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PERIODIC FIBONACCI ALGEBRAS

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Fibonacci algebras are groups equipped with an extra unary operation ϕ that satisfies a Fibonacci-type law. We described in an earlier paper the free objects in the resulting varieties, and here we do the same in the case when ϕ is assumed to be periodic. They turn out to be central extensions of Burnside groups with finite kernels whose orders can be expressed in terms of the resultants of certain polynomials.

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0. Introduction

We are concerned with the variety $\mathfrak{V}(m,n)$ of universal algebras G of the following type: G is a group equipped with a unary operation ϕ that is an automorphism of G and satisfies the one-variable laws

$$xx\phi\ldots x\phi^{m-1}=x\phi^{m}, \tag{1}$$

$$x\phi^{n}=x, \tag{2}$$

where m and n are positive integers and, as in the Polish notation, any occurrence of ϕ or one of its powers is understood to apply only to the symbol immediately preceding it. It seems natural to call such objects Fibonacci algebras, or ϕ-algebras for short, and the $\mathfrak{V}(m,n)$ Fibonacci varieties. Our main object of study is the free object $V_d(m,n)$ of rank d in $\mathfrak{V}(m,n)$. (We assume $m>1$ to avoid triviality.)

1. The problem

The case when $m=2$ is covered comprehensively in [3], where it is shown that the monogenic free algebra $V_1(2,n)$ is isomorphic to the abelianised Fibonacci group $A(2,n)$ studied in [1] and defined as the derived factor group of the Fibonacci group

$$F(2,n)=\langle x_1,x_2,\ldots,x_n \mid x_1x_2=x_3,\ldots,x_{n-1}x_n=x_1,x_nx_1=x_2\rangle. \tag{3}$$

The $A(2,n)$ are all finite and their invariant factors are given explicitly in [3] in terms of the Fibonacci numbers.

The case of arbitrary m is treated in [4], where it is shown (Theorem 3) that the elements $x^{-1}x\phi$, $x\in G$, form a central subgroup, called $G\theta$ here as in [2], and that the quotient group G/Z lies in the Burnside variety $\mathfrak{B}(m-1)$ of groups of exponent dividing $m-1$. There is thus a central extension

$$0\to G\theta\to G\to G^{\phi}\to 1, \tag{4}$$

where $\theta: G\to G$ is given by $x\theta = x^{-1}x\phi$, and G^{ϕ} is the biggest ϕ-trivial factor-algebra of G. In particular, there is a central extension

$$0\to Z_d(m,n)\to V_d(m,n)\overset{\vee}{\to} B_d(m-1)\to 1, \tag{5}$$

where $B_d(m-1)$ is the free object of rank d in $\mathfrak{B}(m-1)$. It is stated in [4] that $Z_d(m,n)$ is free abelian of rank d_m when $n=0$ and finite when $n>0$. The first of these assertions is proved in [2, Corollary 4.4], and the second is proved below.

2. First reduction

We regard $V_d(m,n)$ as the dth free power (in $\mathfrak{B}(m,n)$) of $V_1(m,n)$ and invoke the results of [2] to reduce our problem to the monogenic case.

Corollary 4.2 of [2] asserts that the central extension (4) is well-behaved under the formation of free products: for G, $H\in\mathfrak{B}(m,0)$, there is a central extension

$$0\to G\theta\times H\theta\to G *_{\phi} H\to G^{\phi} *_{B} H^{\phi}\to 1, \tag{6}$$

where $\times$, $*_{\phi}$ and $*_{B}$ denote products in the varieties $\mathfrak{U}$ (of abelian groups), $\mathfrak{B}(m,0)$ and $\mathfrak{B}(m-1)$, respectively.

Now the imposition of (2) has no effect on the right-hand term, and factoring out by the relations it induces commutes with the formation of direct products. (6) can be regarded as a central extension in $\mathfrak{B}(m,n), n>0$, with $*_{\phi}$ suitably re-interpreted. A simple induction on d now gives the following results.

Lemma 1. *For all $m,n,d\in\mathbb{N}$, the kernel $Z_d(m,n)$ is the dth direct power of $Z_1(m,n)$.*

3. Second reduction

The extension (5) is described explicitly in § 3 of [2] in the case when $d=1$ and $n=0$: $V_1(m,0)$ is the result of imposing the law (1) on the free abelian group with basis $\{x_i \mid i\in\mathbb{Z}\}$, where $x_i\phi = x_{i+1}$ for all i. But here, we impose the law (2) *first*, giving the free abelian group on $X=\{x_1,\dots,x_n\}$, and the subsequent imposition of (1) yields the group with presentation

$$\langle X \mid R, C\rangle, \tag{7}$$

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where

$$C=\{x_ix_j=x_jx_i \mid 1\leqq i<j\leqq n\},$$

$$R=\{x_ix_{i+1}\dots x_{i+m-1}=x_{i+m} \mid 1\leqq i\leqq n\},$$

with subscripts reduced modulo n. Noting that $\langle X \mid R\rangle$ is just the Fibonacci group $F(m,n)$, we have proved the following result, which is Theorem 1 of [**4**].

Lemma 2. *The monogenic free ϕ-algebra $V_1(m,n)$ is isomorphic to the abelianized Fibonacci group $A(m,n)$.*

4. The solution

It is shown in [**1**] that $A(m,n)$ is the n-generator group whose relation matrix is the circulant matrix determined by the polynomial

$$f(t)=1+t+\cdots+t^{m-1}-t^m,$$

so that $A(m,n)$ has order

$$\prod_{k=0}^{n-1}|f(\omega^k)|, \quad \omega=e^{2\pi i/n},$$

which is just the resultant $\operatorname{res}(f,g)$ of the polynomials $f(t)$ and $g(t)=1-t^n$. Because f and g have no common root, this is a positive integer (so that $A(m,n)$ is a finite group) which is divisible by $m-1=f(1)$. As the kernel $Z_1(m,n)$ has index $m-1$ in $V_1(m,n)$, this may be combined with Lemmas 1 and 2 to give our main result.

Theorem. *For all $d,m,n\in\mathbb{N}$ with $m>1$, the kernel $Z_d(m,n)$ of the natural map*

$$v: V_d(m,n)\twoheadrightarrow B_d(m-1)$$

is a finite abelian group of order $r_{m,n}^d$, where $r_{m,n}$ is the resultant of the polynomials

$$1+t+\cdots+t^{m-1}-t^m \quad \text{and} \quad 1+t+\cdots+t^{n-1}.$$

The values of $r_{m,n}$ are given explicitly in [**1**] for certain favourable values of m and n.

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