ON THE CRITICAL POINTS OF A POLYNOMIAL

ABDUL AZIZ AND B.A. ZARGAR

Let \( p \) be a complex polynomial, of the form \( p(z) = z \prod_{k=1}^{n-1} (z - z_k) \), where \( |z_k| \geq 1 \) when \( 1 \leq k \leq n - 1 \). Then \( p'(z) \neq 0 \) if \( |z| < 1/n \).

Let \( B(z, r) \) denote the open ball in \( \mathbb{C} \) with centre \( z \) and radius \( r \), and \( B(z, r) \) denote its closure. The Gauss–Lucas theorem states that every critical point of a complex polynomial \( p \) of degree at least 2 lies in the convex hull of its zeros. This theorem has been further investigated and developed. B. Sendov conjectured that, if all the zeros of \( p \) lie in \( \overline{B}(0,1) \), then, for any zero \( \zeta \) of \( p \), the disc \( \overline{B}(\zeta, 1) \) contains at least one zero of \( p' \); see [3, Problem 4.1]. This conjecture has attracted much attention—see, for example, [1], and the papers cited there. In connection with this conjecture, Brown [2] posed the following problem.

Let \( Q_n \) denote the set of all complex polynomials of the form \( p(z) = z \prod_{k=1}^{n-1} (z - z_k) \), where \( |z_k| \geq 1 \) when \( 1 \leq k \leq n - 1 \). Find the best constant \( C_n \) such that \( p' \) does not vanish in \( B(0, C_n) \), for all \( p \) in \( Q_n \).

Brown observed that, if \( p(z) = z(z - 1)^{n-1} \), then \( p'(1/n) = 0 \), and conjectured that \( C_n = 1/n \). We show this here.

**Theorem** For all \( p \) in \( Q_n \), \( p'(z) \neq 0 \) if \( z \in B(0, 1/n) \).

**Proof:** Clearly \( p'(0) = \prod_{k=1}^{n-1} (-z_k) \neq 0 \). If \( 0 < |z| < 1/n \), then \( |z - z_k| > 1 - 1/n \), and so

\[
\left| \frac{p'(z)}{p(z)} \right| = \left| \frac{1}{z} + \sum_{k=1}^{n-1} \frac{1}{z - z_k} \right| \geq \frac{1}{|z|} - \sum_{k=1}^{n-1} \frac{1}{|z - z_k|} > n - \sum_{k=1}^{n-1} \frac{n}{n - 1} = 0.
\]

It follows that \( p' \) does not vanish in \( B(0, 1/n) \).

Similarly, if \( p(z) = z^m \prod_{k=1}^{n-m} (z - z_k) \), where \( |z_k| \geq 1 \) when \( 1 \leq k \leq n - m \), then \( p'(z) \neq 0 \) if \( 0 < |z| < m/n \).

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Postgraduate Department of Mathematics
University of Kashmir
Hazratbal
Srinigar 190006
India