdot(\lfloor mL\rfloor - K_X) + \mathcal{X}(\mathcal{O}_X)$$

is large when $m \in \Pi$ is large. This in turn implies $h^0(\lfloor mL\rfloor)$ is large for such m. Take a divisor $M_1 \in |\lfloor mL\rfloor|$. Then $mL \sim_{\mathbb{R}} M := M_1 + a_mA$. If C is a component of Supp$M$, then $L\cdot C = 0$ which means t is a rational number, a contradiction. ∎

3.10 Mori fiber spaces

PROPOSITION 3.11. *Let (X, B) be a dlt pair of dimension two projective over a field k. Assume $f : X \to Z$ is a Mori fiber structure for (X, B) where* $\dim Z = 1$. *Then the geometric general fibers of f are conics and if char $k > 2$ they are smooth rational curves. In particular, if F is a general fiber, then $(K_X + B)\cdot F \geqslant -2$.*

Proof. Let F be the generic fiber of f which is a regular curve since X is regular in codimension one. Since $-(K_X + B)$ is ample over Z, $-K_F$ is ample. On the other hand, since f is a contraction, $H^0(\mathcal{O}_F) = K$ where K is the function field of Z. The assertions follow from Lemma 3.2 straightforwardly. ∎

3.12 Curves of canonical type

Let X be a regular surface projective over a field k. A connected divisor $D = \sum_1^r d_iD_i \geqslant 0$ is called a curve of canonical type if $D|_D \equiv 0$ and $K_X|_D \equiv 0$. It is called indecomposable if there is no prime number dividing all the d_i. The following result was proved by Mumford [33, p. 332]. Although he assumes the ground field to be algebraically closed his proof works for arbitrary fields. We give the proof for convenience (see also [2, Theorem 7.8]).

PROPOSITION 3.13. *Let D be an indecomposable curve of canonical type. Let L be a Cartier divisor on D such that $L \equiv 0$. If $h^0(L) \neq 0$, then $L \sim 0$.*

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Proof. Assume $\alpha \in H^0(L)$ is nonzero. Then $\alpha|_{D_i}$ is either nowhere vanishing or everywhere vanishing because $L|_{D_i} \equiv 0$. Since D is connected, either α is nowhere vanishing on D or $\alpha|_{\mathrm{Supp}\, D} = 0$. The former implies L is generated by global sections which in turn implies $L \sim 0$. So it is enough to treat the latter. Let n_i be the order of vanishing of α along D_i. Let $N = \sum n_i D_i$. Then by assumption $0 < N < D$.

We claim that for every component D_i such that $n_i < d_i$, $-N|_{D_i}$ is nef. Granted this claim, since D is connected we have that $n_i > 0$ for every $i \in \{1, 2, \ldots, r\}$. Let a be the smallest rational number such that $aN - D \geqslant 0$. Then $aN - D = 0$ because otherwise, since D is connected, there will exist a component D_i of D such that D_i is not contained in $aN - D$, $n_i < d_i$ and $(aN - D) \cdot D_i > 0$, which contradicts that $-N|_{D_i}$ is nef. Since the d_i have no common factor, $N = D$. So $\alpha = 0$, a contradiction.

The claim follows from local analysis. Assume $n_i < d_i$, say for $i = 1$. Consider the exact sequence

$$0 \to \mathcal{O}_{D_1}(L - n_1 D_1) \to \mathcal{O}_{(n_1+1)D_1}(L) \to \mathcal{O}_{n_1 D_1}(L) \to 0.$$

Since $\alpha|_{n_1 D_1} = 0$ by definition of n_1, the section $\alpha|_{(n_1+1)D_1}$ is the image of a section β of $(L - n_1 D_1)|_{D_1}$. If P is the zero divisor of β, then a local computation of intersection numbers shows that $P \geqslant (N - n_1 D_1)|_{D_1}$. More precisely, let $v \in D_1$ be a closed point, let $R = \mathcal{O}_{X,v}$, and let f_i be a local equation of D_i near v. Then locally considering α as an element of $\frac{R}{\langle f_1^{d_1} \cdots f_r^{d_r} \rangle}$, it is easy to see that α is represented by $\lambda f_1^{n_1} \cdots f_r^{n_r}$ for some $\lambda \in R$, and that β is represented by $\lambda f_2^{n_2} \cdots f_r^{n_r}$ which gives the equation of P near v. Therefore, from

$$\mathrm{length}_{R/\langle f_1 \rangle} \frac{R}{\langle f_1, \lambda f_2^{n_2} \cdots f_r^{n_r} \rangle} \geqslant \mathrm{length}_R \frac{R}{\langle f_1, f_2^{n_2} \cdots f_r^{n_r} \rangle}$$

we deduce that locally near v we have $P \geqslant (N - n_1 D_1)|_{D_1}$ because the left hand side of the displayed formula is the coefficient of v in P and the right hand side is nothing but the local intersection number $(N - n_1 D_1) \cdot D_1$ at v which is in turn equal to the coefficient of v in $(N - n_1 D_1)|_{D_1}$. As $P \sim (L - n_1 D_1)|_{D_1}$, we deduce that $\deg N|_{D_1} \leqslant 0$. □

PROPOSITION 3.14. *Let D be an indecomposable curve of canonical type. Then the arithmetic genus $p_a(D) = 1$ and $K_D \sim 0$.*

Proof. By definition of curves of canonical type $K_D = (K_X + D)|_D \equiv 0$. By [31, Chapter 7, Corollary 3.31], $0 = \deg_k K_D = 2(p_a(D) - 1)$. Thus

$p_a(D) = 1$ which means $\mathcal{X}(\mathcal{O}_D) = 0$, hence $h^1(\mathcal{O}_D) = h^0(\mathcal{O}_D) > 0$. So by duality $h^0(K_D) = h^1(\mathcal{O}_D) > 0$ which implies $K_D \sim 0$ by Proposition 3.13. ∎

PROPOSITION 3.15. *Assume char $k > 0$. Let D be an indecomposable curve of canonical type such that $D|_D$ is torsion. Then D is semi-ample on X.*

Proof. Let r be the order of $D|_D$ in $\mathrm{Pic}(D)$. First we want to show $rD|_{rD} \sim 0$. This is trivially true if $r = 1$, so assume $r > 1$. Assume we already know $rD|_{lD} \sim 0$ for some $0 < l < r$. Consider the exact sequence

$$0 \to \mathcal{O}_D(rD - lD) \to \mathcal{O}_{(l+1)D}(rD) \to \mathcal{O}_{lD}(rD) \to 0.$$

Now $h^0(\mathcal{O}_D(rD - lD)) = 0$ by Proposition 3.13, and since $\mathcal{X}(\mathcal{O}_D) = 0$, by Riemann–Roch we get

$$\mathcal{X}(\mathcal{O}_D(rD - lD)) = \deg_k(rD - lD)|_D + \mathcal{X}(\mathcal{O}_D) = 0$$

which implies $h^1(\mathcal{O}_D(rD - lD)) = 0$. So any nowhere vanishing section of $\mathcal{O}_{lD}(rD)$ lifts to $\mathcal{O}_{(l+1)D}(rD)$ which shows $rD|_{(l+1)D} \sim 0$. Inductively one shows $rD|_{rD} \sim 0$. Finally applying [42, Lemma 4.1], we deduce D is semi-ample. ∎

3.16 Abundance

Proof of Theorem 1.5. We can assume $K_X + B$ is not big by Proposition 3.7. Replacing X with its minimal resolution we can assume X is regular. By assumption $m(K_X + B) \sim M$ for some integer $m > 0$ and M is an effective Cartier divisor. Let n be a sufficiently large natural number. We can run an LMMP on $K_X + nM$ because

$$K_X + nM \sim (1 + nm)\left(K_X + \frac{nm}{1 + nm}B\right)$$

and because $(X, \frac{nm}{1+nm}B)$ is klt. Moreover, we claim that M is numerically trivial on each step of the LMMP and the nefness of M is preserved in the process. Indeed assume the first step of the LMMP is a birational contraction $X \to Z$ contracting a curve E. Then $\deg_k K_E = (K_X + E) \cdot E < 0$, hence by Lemma 3.2, if setting $l = H^0(\mathcal{O}_E)$ then

$$-2 = \deg_l K_E = \deg_l (K_X + E)|_E$$

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which implies $\deg_l K_X|_E = -1$. Thus from $\deg_l (K_X + nM)|_E < 0$ we deduce $M \cdot E = 0$, because otherwise, $\deg_l M|_E \geqslant 1$ since M is Cartier, this is impossible since we have assumed $n \gg 0$. On the other hand, if $X \to Z$ is a Mori fiber space, then we stop the LMMP and in this case $M \equiv 0/Z$ by Proposition 3.11 and calculations similar to those above. Applying this argument to every step of the LMMP proves the claim. Note that the regularity of X is also preserved by the LMMP because the LMMP is a K_X-MMP; hence X remains with terminal singularities which implies regularity.

Replacing X with the end product of the LMMP we can assume either $K_X + nM$ is nef or that there is a Mori fiber structure $X \to Z$ so that $M \equiv 0/Z$. First assume $K_X + nM$ is nef. There is a divisor $0 \leqslant D = \sum d_i D_i \leqslant M$ such that D is connected, the d_i have no common prime factor, and $M = aD$ in a neighborhood of D for some number a. In particular, D is nef. We show D is an indecomposable curve of canonical type. It is enough to show $K_X|_D \equiv 0$ because M not being big implies $D|_D \equiv 0$. Since M is not big and since $m(K_X + B) \sim M$, we deduce $K_X + n'M$ is not big for any n'. Therefore, $(K_X + nM + M)^2 = 0$ from which we deduce $(K_X + nM) \cdot M = 0$, hence $(K_X + nM)|_D \equiv 0$, so $K_X|_D \equiv 0$.

In order to apply Proposition 3.15 we need to show $D|_D$ is torsion. By construction, $B|_D \equiv 0$ which implies $B = bD$ in some neighborhood of D because D is connected, where $b < 1$ is a rational number. Taking positive integer m' so that $m'b \in \mathbb{Z}$ we get

$$
\begin{aligned}
0 &\sim m'K_D = m'(K_X + D)|_D = m'(K_X + B + D)|_D - m'B|_D \\
&\sim (a + m')D|_D - m'bD|_D
\end{aligned}
$$

which implies $D|_D$ is torsion because $a + m' - m'b > 0$. Therefore, D is semi-ample, hence $\kappa(M) = 1$ which implies M is semi-ample by Lemma 2.16.

Now assume we have a Mori fiber structure $X \to Z$ with $M \equiv 0/Z$. If F is the generic fiber, then $M|_F \sim 0$. This implies M is the pullback of some effective divisor N on Z. Either N is ample or $N = 0$, hence in any case M is semi-ample. ∎

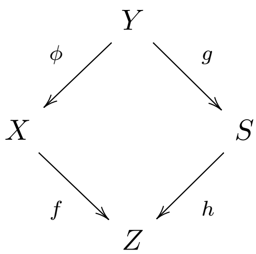
Remark 3.17. Here we explain what we need from this section for the proof of Theorem 1.2. We need Theorem 1.5 for the proof of Theorem 1.6. In turn we use Theorem 1.6 in the proofs of Corollary 4.1 and Proposition 5.3 (steps 1 and 5) in two situations: when (1) F is smooth and when (2) $\kappa(K_F + B_F := (K_X + B)|_F) = 0$ and there is a surjective map $F \to C$ onto an elliptic curve defined over the function field K of Z and such that $K_F + B_F$ is big over C. In each case it is enough to know that $K_F + B_F$ is semi-ample. In case (1), we can pass to the algebraic closure $\bar{K}$ and deduce that $K_F + B_F$ is semi-ample. In case (2), $m(K_F + B_F) \sim M_F \geqslant 0$ for some $m > 0$, and using the map $F \to C$ it is relatively easy to show $M_F = 0$: if not then each connected component of M_F is irreducible; let D be the reduction of such a component; then there is one such D which maps onto C and one can show that D is an elliptic curve with $K_F \cdot D = 0$ and $D|_D$ torsion; one then applies Proposition 3.15 to deduce that D is semi-ample, hence $\kappa(M_F) \geqslant 1$, a contradiction.

§4. Relative good minimal models of 3-folds

Proof of Theorem 1.6. By [5], (X, B) has a log minimal model over Z. Replacing (X, B) with the minimal model, we can assume $K_X + B$ is nef/Z. Let F be the generic fiber of $X \to Z$ and let $K_F + B_F = (K_X + B)|_F$. Then (F, B_F) is klt and $K_F + B_F$ is nef with $\kappa(K_F + B_F) \geqslant 0$. By Theorem 1.5, $K_F + B_F$ is semi-ample.

If $\kappa(K_F + B_F) = 0$, then $K_F + B_F \sim_{\mathbb{Q}} 0$, hence $K_X + B \sim_{\mathbb{Q}} 0/Z$, by Lemma 2.18. On the other hand, if $\kappa(K_F + B_F) = 2$, then $K_X + B$ is big/Z, hence it is semi-ample over Z by Theorem 2.7. So we assume $\kappa(K_F + B_F) = 1$.

Since $K_F + B_F$ is semi-ample and $\kappa(K_F + B_F) = 1$, there is a diagram



where ϕ is birational, S is a smooth projective surface, and $\phi^*(K_X + B)|_G \sim_{\mathbb{Q}} 0$ on the generic fiber G of g. By Lemma 2.18, we can actually assume $\phi^*(K_X + B) \sim_{\mathbb{Q}} 0/S$. So $\phi^*(K_X + B) \sim_{\mathbb{Q}} g^*D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on S. On the other hand, let H be an ample divisor on Z. Then since D is nef and big over Z, $D + nh^*H$ is nef and big for any $n \gg 0$. Since we are working over $\bar{\mathbb{F}}_p$, $D + nh^*H$ is semi-ample (this follows from [26]) which implies D is semi-ample over Z from which we deduce $K_X + B$ is semi-ample over Z. ∎

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COROLLARY 4.1. *Let W be a smooth projective 3-fold over an algebraically closed field k of char $p > 5$. Assume $\kappa(K_W) = 1$. If X is a minimal model of W, then the Iitaka fibration $X \dashrightarrow C$ is a morphism where C is a smooth projective curve, and $K_X \sim_{\mathbb{Q}} 0/C$.*

Proof. Denote by $g\colon W \dashrightarrow C$ the Iitaka fibration, which is assumed to be a morphism by blowing up W. Let $r : Y \to C$ be a minimal model of W over C which has at most terminal singularities. Let R be the generic fiber of r. Since Y is regular in codimension two, R is a regular surface. Since $\kappa(K_W) = 1$ and since g is the Iitaka fibration of K_W, $\kappa(K_R) = 0$. Moreover, as K_Y is nef over C, K_R is nef too. Therefore, by Theorem 1.5, $K_R \sim_{\mathbb{Q}} 0$. This implies $K_Y \sim_{\mathbb{Q}} 0/C$ as K_Y is nef over C, by Lemma 2.18. Thus K_Y is the pullback of an ample $\mathbb{Q}$-divisor on C. In particular, this means that Y is a minimal model of W globally, not just over C.

From the above arguments, we have that K_Y is semi-ample. So if X is a minimal model of W, then K_X is semi-ample too by a standard argument using the negativity lemma, the pullback of K_X and K_Y coincide on any common resolution of X and Y (cf. [4, Remark 2.7]). Therefore, the Iitaka fibration $X \dashrightarrow C$ is a morphism, and $K_X \sim_{\mathbb{Q}} 0/C$ as claimed. ∎

§5. Kodaira dimensions

In this section we prove some results on Kodaira dimensions which will be used in the proof of Theorem 1.2.

PROPOSITION 5.1. *Let $f\colon X \to Z$ be a contraction from a smooth projective variety onto a smooth projective curve over an algebraically closed field k of char $p > 0$. Assume there is an integer $m > 1$ such that $f_*\mathcal{O}_X(mK_{X/Z})$ is a nonzero nef vector bundle. If either*

(1) *$g(Z) > 1$; or*
(2) *$g(Z) = 1$ and $\deg f_*\mathcal{O}_X(mK_{X/Z}) > 0$;*

then $\kappa(K_X) \geqslant \kappa(K_F) + 1$ where F denotes the generic fiber.

Proof. (1) Since $f_*\mathcal{O}_X(mK_{X/Z})$ is nef, $f_*\mathcal{O}_X(mK_{X/Z}) \otimes \mathcal{O}_Z(P)$ is ample where P is a closed point on Z. So for sufficiently divisible positive integer l, the sheaf $S^l(f_*\mathcal{O}_X(mK_{X/Z}) \otimes \mathcal{O}_Z(P))$ is globally generated. Considering the natural homomorphism $S^l(f_*\mathcal{O}_X(mK_{X/Z} + f^*P)) \to f_*\mathcal{O}_X(l(mK_{X/Z} + f^*P))$ which is nonzero, we conclude that the divisor $l(mK_{X/Z} + f^*P)$ is linearly equivalent to an effective Cartier divisor.

As $g(Z) > 1$, we can write $K_Z \sim P + N$ where N is an effective divisor. Then

$$lmK_X \sim l(mK_{X/Z} + f^*P) + (m-1)lf^*P + lmN.$$

Applying Lemma 2.20 we are done in this case.

(2) Let $r = \operatorname{rank} f_*\mathcal{O}_X(mK_{X/Z})$. Let $d > r$ be an integer not divisible by p. With Z seen as an abelian variety, the morphism $Z' = Z \xrightarrow{\times d} Z$ is an étale cover of degree d^2 by [34, Section 6]. Consider the base change

$$\begin{array}{ccc} X' = X \times_Z Z' & \xrightarrow{\ g\ } & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\ \pi\ } & Z. \end{array}$$

Since π is flat, we have $K_{X'/Z'} = g^*K_{X/Z}$ by [18, Theorem 8.7] and $f'_*\mathcal{O}_{X'}(mK_{X'/Z'}) \cong \pi^* f_*\mathcal{O}_X(mK_{X/Z})$ by [19, Proposition 9.3]. By Riemann–Roch for vector bundles over a curve, we have

$$\begin{aligned} & h^0(f'_*\mathcal{O}_{X'}(mK_{X'/Z'}) \otimes \mathcal{O}_{Z'}(-P')) \\ & \quad \geqslant \deg(f'_*\mathcal{O}_{X'}(mK_{X'/Z'}) \otimes \mathcal{O}_{Z'}(-P')) \\ & \quad = \deg f'_*\mathcal{O}_{X'}(mK_{X'/Z'}) - r \\ & \quad = d \deg f_*\mathcal{O}_X(mK_{X/Z}) - r > 0, \end{aligned}$$

where $P' \in Z'$ is a closed point. So $mK_{X'} = mK_{X'/Z'} \sim f'^*P' + E$ for some effective divisor E on X'. Then

$$\kappa(K_X) = \kappa(K_{X'}) = \kappa(K_{X'} + f'^*P') \geqslant \kappa(K_{F'}) + 1 = \kappa(K_F) + 1$$

by Lemma 2.20 where F' is the generic fiber of f'. ∎

PROPOSITION 5.2. *Let $f\colon X \to Z$ be a contraction from a normal $\mathbb{Q}$-factorial projective variety to an elliptic curve over $\bar{\mathbb{F}}_p$. Assume that $f_*\mathcal{O}_X(lK_{X/Z})$ is a nonzero nef vector bundle for some $l > 0$. Then $\kappa(K_X) \geqslant 0$.*

Proof. If $\deg f_*\mathcal{O}_X(lK_{X/Z}) > 0$, then

$$h^0(X, \mathcal{O}_X(lK_{X/Z})) = h^0(f_*\mathcal{O}_X(lK_{X/Z})) \geqslant \chi(f_*\mathcal{O}_X(lK_{X/Z})) > 0.$$

So we can assume $\deg f_*\mathcal{O}_X(lK_{X/Z}) = 0$. By [1, Part II, Theorem 5], the vector bundle $f_*\mathcal{O}_X(lK_{X/Z})$ can be decomposed into a direct sum $\bigoplus_i V_i \otimes L_i$

where V_i are nef indecomposable vector bundles with $h^0(V_i)=1$ and L_i are line bundles with $\deg L_i=0$. Since we work over $\bar{\mathbb{F}}_p$, L_1 is torsion, say of order n. We have a cover $\pi\colon Z'\cong Z\to Z$ between elliptic curves induced by the dual map $\mathrm{Pic}^0(Z)\xrightarrow{\times n}\mathrm{Pic}^0(Z)$. Then $\pi^*L_1\sim\mathcal{O}_{Z'}$, and $X\times_Z Z'$ is integral since π is flat and f is separable. Let X' be the normalization of the fiber product $X\times_Z Z'$. Consider the natural morphisms $f'\colon X'\to Z'$ and $\pi':X'\to X$. Then we have a natural inclusion $\pi^*f_*\mathcal{O}_X(lK_{X/Z})\subseteq f'_*(\pi'^*\mathcal{O}_X(lK_{X/Z}))$ by [19, Proposition 9.3]. So $f'_*(\pi'^*\mathcal{O}_X(lK_{X/Z}))$ contains $\pi^*V_1\otimes\pi^*L_1\cong\pi^*V_1$. Hence

$$h^0(X',\pi'^*\mathcal{O}_X(lK_{X/Z}))=h^0(Z',f'_*(\pi'^*\mathcal{O}_X(lK_{X/Z})))\geqslant h^0(Z',\pi^*V_1)\geqslant 1.$$

By Theorem 2.22, we have

$$\kappa(X,K_X)=\kappa(X,lK_{X/Z})=\kappa(X',\pi'^*\mathcal{O}_X(lK_{X/Z}))\geqslant 0.\qquad\square$$

PROPOSITION 5.3. *Let $f\colon X\to Z$ be a contraction from a projective 3-fold with $\mathbb{Q}$-factorial terminal singularities to an elliptic curve over $\bar{\mathbb{F}}_p$ with $p>5$. Assume that K_X is big over Z and that the generic fiber of f is smooth. Then $\kappa(X)\geqslant 2$.*

Proof. We break the proof into several steps.

Step 1. By Theorem 2.7, there is a minimal model Y of X over Z. Since Y has terminal singularities, the generic fiber of $Y\to Z$ is also smooth. Replacing X with Y, we can assume K_X is nef$/Z$. Since Z is an elliptic curve, K_X is actually globally nef by the cone theorem [6, Theorem 1.1], because otherwise there will exist a rational curve Γ such that $K_X\cdot\Gamma<0$, and Γ must be contained one fiber of f since $g(Z)=1$, which, however, contradicts that X is minimal over Z.

Step 2. Let X' be the relative canonical model of X over Z which exists by Theorem 2.7. So $K_{X'}$ is ample over Z. Since the geometric generic fiber $X_{\bar\eta}$ of f is smooth, the geometric generic fiber $X'_{\bar\eta}$ of f' coincides with the canonical model of $X_{\bar\eta}$, and has canonical singularities which are strongly F-regular since $\mathrm{char}k=p>5$ by [17]. Therefore, by Theorem 2.12, $f'_*\mathcal{O}_{X'}(mK_{X'})=f_*\mathcal{O}_X(mK_X)$ is a nef vector bundle for any sufficiently divisible $m>0$.

If $\nu(K_X)=3$, then $\kappa(K_X)=3$, so there is nothing to prove. On the other hand, if $f_*\mathcal{O}_X(mK_X)$ is nef with $\deg f_*\mathcal{O}_X(mK_X)>0$ for some $m>0$, then we are done by applying Proposition 5.1 to a resolution of X. So in the following we assume $\nu(K_X)=2$ and that $\deg f_*\mathcal{O}_X(mK_X)=0$ for any sufficiently divisible positive integer m.

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Step 3. Applying Proposition 5.2 and Step 2, we find a positive integer l and a divisor $M \geqslant 0$ such that $lK_X \sim M$ and that $f_*\mathcal{O}_X(klK_X)$ is nef for any $k \geqslant 1$. We prove that $K_X|_M$ is semi-ample.

Let T be a horizontal/Z component of M. We show that $K_X|_T$ is semi-ample. Take the normalization $S \to T$, and let $C \subset S$ be the reduction of the conductor. Write $M = nT + T'$ where T is not a component of T'. By adjunction [26, 5.3], we have

$$\left(K_X + \frac{M}{n}\right)\bigg|_S \sim_{\mathbb{Q}} K_S + C + D$$

where D is a canonically defined effective $\mathbb{Q}$-divisor and $|_S$ means pullback to S. Then $K_X|_S$ is semi-ample on S by Theorem 2.10.

We want to argue that semi-ampleness of $K_X|_S$ implies semi-ampleness of $K_X|_T$. Since K_X is nef and $K_X^3 = 0$, we have $K_X^2 \cdot T = 0$, that is, $(K_X|_T)^2 = 0$, thus $(K_X|_S)^2 = 0$.

If $\nu(K_X|_S) = 0$, then $K_X|_S \sim_{\mathbb{Q}} 0$, that is, the associated map is trivial, hence $K_X|_T$ is semi-ample by [26, Corollary 2.14].

If $\nu(K_X|_S) = 1$. We denote by $h\colon S \to V$ the map associated to $K_X|_S$ and denote by H a general fiber, which has genus $g(H) \geqslant 1$ because it dominates Z. As $K_X|_S \cdot H = 0$, we have $(K_S + C + D) \cdot H = 0$, hence

$$0 \leqslant \deg K_H = (K_S + H) \cdot H = K_S \cdot H = -(C + D) \cdot H \leqslant 0.$$

Therefore, $C \cdot H = D \cdot H = 0$, and H is smooth with arithmetic genus $p_a(H) = 1$. Applying [26, Corollary 2.14] again, we conclude that $K_X|_T$ is semi-ample, and the associated map $\bar{h}\colon T \to \bar{V}$ is an elliptic fibration. In particular, this means that no component of T' intersects the general fibers of $\bar{h}$.

Let R_0 be the union of the vertical/Z components of M. Then since K_X is f-semi-ample by Theorem 2.7, the restriction $K_X|_{R_0}$ is semi-ample. Let $R \leqslant M$ be a reduced divisor containing R_0 and assume that $K_X|_R$ is semi-ample. If $R = \operatorname{Supp} M$, then $K_X|_M$ is semi-ample by [26, Lemma 1.4]. If not, pick a horizontal/Z component T of M which is not a component of R. As noted above, $K_X|_T$ is semi-ample defining a contraction $\bar{h}\colon T \to \bar{V}$ such that, either $\bar{V} = \operatorname{spec} k$, or $\dim \bar{V} = 1$ and general fibers of $\bar{h}$ do not intersect any component of R. Applying [26, Corollary 2.12], we deduce that $K_X|_{T \cup R}$ is semi-ample. Inductively, we extend R to the support of M.

Step 4. In this step we prove $\kappa(K_X)\geqslant 1$. Consider the following exact sequence

$$0\to \mathcal{O}_X((k-1)M)\to \mathcal{O}_X(kM)\to \mathcal{O}_M(kM)\to 0. \tag{5.3.1}$$

For $k\geqslant 2$, by assumptions in Step 2, both $f_*\mathcal{O}_X(kM)$ and $f_*\mathcal{O}_X((k-1)M)$ are nef vector bundles with

$$\deg f_*\mathcal{O}_X(kM)=\deg f_*\mathcal{O}_X((k-1)M)=0.$$

If

$$h^0(f_*\mathcal{O}_X(kM))=h^0(f_*\mathcal{O}_X((k-1)M))=1$$

for all $k\geqslant 2$, then $h^1(f_*\mathcal{O}_X((k-1)M))=1$ for such k by Riemann–Roch, and by taking cohomology of the exact sequence (5.3.1), we conclude that $h^0(\mathcal{O}_M(kM))\leqslant 1$. However, this contradicts semi-ampleness of $M|_M$ and the property $\nu(M|_M)\geqslant 1$. Therefore, $\kappa(K_X)\geqslant 1$.

Step 5. Assume $\kappa(K_X)=1$. We derive a contradiction. Let $W\to X$ be a resolution so that the Iitaka fibration $W\to C$ is a morphism. By Corollary 4.1, the induced map $X\dashrightarrow C$ is a morphism and $K_X\sim_{\mathbb{Q}} 0/C$. In particular, $\nu(K_X)=1$ which contradicts the assumption $\nu(K_X)=2$. ∎

§6. Proof of Theorem 1.2

Proof of Theorem 1.2. We can assume $\kappa(K_Z)\geqslant 0$ and $\kappa(K_F)\geqslant 0$. As pointed out in the introduction $C_{3,2}$ follows from [13], so we assume $n=3$ and $m=1$. Replacing X with a minimal model over Z, we can assume K_X is nef$/Z$. Of course X may not be smooth any more but it has $\mathbb{Q}$-factorial terminal singularities. The generic fiber stays smooth by the arguments in Step 1 of the proof of Theorem 5.3.

If $\kappa(K_F)=0$, then by Theorem 1.6 and Lemma 2.18, $K_{X/Z}\sim_{\mathbb{Q}} f^*M$ for some $\mathbb{Q}$-divisor M. Moreover, by Theorem 2.12, $K_{X/Z}$ is nef, hence $\deg M\geqslant 0$ which implies $\kappa(M)\geqslant 0$ as we are working over $\bar{\mathbb{F}}_p$. Thus $\kappa(K_{X/Z})\geqslant 0$ and

$$\kappa(K_X)=\kappa(K_{X/Z}+f^*K_Z)\geqslant \kappa(K_Z).$$

If $\kappa(K_F)=1$, then K_F is semi-ample by Theorem 1.5. We claim that the geometric generic fiber of the Iitaka fibration $I:F\to C$ is a smooth elliptic curve. Indeed, denote by $\bar{F}=F\otimes_{K(Z)}\overline{K(Z)}$ the geometric generic fiber of f. Since char $k=p>5$, the geometric generic fiber of the Iitaka fibration $\bar{I}:\bar{F}\to\bar{C}$ is a smooth elliptic curve by [2, Theorem 7.18]. For any sufficiently

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divisible positive integer n, since $H^0(\bar{F}, nK_{\bar{F}}) \cong H^0(F, nK_F) \otimes_{K(Z)} \overline{K(Z)}$, we see that $\bar{I}: \bar{F} \to \bar{C}$ coincides with the base change of the morphism $I: F \to C$ via spec $\overline{K(Z)} \to$ spec $K(Z)$. Thus the geometric generic fiber of I is a smooth elliptic curve.

Considering the relative Iitaka fibration, blowing up X if necessary, with the help of Theorem 1.6 and Lemma 2.18, we get a smooth resolution of $\sigma: X' \to X$ and a smooth surface Y fitting into the following commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xleftarrow[g]{} & Y \end{array}$$

such that

- the geometric generic fiber of h is a smooth elliptic curve;
- $\sigma^* K_X \sim_{\mathbb{Q}} h^* D$ where D is a g-big divisor on Y.

By flattening trick (cf. [46, Lemma 7.3]), we can assume that every h-exceptional divisor is also σ-exceptional. By Theorem 2.14, we have effective vertical (w.r.t. h) divisors D_1, D_2 on X' such that

$$\sigma^* K_X + D_1 \sim_{\mathbb{Q}} K_{X'} \sim_{\mathbb{Q}} h^* K_Y + D_2.$$

Then $D_2 - D_1 \sim_{\mathbb{Q}} h^*(K_Y - D)$. So there exists a $\mathbb{Q}$-divisor Δ on Y such that $D_1 - D_2 = h^*\Delta$. We can write that

$$\Delta = \Delta_1 - \Delta_2$$

where Δ_1 and Δ_2 are effective divisors on Y having no common components, thus are supported in $h(D_1)$ and $h(D_2)$, respectively. It follows that

$$h^* D \sim_{\mathbb{Q}} h^*(K_Y + \Delta_2 - \Delta_1) \text{ and thus } D_2 - D_1 \sim_{\mathbb{Q}} h^*(\Delta_2 - \Delta_1).$$

Therefore,

$$\sigma^* K_X + h^*\Delta_1 \sim_{\mathbb{Q}} h^*(D + \Delta_1) \sim_{\mathbb{Q}} h^*(K_Y + \Delta_2).$$

Consider the pair (Y, Δ_2). By Theorem 2.10, we obtain a minimal model (Y', Δ') of (Y, Δ_2) such that $K_{Y'} + \Delta'$ is semi-ample. Denote by $\mu: Y \to Y'$

the natural morphism. There exists an effective μ-exceptional divisor E such that

$$K_Y + \Delta_2 = \mu^*(K_{Y'} + \Delta') + E.$$

Since $D + \Delta_1 \sim_{\mathbb{Q}} K_Y + \Delta_2$ and D is nef and g-big, we have

$$\nu(Y', K_{Y'} + \Delta') = \nu(Y, K_Y + \Delta_2) \geqslant \nu(Y, D) \geqslant 1$$

where the first "=" is obtained by the proof of [9, Proposition 2.7] and the fact that $\mu_*(\mathcal{O}_Y(\llcorner nE \lrcorner)) = \mathcal{O}_{Y'}$ for any integer $n > 0$. So

$$\kappa(Y, K_Y + \Delta_2) = \kappa(Y', K_{Y'} + \Delta') = \nu(Y', K_{Y'} + \Delta') \geqslant 1.$$

Observe that every component of $h^*\Delta_1$ is either contained in D_1 or h-exceptional, thus is σ-exceptional. Then applying Theorem 2.22, we conclude that

$$\kappa(X) = \kappa(X', \sigma^* K_X + h^*\Delta_1) = \kappa(Y, K_Y + \Delta_2) \geqslant 1.$$

If $g(Z) = 1$, then we are done by the above inequality.

So assume that $g(Z) \geqslant 2$. If $\kappa(X) = 1$, then by Corollary 4.1 K_X is semi-ample, and the Iitaka fibration $I : X \to C$ is a morphism. Let G be a general fiber of I, which is integral since I is separable [2, Lemma 7.2]. Then G is dominant over Z and $K_G \sim_{\mathbb{Q}} 0$. Let $\tilde{G}$ be a smooth resolution of G. By the construction above, the divisor $h^*D \sim_{\mathbb{Q}} \sigma^* K_X$ is semi-ample, and induces a morphism coinciding with the composite morphism $X' \to X \to C$. Since $h : X' \to Y$ is fibered by elliptic curves, we conclude that the Stein factorization of the natural morphism $\tilde{G} \to Z$ induces an elliptic fibration $\tilde{G} \to \tilde{Z}$ with $g(\tilde{Z}) \geqslant 2$, hence $\kappa(\tilde{G}) \geqslant 1$. However, this contradicts that $K_G \sim_{\mathbb{Q}} 0$.

Finally assume $\kappa(K_F) = 2$. If $\kappa(K_Z) = 0$ we apply Proposition 5.3. But if $\kappa(K_Z) = 1$, we replace X with its canonical model over Z so that K_X is ample over Z, and we use Theorem 2.12 to deduce $f_*\mathcal{O}_X(mK_{X/Z})$ is a nef vector bundle for sufficiently divisible integer $m > 0$ as in the proof of 5.3; next we apply Proposition 5.1 to a resolution of X. ∎

Proof of Corollary 1.3. By [2, Corollary 7.3] f has integral generic geometric fiber. Let F be the generic fiber of f, let $F_1 = F \times_{K(Z)} K(Z)^{1/p^\infty}$, and let $\tilde{F}_1 \to F_1$ be a desingularization. Since $K(Z)^{1/p^\infty}$ is perfect, $\tilde{F}_1$ is smooth over $K(Z)^{1/p^\infty}$ by [31, Chapter 4 Corollary 3.33]. Therefore, there exists a natural number e such that $\tilde{F}_1$ can be descent to $\tilde{F}_2$, which is a desingularization of $F_2 = F \times_{K(Z)} K(Z)^{1/p^e}$ and smooth over $K(Z)^{1/p^e}$.

Denote by $F^e : Z' \to Z$ the eth absolute Frobenius iteration. Then $X \times_Z Z'$ is integral since F^e is flat and f is separable. We have the following commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\ \pi\ } & X \times_Z Z' & \xrightarrow{\ g\ } & X \\ & \searrow^{f'} & \downarrow & & \downarrow f \\ & & Z' & \xrightarrow[F^e]{} & Z \end{array}$$

where $\pi : X' \to X \times_Z Z'$ is a resolution. By the above argument, the generic fiber F' of f' is a smooth curve over $K(Z')$. By Theorem 1.2,

$$\kappa(K_{X'}) \geqslant \kappa(K_{F'}) + \kappa(K_{Z'}) = \kappa(K_{\tilde{F}}) + \kappa(K_Z).$$

Let $\sigma = g\pi : X' \to X$ be the natural composite morphism. By [13, Theorem 2.4], there exists an effective σ-exceptional divisor E on X' such that

$$K_{X'/Z'} \leqslant \sigma^* K_{X/Z} + E.$$

Thus

$$K_{X'} + (p^e - 1) f'^* K_{Z'} \leqslant \sigma^* K_X + E.$$

We can assume $K_{Z'}$ is effective, thus

$$\begin{aligned} \kappa(K_X) &= \kappa(\sigma^* K_X + E) \geqslant \kappa(K_{X'} + (p^e - 1) K_{Z'}) \\ &\geqslant \kappa(K_{X'}) \geqslant \kappa(K_{\tilde{F}}) + \kappa(K_Z), \end{aligned}$$

where the first "=" is from Theorem 2.22. ∎

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