Can. J. Math., Vol. XXXVII, No. 3, 1985, pp. 442-451

THE LARGEST IRREDUCIBLE CHARACTER DEGREE OF A FINITE GROUP

DAVID GLUCK

Introduction. Much information about a finite group is encoded in its character table. Indeed even a small portion of the character table may reveal significant information about the group. By a famous theorem of Jordan, knowing the degree of one faithful irreducible character of a finite group gives an upper bound for the index of its largest normal abelian subgroup.

Here we consider $b(G)$, the largest irreducible character degree of the group G. A simple application of Frobenius reciprocity shows that $b(G) \leqq |G:A|$ for any abelian subgroup A of G. In light of this fact and Jordan's theorem, one might seek to bound the index of the largest abelian subgroup of G from above by a function of $b(G)$. If is G is nilpotent, a result of Isaacs and Passman (see [**7**, Theorem 12.26]) shows that G has an abelian subgroup of index at most $b(G)^4$.

In this paper, we show that there is a constant K such that any finite group G has an abelian subgroup of index at most $b(G)^K$. We use the classification of simple groups to handle the nonsolvable case, but the existence of such a polynomial bound is a new result even for solvable groups.

The first half of this paper is devoted to solvable groups. We show first that if H is solvable and V is a completely reducible H-module, then V contains a vector v such that

$$|C_H(v)| \leqq |H|^{11/13}.$$

We deduce that if G is a solvable group with Fitting subgroup $F(G)$, then

$$|G:F(G)| \leqq b(G)^{13/2}.$$

Combining this with the result of Isaacs and Passman mentioned above, we obtain an abelian subgroup $A \leqq F(G) \leqq G$ with

$$|G:A| \leqq b(G)^{21/2}.$$

These inequalities for solvable groups depend on several substantial

Received August 10, 1983 and in revised form June 25, 1984.

known results, including T. R. Wolf's upper bound for the order of a completely reducible solvable subgroup of $GL(n, p)$.

In the nonsolvable case, it is the nonsolvable composite groups, rather than the simple groups, which cause difficulties. Indeed there is a constant C such that $|G| < b(G)^C$ for every nonabelian simple group G. This constant C exists for the trivial reason that $|\mathrm{Irr}(G)|$ is much smaller than $|G|$ for any such G.

Our methods are not strong enough to give best possible bounds, even for solvable groups. We indicate why this is so at the end of Section 1.

I would like to thank I. M. Isaacs for many helpful conversations related to this paper. I am using his simplified version of my original proof of Theorem 2.10.

1. Solvable groups. Throughout this paper all groups are assumed to be finite and all modules are assumed to be finite dimensional right modules.

We begin our discussion of solvable groups with precise statements of three results mentioned in the introduction.

THEOREM A. *Let G be solvable and let V be a faithful completely reducible $F[G]$-module, for a finite field F. Then*

$$|C_G(v)| \leqq |G|^{11/13} \quad \textit{for some } v \in V.$$

THEOREM B. *Let G be solvable with largest irreducible character degree $b(G)$ and Fitting subgroup $F(G)$. Then $|G:F(G)| \leqq b(G)^{13/2}$.*

THEOREM C. *Let G be solvable with largest irreducible character degree $b(G)$. Then G has an abelian subgroup A such that $|G:A| \leqq b(G)^{21/2}$.*

To prove these theorems, we assemble several known results and consequences of known results.

LEMMA 1.1. *Let H be a solvable group and let V be a faithful completely reducible $F[H]$-module, for a finite field F. Then $|H| \leqq |V|^{9/4}$.*

Proof. This is a slightly weaker version of [**11**, Theorem 3.1].

COROLLARY 1.2. *Let G be solvable with Fitting subgroup $F(G)$. Then*

$$|G:F(G)| \leqq |F(G)|^{9/4}.$$

Proof. Let $\Phi(G)$ be the Frattini subgroup of G and let $\bar{G} = G/\Phi(G)$. By [**6**, III, Satz 4.5],

$$|G:F(G)| = |\bar{G}:F(\bar{G})| \quad \text{and} \quad F(\bar{G}) = \bar{N}_1 \times \ldots \times \bar{N}_k,$$

where each $\bar{N}_i$ is a minimal normal subgroup of $\bar{G}$. For $1 \leqq i \leqq k$, let $\bar{C}_i$ be the centralizer in $\bar{G}$ of $\bar{N}_i$. By Lemma 1.1, each $|\bar{G}:\bar{C}_i| \leqq |\bar{N}_i|^{9/4}$, and an obvious subdirect product argument yields

$$|\bar{G}:F(\bar{G})| \leqq |F(\bar{G})|^{9/4}.$$

Hence

$$|G:F(G)| \leqq |F(\bar{G})|^{9/4} = |\overline{F(G)}|^{9/4} \leqq |F(G)|^{9/4}.$$

LEMMA 1.3. *Let P be a p-group and let V be a faithful $F[P]$-module for a finite field F of characteristic different from p. Then there exists $v \in V$ such that $|C_P(v)| \leqq |P|^{1/2}$.*

Proof. This follows from [**9**, Corollary 2.4].

COROLLARY 1.4. *Let N be a nilpotent group and let V be a faithful $F[N]$-module for a finite field F of characteristic not dividing $|N|$. Then there exists $v \in V$ such that $|C_N(v)| \leqq |N|^{1/2}$.*

Proof. We proceed by induction on the number of prime divisors of $|N|$.

The hypotheses imply that V is completely reducible. Suppose that $V = V_1 \oplus V_2$, where V_1 and V_2 are $F[N]$-submodules of V. Let $C = C_N(V_2)$. Then C acts faithfully on V_1 and N/C acts faithfully on V_2. Suppose we could find $v_1 \in V_1$ and $v_2 \in V_2$ so that

$$|C_C(v_1)| \leqq |C|^{1/2} \quad \text{and} \quad |C_{N/C}(v_2)| \leqq |N/C|^{1/2}.$$

Let v be the vector (v_1, v_2) in V. Then

$$C_N(v)C/C \leqq C_{N/C}(v_2),$$

so that

$$|C_N(v):C \cap C_N(v)| = |C_N(v)C:C| \leqq |N/C|^{1/2}.$$

Since

$$|C \cap C_N(v)| = |C_C(v_1)| \leqq |C|^{1/2},$$

it follows that

$$|C_N(v)| = |C_N(v):C \cap C_N(v)|\,|C \cap C_N(v)| \leqq |N|^{1/2}.$$

Thus we may assume V is irreducible.

Let P be a fixed Sylow subgroup of N. We may consider V an $F[P]$-module by restriction. Let $\mathscr{E}$ be the endomorphism ring $\mathrm{End}_{F[P]}V$. Let R be the p-complement of N, so that $N = R \times P$. We view R as a subgroup of the group of units of $\mathscr{E}$. Since $R \leqq \mathscr{E}$, R stabilizes each homogeneous component of $V|_P$. Therefore V is a homogenous $F[P]$-module. Let W be a fixed simple $F[P]$-submodule of V and let

$$E = \mathrm{End}_{F[P]}W.$$

Thus E is a finite field containing F. Let

$$m = \dim_F V/\dim_F W.$$

By [**1**, Lemma 2.2], $\mathscr{E}$ is isomorphic to the ring of all $m \times m$ matrices over E. Let E^m denote the vector space of all row vectors of size m with entries in E. The embedding of R in $\mathscr{E}$ and the isomorphism of $\mathscr{E}$ with the ring of $m \times m$ matrices over E makes E^m a faithful $E[R]$-module. Since $|R|$ has fewer prime divisors than $|N|$, we may choose a vector $x \in E^m$ so that

$$|C_R(x)| \leqq |R|^{1/2}.$$

There is a simple $F[P]$-submodule X of V such that $C_R(x) = C_R(X)$, the set of elements of R which fix and centralize X. By Lemma 1.3 we may choose $v \in X$ so that

$$|C_P(v)| \leqq |P|^{1/2}.$$

Since $C_N(v) = C_P(v) \times C_R(v)$, it remains to show that $C_R(v) = C_R(X)$.

Since $C_R(v) \leqq \mathscr{E}$ and X is the unique simple $F[P]$-submodule of V containing v, it follows that $C_R(v)$ stabilizes X. Since nonidentity elements of $\mathrm{End}_{F[P]}X \cong E$ have no fixed points in X, it then follows that

$$C_R(v) \leqq C_R(X).$$

Thus $C_R(v) = C_R(X)$, completing the proof.

LEMMA 1.5. *Let G be a nilpotent group with largest character degree $b(G)$. Then G has an abelian subgroup A with $|G:A| \leqq b(G)^4$.*

Proof. This is a special case of [**7**, Theorem 12.26].

Proof of Theorem A. Let N be the Fitting subgroup of G. Since $N \lhd G$, V is a completely reducible $F[N]$-module. Thus Corollary 1.4 applies and we may choose $v \in V$ so that $|C_N(v)| \leqq |N|^{1/2}$.

We have $|C_G(v)| \leqq |G:N|\,|C_N(v)|$. Thus

$$|G:C_G(v)| \geqq |G|/(|G:N|\,|C_N(v)|) = |N:C_N(v)| \geqq |N|^{1/2}.$$

By Corollary 1.2, $|G| \leqq |N|^{13/4}$ and so $|N|^{1/2} \geqq |G|^{2/13}$. Therefore

$$|G:C_G(v)| \geqq |N|^{1/2} \geqq |G|^{2/13},$$

and thus

$$|C_G(v)| \leqq |G|^{11/13}.$$

Before proving Theorem B, we must extend Theorem A to modules in mixed characteristic.

LEMMA 1.6. *Let G be solvable and let $V_i (1 \leqq i \leqq k)$ be completely reducible $F_i[G]$-modules, for various finite fields F_i. Let $V = V_1 \times \ldots \times V_k$, and let G act componentwise on V. Let $C = C_G(V)$. Then there exists $v \in V$ such that*

$$|C_G(v):C| \leqq |G:C|^{11/13}.$$

Proof. We proceed by induction on k. Let $\bar{G} = G/C$ and let $\bar{D} = C_{\bar{G}}(V_1)$. Then $\bar{G}/\bar{D}$ acts faithfully on V_1. By Theorem A we may choose $v_1 \in V_1$ so that

$$|C_{\bar{G}/\bar{D}}(v_1)| \leqq |\bar{G}/\bar{D}|^{11/13}.$$

Since $\bar{D} \lhd \bar{G}$, $\bar{D}$ acts faithfully and completely reducibly on $V_2 \times \ldots \times V_k$, so by induction we may choose $(v_2, \ldots, v_k) \in V_2 \times \ldots \times V_k$ so that

$$|C_{\bar{D}}(v_2, \ldots, v_k)| \leqq |\bar{D}|^{11/13}.$$

As in the second paragraph of the proof of Corollary 1.4, we have

$$|C_{\bar{G}}(v_1, \ldots, v_k)| \leqq |\bar{G}|^{11/13}.$$

The conclusion of the lemma follows.

Proof of Theorem B. Let $\bar{G} = G/\Phi(G)$ as in the proof of Corollary 1.2. Since $|\bar{G}:F(\bar{G})| = |G:F(G)|$ and $b(\bar{G}) \leqq b(G)$, we may assume that $\Phi(G) = 1$. By [**6**, III, Satz 4.5] we may write

$$F(G) = P_1 \times \ldots \times P_k,$$

where the P_i are the nonidentity Sylow subgroups of $F(G)$, and each P_i is a direct product of minimal normal subgroups of G. Let $V = \mathrm{Irr}(F(G))$, the multiplicative group of linear characters of $F(G)$. Then

$$V = \mathrm{Irr}(P_1) \times \ldots \times \mathrm{Irr}(P_k)$$

and the usual conjugation action of G on V leaves each $\mathrm{Irr}(P_i)$ invariant. Moreover, for $1 \leqq i \leqq k$, P_i and $\mathrm{Irr}(P_i)$ are dual $GF(p_i)[G]$-modules, where p_i denotes the prime divisor of $|P_i|$.

In particular, each $\mathrm{Irr}(P_i)$ is a completely reducible $GF(p_i)[G]$-module and $C_G(V) = F(G)$. We may apply the preceeding lemma to obtain $\lambda \in V$ such that

$$|C_G(\lambda):F(G)| \leqq |G:F(G)|^{11/13}.$$

Since $C_G(\lambda)$ is the inertia subgroup of λ in G, Clifford's theorem implies that

$$b(G) \geqq |G:C_G(\lambda)| = |G:F(G)|/|C_G(\lambda):F(G)| \geqq |G:F(G)|^{2/13}.$$

Hence $|G:F(G)| \leqq b(G)^{13/2}$, as required.

Proof of Theorem C. By Lemma 1.5 we may choose $A \leqq F(G)$ with

$$|F(G):A| \leqq b(F(G))^4.$$

Theorem B yields

$$|G:A| = |G:F(G)|\,|F(G):A| \leqq b(G)^{13/2} b(F(G))^4$$
$$\leqq b(G)^{13/2} b(G)^4 = b(G)^{21/2}.$$

Remarks. Of course the bounds in Theorem A, B, and C are not the best possible. In fact we know of no solvable group G in which $|G:F(G)| > b(G)^2$. Moreover, if it were true that $|G:F(G)| \leqq b(G)^2$ for all solvable groups G, we would not be able to prove it by sharpening the exponent in Theorem A.

Indeed to obtain $|G:F(G)| \leqq b(G)^2$ by the method we used to prove Theorem B, we would have to show in the situation of Theorem A, that

$$|C_G(v)| \leqq |G|^{1/2} \quad \text{for some } v \in V.$$

However, the natural imprimitive action of the wreath product $GL(2, 2)$ Wr S_4 on an 8-dimensional vector space over $GF(2)$ shows that this stronger version of Theorem A does not hold.

2. Nonsolvable groups. We begin this section by stating the analogs of Theorems B and C for arbitrary finite groups.

THEOREM D. *There exists a constant L such that $|G:F(G)| \leqq b(G)^L$ for every finite group G.*

THEOREM E. *There exists a constant K such that every finite group G contains an abelian subgroup A with $|G:A| \leqq b(G)^K$.*

We will prove the existence of L and K without giving specific values for them. We will use the classification of simple groups only in Lemma 2.4 below.

LEMMA 2.1. *Let G be a primitive permutation group of degree n which does not contain the alternating group A_n. Then $|G| < 4^n$.*

Proof. This is [**10**, Theorem].

LEMMA 2.2. *Let G be a permutation group of degree n. Suppose that no nonabelian simple alternating group is involved in G. Then $|G| < 16^n$.*

Proof. Since G is a subdirect product of transitive groups, we may assume that G is transitive on n points. If G is primitive, the result follows from Lemma 2.1. Otherwise there is an integer m with $1 < m < n$ and a partition of the set permuted by G into m blocks of imprimitivity, such that the stabilizer in G of each block induces a primitive group on that block. Let G_0 be the normal subgroup of G consisting of those elements of G which stabilize all m blocks. Then G_0 is isomorphic to a subgroup of a direct product of m primitive groups which satisfy the hypotheses of Lemma 2.1. Thus

$$|G_0| < (4^{n/m})^m = 4^n.$$

Since G/G_0 is a transitive group of degree m which satisfies the hypotheses of Lemma 2.2, induction yields that $|G/G_0| < 16^m$. Thus

$$|G| < 4^n 16^m \leqq 4^n 16^{n/2} = 16^n.$$

LEMMA 2.3. *Let b_n denote the largest irreducible character degree of the symmetric group S_n. Then for all sufficiently large n, $b_n > |S_n|^{1/3}$.*

Proof. Let $p(n)$ be the number of partitions of n. If the lemma is false, then

$$p(n)\,|S_n|^{2/3} \geqq |S_n|$$

for infinitely many values of n. Then

$$\log p(n) \geqq (\log n!)/3 > n/3$$

for infinitely many values of n. But the asymptotic formula for $\log p(n)$ [**5**, p. 40] shows that this is not the case.

LEMMA 2.4. *There exists a constant $N > 0$ such that*

$$b(S)/|\mathrm{Out}(S)| > |\mathrm{Aut}(S)|^{1/N}$$

for any nonabelian simple group S. Also

$$|\mathrm{Aut}(S)| \leqq |S|^2 \textit{ for any such group } S.$$

Proof. First suppose S is an adjoint group of Lie type, of characteristic p. Let $|S|_p = p^a$. The order formulas [**2**, p. 491] show that

$$|S| < p^{3a} \quad \text{and} \quad |\mathrm{Out}(S)| < p^{2a/3}.$$

Since the Steinberg character of S has degree p^a, it follows that

$$b(S)/|\mathrm{Out}(S)| > p^{a/3} = (p^{4a})^{1/12} > |\mathrm{Aut}(S)|^{1/12}.$$

Next suppose that S is an alternating group A_n. Since A_5 and A_6 are isomorphic to groups of Lie type, we may assume $n \geqq 7$. Then

$$|\mathrm{Out}(S)| = 2 \quad \text{and} \quad b(S) \geqq b(S_n)/2 > |\mathrm{Out}(S)|.$$

It follows from Lemma 2.3 that there exists $N_2 > 0$ such that

$$b(S)/|\mathrm{Out}(S)| > |\mathrm{Aut}(S)|^{1/N_2}$$

for all such groups S.

Finally suppose that S is a sporadic group or the Tits simple group. Then $|\mathrm{Out}(S)| \leqq 2$ ([**3**, Theorem 4.239] and [**4**]) and $b(S) > 2$. Thus there exists $N_3 > 0$ such that

$$b(S)/|\mathrm{Out}(S)| > |\mathrm{Aut}(S)|^{1/N_3}$$

for all such groups S.

We may let N be the maximum of 12, N_2, and N_3. The assertion that

$$|\mathrm{Aut}(S)| \leqq |S|^2$$

is clear.

Notation. We denote by Sol(G) the solvable radical of a group G.

PROPOSITION 2.5. *Let $M \lhd G$ with G/M solvable. Then*

$$|G:\mathrm{Sol}(G)| \leqq |M|^3.$$

Proof. Use induction on $|M|$. If there exists $K \lhd G$ with $1 < K < M$, let $U/K = \mathrm{Sol}(G/K)$. Then

$$|G:U| \leqq |M/K|^3 \quad \text{and} \quad |U:\mathrm{Sol}(U)| \leqq |K|^3$$

by the inductive hypothesis. Thus

$$|G:\mathrm{Sol}(G)| \leqq |G:\mathrm{Sol}(U)| \leqq |M|^3.$$

Now assume M is minimal normal in G. We may assume M is nonabelian, so that M is the direct product of isomorphic simple groups $S_i (1 \leqq i \leqq r)$. Let N be the kernel of the permutation action of G on the S_i. Thus $|G:N| \leqq 16^r$ by Lemma 2.2. Also, $N/C_N(M)$ injects into $\prod \mathrm{Aut}(S_i)$ and so

$$|N:C_N(M)| \leqq |M|^2$$

by Lemma 2.4.

Since $C_N(M) \cap M = 1$, $C_N(M)$ is solvable. Since $16 < |S_1|$, we have

$$|G:\mathrm{Sol}(G)| \leqq |G:C_N(M)| \leqq 16^r|M|^2 \leqq |M|^3.$$

LEMMA 2.6. *Let $M = S_1 \times \ldots \times S_r$, where the S_i constitute a conjugacy class of subgroups of G. Let $\theta \in \mathrm{Irr}(M)$ be G-invariant and let $N = N_G(S_1)$. Write*

$$\theta = \alpha_1 \times \ldots \times \alpha_r \quad \text{and} \quad \gamma = \alpha_1 \times 1 \times \ldots \times 1 \in \mathrm{Irr}(M),$$

where $\alpha_i \in \mathrm{Irr}(S_i)$. Let $\beta \in \mathrm{Irr}(N|\gamma)$. Then $(\beta^{\otimes G})_M$ is a multiple of θ.

Proof. Observe that $\beta_M = e\gamma$ for some integer e. We have for $m \in M$

$$\beta^{\otimes G}(m) = \prod \beta(tmt^{-1}) = e^r \prod \gamma(tmt^{-1}) = e^r\theta(m)$$

where t runs over a right transversal for N in G; see [**8**, Lemma 4.1].

Notation. If $M = S_1 \times \ldots \times S_r$, where the S_i are nonabelian simple groups, write $a(M) = \prod |\mathrm{Out}(S_i)|$.

COROLLARY 2.7. *Let $M \lhd G$ be a direct product of nonabelian simple groups and let $\theta \in \mathrm{Irr}(M)$ be G-invariant. Then there exists $\chi \in \mathrm{Irr}(G|\theta)$ with*

$$\chi(1)/\theta(1) \leqq a(M).$$

Proof. Use induction on $|M|$. If $M = M_1 \times M_2$, with both factors nontrivial normal subgroups of G, write $\theta = \theta_1 \times \theta_2$. Choose $\chi_i \in \mathrm{Irr}(G|\theta_i)$ with

$$\chi_i(1)/\theta_i(1) \leqq a(M_i) \quad \text{for } i = 1, 2.$$

Then let χ be an irreducible constituent of $\chi_1\chi_2$.

We may thus assume M is minimal normal. Write

$$M = S_1 \times \ldots \times S_r$$

with the S_i simple and G-conjugate. Write

$$\theta = \alpha_1 \times \ldots \times \alpha_r \quad \text{and} \quad \gamma = \alpha_1 \times 1 \times \ldots \times 1.$$

Let $N = N_G(S_1)$ and $C = C_G(S_1)$. Observe that γ extends to S_1C and so there exists $\beta \in \mathrm{Irr}(N|\gamma)$ with

$$\beta(1)/\gamma(1) \leqq |N:S_1C| \leqq |\mathrm{Out}(S_1)|.$$

Thus

$$\beta^{\otimes G}(1) \leqq \gamma(1)^r|\mathrm{Out}(S_1)|^r = \theta(1)a(M).$$

Take χ to be any irreducible constituent of $\beta^{\otimes G}$. By Lemma 2.6, $\chi \in \mathrm{Irr}(G|\theta)$.

LEMMA 2.8. *Let $M \lhd G$ and $\theta \in \mathrm{Irr}(M)$. Let $T = I_G(\theta)$, the inertia group. Let $\chi \in \mathrm{Irr}(T|\theta)$. Then*

$$b(G/M) \leqq b(G)\chi(1)/\theta(1)^2.$$

Proof. Observe that $b(G/M) \leqq |G:T|b(T/M)$. Let $\beta \in \mathrm{Irr}(T/M)$ with $\beta(1) = b(T/M)$. Let ψ be any irreducible constituent of $\beta\chi$. Now [**7**, Theorem 12.7] yields

$$\psi(1)\chi(1) \leqq \beta(1)\theta(1)^2.$$

Since $\psi \in \mathrm{Irr}(T|\theta)$, ψ^G is irreducible, so that

$$\psi(1)\,|G:T| \leqq b(G).$$

Now

$$b(G/M) \leqq |G:T|b(T/M) = |G:T|\beta(1) \leqq |G:T|\psi(1)\chi(1)/\theta(1)^2$$
$$\leqq b(G)\chi(1)/\theta(1)^2.$$

PROPOSITION 2.9. *Let $M \lhd G$ be nonabelian and minimal normal. Then*

$$b(G/M) \leqq b(G)/|M|^{1/N},$$

where N is as in Lemma 2.4.

Proof. Choose $\theta \in \mathrm{Irr}(M)$ with $\theta(1) = b(M)$. Let $T = I_G(\theta)$. By Corollary 2.7, choose

$$\chi \in \mathrm{Irr}(T|\theta) \quad \text{with } \chi(1)/\theta(1) \leqq a(M).$$

Lemma 2.8 yields

https://doi.org/10.4153/CJM-1985-026-8 Published online by Cambridge University Press

$$b(G/M) \leqq b(G)\chi(1)/\theta(1)^2 \leqq b(G)a(M)/\theta(1)$$
$$= b(G)a(M)/b(M).$$

By Lemma 2.4, $b(M) \geqq a(M)\,|M|^{1/N}$ and so $b(G/M) \leqq b(G)/|M|^{1/N}$.

THEOREM 2.10. *Let G be arbitrary and N as in Lemma* 2.4. *Then*

$$|G:\mathrm{Sol}(G)| \leqq b(G)^{3N}.$$

Proof. We may assume $\mathrm{Sol}(G) = 1$. Let M be minimal normal in G and let $U/M = \mathrm{Sol}(G/M)$. Then $\mathrm{Sol}(U) = 1$ and $|U| \leqq |M|^3$ by Proposition 2.5. Induction on $|G|$ and Proposition 2.9 yield

$$|G:U| \leqq b(G/M)^{3N} \leqq b(G)^{3N}/|M|^3.$$

Thus $|G| \leqq b(G)^{3N}$.

Proof of Theorem D. Let N be as above. Theorem 2.10 and Theorem B yield

$$|G:F(G)| = |G:\mathrm{Sol}(G)|\,|\mathrm{Sol}(G):F(G)|$$
$$\leqq b(G/\mathrm{Sol}(G))^{3N}b(\mathrm{Sol}(G))^{13/2} \leqq b(G)^{3N}b(G)^{13/2} = b(G)^{3N+13/2}.$$

Proof of Theorem E. Theorem E follows from Theorem D and Lemma 1.5 in the same way that Theorem C followed from Theorem B and Lemma 1.5.

REFERENCES

1. L. Dornhoff, *Group representation theory*, Part A (Marcel Dekker, New York, 1971).
2. D. Gorenstein, *Finite groups* (Harper and Row, New York, 1968).
3. ——— *Finite simple groups* (Plenum Press, New York, 1982).
4. R. L. Griess, Jr. and R. Lyons, *The automorphism group of the Tits simple group*, Proc. A.M.S. *52* (1975), 75-78.
5. M. Hall, Jr., *Combinatorial theory* (Blaisdell, New York, 1967).
6. B. Huppert, *Endliche gruppen I* (Springer Verlag, Berlin, 1967).
7. I. M. Isaacs, *Character theory of finite groups* (Academic Press, New York, 1976).
8. ——— *Character correspondences in solvable groups*, Advances in Math. *43* (1982), 284-306.
9. D. S. Passman, *Groups with normal solvable Hall p′-subgroups*, Trans. A.M.S. *123* (1966), 99-111.
10. C. Praeger and J. Saxl, *On the orders of primitive permutation groups*, Bull. L.M.S. *37* (1980), 303-307.
11. T. R. Wolf, *Solvable and nilpotent subgroups of GL*(n, q^m), Can. J. Math. *34* (1982), 1097-1111.

Wayne State University,
Detroit, Michigan