Hyperbolic sets for twist maps

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Abstract. An example is given of an area-preserving monotone twist map such that a uniformly hyperbolic structure exists on the closure of its Birkhoff maximizing orbits.

This note provides a rigorous example of an area preserving monotone twist map \( f \) with the property that \( Df|_B \) has a uniformly hyperbolic structure, where \( B \) denotes the closure of the Birkhoff maximizing orbits. As shown by Mather [8] and by Aubry, La Daeron, and André [3], the set \( B \) associated with \( f \) contains invariant Cantor sets of all possible rotation numbers.

A result which would imply the hyperbolicity of these invariant Cantor sets was announced by Aubry in [1]. The heuristic justification given there is discussed further in [2]. Nevertheless, Katok raises the hyperbolicity question again in [4] and [5]. The construction below gives a rigorous answer based on an estimate first due to Aubry. Another proof that hyperbolic Cantor sets \( \lambda \) can exist in \( B \) was obtained independently by Michel Herman.

Consider the 'standard' one parameter family of area preserving monotone twist maps of the cylinder \( T^1 \times \mathbb{R} \). One lift to \( \mathbb{R}^2 \) of the map in this family corresponding to the parameter \( k \) has the form

\[
f(x, y) = \left( x + y - \frac{k}{2\pi} \sin 2\pi x, y - \frac{k}{2\pi} \sin 2\pi x \right).
\]

The function

\[
h(x, x') = -\frac{1}{2} (x - x')^2 - \frac{k}{4\pi^2} \cos 2\pi x
\]

generates \( f \) in the sense that \( f(x, y) = (x', y') \) if and only if

\[
y = \frac{\partial h}{\partial x}(x, x') \quad \text{and} \quad y' = -\frac{\partial h}{\partial x'}(x, x').
\]

Thus, given a sequence \( \{x_n\} \), there exists a sequence \( \{y_n\} \) such that \( (x_m, y_n) = f^n(x_0, y_0) \) if and only if \( \{x_n\} \) satisfies

\[
\frac{\partial h}{\partial x'}(x_{n-1}, x_n) + \frac{\partial h}{\partial x}(x_m, x_{n+1}) = 0,
\]

(1)
in which case
\[ y_n = \frac{\partial h}{\partial x}(x_n, x_{n+1}). \]  
(2)

Following Birkhoff, this generating function for \( f \) can be used to prove the existence of certain periodic orbits. Given integers \( p \) and \( q > 0 \), define the action \( w : \mathbb{R}^q \to \mathbb{R} \) by

\[ w(x_1, \ldots, x_q) = \sum_{n=1}^{q} h(x_{n-1}, x_n), \]

where \( x_0 + p = x_q \). It can be shown that the maximum of \( w \) is achieved at a critical point, which therefore satisfies (1). If we associate second coordinates to this sequence using (2), the resulting union of \( q \) points in \( \mathbb{R}^2 \) is called a Birkhoff maximizing orbit of type \((p, q)\) and its projection to \( \mathbb{T}^1 \times \mathbb{R} \) is a periodic trajectory. Let \( B \) denote the union of all the orbits obtained in this way as \( p \) and \( q \) vary, and let \( \bar{B} \) denote the closure of \( B \) in \( \mathbb{R}^2 \).

For details concerning the construction and properties of \( B \) and \( \bar{B} \), see [3], [5], [6] and [8]. Although the last three references treat boundary preserving twist maps of an annulus, straightforward modifications allow the relevant results to apply to the standard family on the cylinder. See [7] in this regard.

**Lemma (Aubry, [2]).** Let \( f \) be a standard map with parameter \( k > 2\sqrt{1 + \pi^2} \). If \((x_0, y_0)\) belongs to the set \( B \) associated with \( f \) then \(-k \cos 2\pi x_0 \geq 2\).

**Proof.** Suppose \((x_0, y_0)\) belongs to a Birkhoff maximizing orbit of type \((p, q)\). By choosing an appropriate lift \( \tilde{f} \), we can assume without loss of generality that \( 0 < p/q \leq 1 \). Let \((x_n, y_n) = f^n(x_0, y_0)\). Then \( 0 < x_{n+1} - x_n \leq 1 \) and, from (1), we have

\[ (x_{n+1} - x_n) - (x_n - x_{n-1}) + \frac{k}{2\pi} \sin 2\pi x_n = 0. \]

This condition, which is independent of the lift chosen, implies \(|\sin 2\pi x_n| \leq 2\pi/k\), and hence, for \( k \geq 2\pi \),

\[ |\cos 2\pi x_n| \geq \sqrt{1 - \frac{4\pi^2}{k^2}}. \]  
(3)

Now from the definition of \( B \) as sequences which maximize the function \( w \), we also have

\[ \frac{\partial^2 w}{\partial x_n^2}(x_1, \ldots, x_q) = -2 + k \cos 2\pi x_n \leq 0. \]  
(4)

For \( k > 2\sqrt{1 + \pi^2} \), (3) and (4) together ensure that

\[ -k \cos 2\pi x_n \geq k \sqrt{1 - \frac{4\pi^2}{k^2}} > 2, \]

which completes the proof. \( \square \)

**Theorem.** Let \( f \) be a standard map with parameter \( k > 2\sqrt{1 + \pi^2} \). Then the set \( \bar{B} \) associated with \( f \) has a uniform hyperbolic structure.
Proof. Using the characterization of hyperbolicity given by Newhouse and Palis [10], it suffices to find a cone $C$ in $\mathbb{R}^2$ and a positive integer $m$ such that for each $(x, y)$ in $\bar{B}$, the derivative of $f$ at $(x, y)$, $D_{(x,y)}f$, maps $C$ into itself and such that $D_{(x,y)}f^m$ expands $C$ and $D_{(x,y)}f^{-m}$ expands $\mathbb{R}^2 \setminus C$ (see also [9]).

From the lemma, we have that for $(x, y)$ in $\bar{B}$,

$$D_{(x,y)}f = \begin{bmatrix} 1 - k \cos 2\pi x & 1 \\ -k \cos 2\pi x & 1 \end{bmatrix} = \begin{bmatrix} 1 + \epsilon_0 & 1 \\ \epsilon_0 & 1 \end{bmatrix}$$

for some $\epsilon > 2$. Let $C = \{(v_1, v_2) \in \mathbb{R}^2 | v_1, v_2 \geq 0\}$. Then $D_{(x,y)}f$ not only maps each $v$ in $C$ back into $C$, but also expands its Euclidean norm in the sense that

$$\|D_{(x,y)}f[v]\| \geq \sqrt{2}\|v\|$$

for $v$ in $C$. Similarly, we find

$$D_{(x,y)}f = \begin{bmatrix} 1 & -1 \\ -\epsilon & 1 + \epsilon \end{bmatrix},$$

for some $\epsilon > 2$, and so

$$\|D_{(x,y)}f[v]\| \geq \sqrt{5}\|v\|,$$

whenever $v$ belongs to $\mathbb{R}^2 \setminus C$. This completes the proof.

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REFERENCES