ON COUNTING ROOTED TRIANGULAR MAPS

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1. Triangular maps. Let R be a simply connected closed region in the Euclidean plane E^2 whose boundary is a simple closed curve C. A *triangular map*, or simply "map," is a representation of R as the union of a finite number of disjoint point sets called cells, where the cells are of three kinds, vertices, edges, and faces (said to be of dimension 0, 1, and 2, respectively), where each vertex is a single point, each edge is an open arc whose ends are distinct vertices, and each face is a simply connected open region whose boundary consists of the closure of the union of three edges. Two cells of different dimension are incident if one is contained in the boundary of the other.

Vertices and edges are *external* if they are contained in the closure of the complement of R. Otherwise they are *internal.*

A *rooted triangular map* is a triangular map in which one *external vertex* is distinguished as the *root vertex* and an *external edge* incident with the root vertex is distinguished as the *root edge.* Two triangular rooted maps T and T^* are *isomorphic* if there exists a bi-unique mapping f of the cells of T onto the cells of T^* which preserves dimension and rooting, and both f and f^{-1} preserve incidence.

Isomorphism is clearly an equivalence relation and, as usual, we enumerate only the number of isomorphism classes of such maps. A map is said to be of type $[n, m]$ if it contains precisely $m + 3$ external vertices and n internal vertices.

We use the symbol $t_{n,m}$ to represent the number of rooted triangular maps of type $[n, m]$ and define the generating function $T(x, y)$ as the formal power series,

$$T(x, y) = \sum_{n=0}^{\infty} \sum_{m=-1}^{\infty} t_{n,m}\, x^n y^{m+1}. \tag{1.1}$$

2. An equation for $T(x, y)$. Every rooted triangular map falls into one of two classes, those of the *first kind* in which the internal triangle incident with the root edge is incident with three external vertices, and those of the *second kind* in which one of the vertices is internal.

Every triangular map of the first kind may be obtained from two rooted triangular maps by identifying the non-root vertex of the root edge of one with the root vertex of the other and joining the remaining ends of the root edges (cf. Fig. 1). The map is rooted by taking the adjoined edge as root edge and

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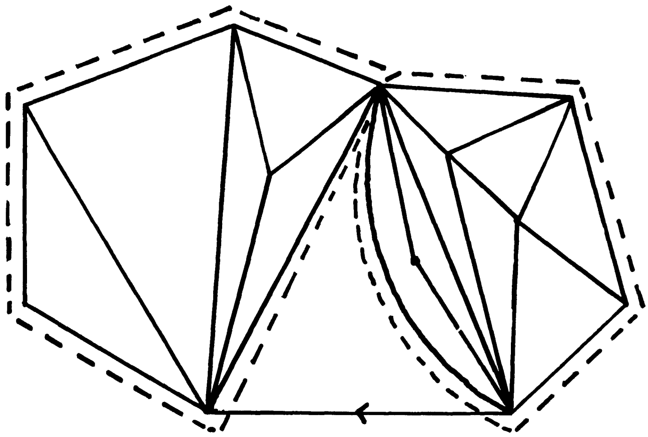

FIGURE 1

choosing the root vertex incident with this as the root vertex of the resultant map. Thus the generating function for maps of the first kind is

$$y(1 + T(x, y))^2.$$

The 1 in the above expression is to take into account the fact that a single edge may be used in the construction as a degenerate map.

Every map of the second kind of type $[n, m]$ may be obtained from a map of type $[n - 1, m + 1]$ by adjunction of an edge, and therefore the adjunction of a new triangle. Indeed, since the ends of every edge are distinct, there is an edge of the boundary of any map which is incident with the non-root vertex of the root edge. We shall call the remaining end of this edge the free vertex. The free vertex is also the root vertex if and only if the triangular map is of type $[n, -1]$. Therefore, by adjoining an edge connecting the root to the free vertex in any graph of type $[n, m]$, $m \geqslant 0$, and taking the new edge as root edge, and retaining the root vertex, we obtain a map of the second kind. Conversely, we obtain a unique map by reversing the process (cf. Fig. 2). Therefore, the enumerator for maps of the second kind is

$$\frac{x}{y}(T(x, y) - L(x)),$$

where $L(x)$ enumerates maps of type $[n, -1]$.

Since every map is of the first or second kind,

$$T(x, y) = y(1 + T(x, y))^2 + \frac{x}{y}(T(x, y) - L(x)).$$

That is

$$y^2T^2(x, y) + (2y^2 - y + x)T(x, y) + y^2 - xL(x) = 0. \tag{2.1}$$

We denote $T(0, y)$ by $T(y)$, and setting $x = 0$ we obtain

$$y^2T^2(y) + (2y^2 - y)T(y) + y^2 = 0.$$

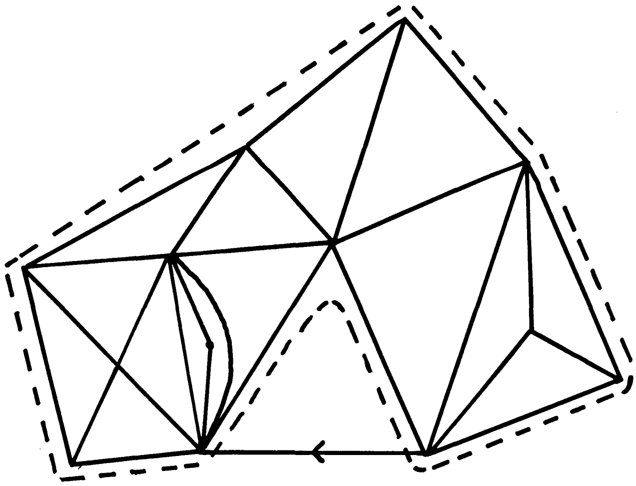

FIGURE 2

This yields

$$(2.1)\qquad T(y) = \frac{1 - 2y - \sqrt{1-4y}}{2y} = \sum_{m=0}^{\infty} \frac{(2m+2)!}{(m+1)!\,(m+2)!} y^{m+1},$$

the negative sign before the radical being chosen to refer to the series with constant term -1. Therefore the number of maps of type $[0, m]$ is

$$\frac{(2m+2)!}{(m+1)!\,(m+2)!},$$

a result which has been obtained in many combinatorial investigations. A direct interpretation of our results shows that the number of ways of dividing a rooted convex polygon into triangles by non-intersecting diagonals is

$$\frac{1}{m+1}\binom{2m}{m},$$

where m is the number of internal triangular faces.

Setting $y = 0$, we obtain the expected result that $T(x, 0) = L(x)$.

Equation (2.1) uniquely determines the function $T(x, y)$, since it can be shown that the equation

$$(2.2)\qquad y^2 T^{2*}(x, y) + (2y^2 - y + x)T^*(x, y) + y^2 - xL^*(x) = 0$$

has a power series solution (that is, a series in positive powers only) for only one choice of function $L^*(x)$. This solution, of course, will be $T(x, y)$.

By the quadratic formula,

$$(2.3)\ T^*(x, y) = \frac{-2y^2 + y - x \pm \sqrt{x^2 - 2xy + [1 + 4x + 4xL^*(x)]y^2 - 4y^3}}{2y^2}$$

The ambiguity above will be removed later in accordance with the power series condition on $T^*(x, y)$. Let us examine the discriminant D of equation (2.2). If we write $1 + 4x + 4L^*(x) = A(x)$,

$$D = x^2 - 2xy + A(x)y^2 - 4y^3.$$

It has been shown by Brown **(4)** that D must have a repeated factor if $T(x, y)$ is to be a power series. Experimentation suggests that we define a parameter u by the relation $x = uw^{-3}$, where w is a function of u which is yet to be determined. Consider the expression

$$\left(y - \frac{u}{w}\right)^2 \left(\frac{1}{w^2} - 4y\right) = \frac{u^2}{w^6} - \frac{2u(1 + 2u)}{w^4} y + \frac{1 + 8u}{w^2} y^2 - 4y^3.$$

This is identical with D if

$$w = 1 + 2u, \text{ and } A(x) = \frac{1 + 8u}{w^2}.$$

We note that the function w defined by

$$x = uw^{-3}, \qquad w = 1 + 2u,$$

may be expanded as a power series in x by Lagrange's theorem; hence we can find power series expansions for $A(x)$, $1/w$ and w^2 as well. Writing (2.3) in terms of u and w,

$$(2.4) \qquad T^*(x, y) = \left\{-2y^2 + y - x - \frac{1}{w}\left(y - \frac{u}{w^2}\right) \sqrt{1 - 4yw^2}\right\} \Big/ 2y^2,$$

which clearly may be expanded as a power series in x and y. This must be the generating function $T(x, y)$.

Expanding (2.4), we obtain

$$(2.5) \qquad T(x, y) = -1 + \sum_{m=-1}^{\infty} \frac{(2m + 2)!}{(m + 1)!\,(m + 2)!} y^{m+1} w^{2m+3} - u \sum_{m=-1}^{\infty} \frac{(2m + 4)!}{(m + 3)!\,(m + 2)!} y^{m+1} w^{2m+3}.$$

Applying the Lagrange expansion to $u = xw^3$, we obtain

$$w^k = \sum_{n=0}^{\infty} \frac{2^n k(3n + k - 1)!}{n!\,(2m + k)!} x^n.$$

Therefore

$$T(x, y) = -1 + \sum_{m=-1}^{\infty} y^{m+1} \left\{ \frac{(2m + 2)!}{(m + 1)!\,(m + 2)!} \sum_{n=0}^{\infty} \frac{2^n (3n + 2m + 2)!\,(2m + 3)}{n!\,(2n + 2m + 3)!} x^n - \frac{(2m + 4)}{(m + 2)!\,(m + 3)!} \sum_{n=0}^{\infty} \frac{2^n (3n + 2m + 5)!\,(2m + 6)}{n!\,(2n + 2m + 6)!} x^{n+1} \right\}.$$

After simplification, for $n \geqslant 0$ and $m \geqslant -1$,

$$t_{n,m} = \frac{2^{n+1}(2m+3)!\,(3n+2m+2)!}{(m+1)!^2\,n!\,(2n+2m+4)!}$$

with the exception that $t_{0,-1} = 0$. For fixed m, we observe by Stirling's formula that as $n \to \infty$

$$t_{n,m} \sim \frac{e^2 2^{n+1}(2m+3)!\,(3n+2m+2)^{3n+2m+2}}{(m+1)!^2\,n^n(2n+2m+4)^{2n+2m+4}} \sqrt{\frac{3n+2m+2}{2n(2n+2m+4)\pi}}$$
$$\sim \frac{1}{4}\frac{(2m+3)!}{(m+1)!^2}\left(\frac{27}{2}\right)^n\left(\frac{9}{4}\right)^{m+1} n^{-5/2}\sqrt{\frac{3}{\pi}}.$$

3. Simple triangular maps. As shown in Section 2 all rooted triangular maps with $\alpha + 1$ internal faces can be obtained from those of α faces by one of two simple operations. Further, there is an obvious connection between triangulations and triangular maps, namely all of the former may be obtained from the latter by contracting triangulated digons to single edges. With respect to the four-colour problem, the statements

(1) every triangulation is vertex four-colourable,

(2) every triangular map is vertex four-colourable,

are clearly equivalent.

A map is *simple triangular* if it is the degenerate map of type $[0, -1]$ or if it cannot be obtained from a map (other than that of type $[0, 0]$) of fewer regions by replacing edges by maps of type $[n, -1]$, then replacing triangular faces by simple border maps of type $[n, 0]$. It can be shown that every map may be obtained from a simple map by applying either or both of the operations mentioned above. Moreover, no map may be thus generated from two distinct simple maps. Simple triangular maps could justifiably be called *simple triangulations* since no pair of edges have the same ends. They are triangulations in the sense of Brown **(3)** since an interior edge may have exterior vertices for both its ends, a possibility not admitted by Tutte **(6)**. The statement

(3) every simple triangular map is vertex four-colourable,

can be shown to be equivalent to (1) or (2) above.

A map is called a simple border map if no pair of external vertices is joined by more than one edge. This clearly rules out the possibility of simple border maps of type $[n, -1]$. If $s_{n,m}$ represents the number of simple border maps of type $[n, m]$ we define

$$S(x, y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} s_{n,m}\,x^n y^m. \tag{3.1}$$

Every map of type $[n, m]$ $(m \geqslant 0)$ can be obtained uniquely from a simple border map by replacing some of the external edges by *lunes*, that is, maps of type $[n, -1]$. Hence we have

$$T_1(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} s_{n,m}\, x^n y^m [1 + L(x)]^{m+3},$$

where

$$T_1(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_{n,m}\, x^n y^m, \qquad \text{cf (1.1)},$$

and $L(x)$ enumerates lunes. If

$$M(x) = 1 + L(x) \quad \text{and} \quad Y = y[M(x)],$$
$$S(x, Y) = [M(x)]^{-3} T[x, Y/M(x)].$$

We set $x = uw^{-3}$, where $w = 1 + 2u$, and $v = 1 - u$. By (2.5)

(3.2) $$1 + L(x) = vw;$$

therefore

$$S(x, y) = \sum_{m=0}^{\infty} \frac{(2m+2)!}{(m+1)!\,(m+2)!} \frac{w^m}{v^{m+3}} y^m - u \sum_{m=0}^{\infty} \frac{(2m+4)!}{(m+2)!\,(m+3)!} \frac{w^m}{v^{m+3}} y^m.$$

In particular, if

$$S(x) = \sum_{n=0}^{\infty} s_{n,0}\, x^n,$$

$S(x)$ is represented parametrically by

(3.3) $$S(x) = (1 - 2u)v^{-3}, \qquad x = uw^{-3}.$$

We define the generating function $K(x, y)$ by

(3.4) $$K(x, y) = \sum_{n=0}^{\infty} \sum_{m=-1}^{\infty} k_{n,m}\, x^n y^{m+1},$$

where $k_{n,m}$ is the number of rooted simple triangular maps of type $[n, m]$ for $[n, m] \neq [0, 0]$. We define $k_{0,0} = 0$. Every map of type $[m, n]$ contains $3n + 2m + 3$ edges and $2n + m + 1$ triangular faces, and every map can be obtained from a unique triangular map by subdividing triangular faces and replacing edges by lunes; hence we have the relation

$$\sum_{n=0}^{\infty} \sum_{m=-1}^{\infty} k_{n,m}\, x^n y^{m+1} M(x)^{3n+2m+3} S(x)^{2n+m+1} = T(x, y) + 1 - y[M(x)]^3.$$

In the above expression $L(x)$, $S(x)$, and $T(x, y)$ enumerate lunes, simple border maps of type $[n, 0]$, and rooted triangular maps respectively. The term $y[M(x)]^3$ is subtracted to account for the fact that since $k_{0,0}$ is defined to be zero, we cannot include maps which are derived from the map of type $[0, 0]$ by substitution of link maps for edges. The term 1 is present to enumerate the degenerate map of type $[0, -1]$. This relation may be written

$$[M(x)]K\{x[M(x)]^3 S^2(x),\, yM^2(x)S(x)\} = T(x, y) + 1 - y[M(x)]^3.$$

We define

$$X = x[M(x)]^3 S^2(x), \qquad Y = y[M(x)]^2 S(x),$$

and hence

$$K(X, Y) = [M(x)]^{-1}\{T(x, Y/S(x)[M(x)]^2) + 1\} - Y/S(x).$$

We define u, v, w, and t by the relations

$$x = uw^{-3}, \qquad w = 1 + 2u, \qquad v = 1 - u, \qquad t = 1 - 2u.$$

Let $\delta = vt^{-1} - 1$; then by (3.2), (3.3), and (2.5), $\delta = X(1+\delta)^3$,

(3.5) $K(X, Y)$

$$= -Y\frac{(1+\delta)}{(1+2\delta)^2} + \sum_{m=-1}^{\infty} \frac{(2m+2)!}{(m+1)!\,(m+2)!}(1+\delta)^m(1+2\delta)Y^{m+1}$$

$$-\delta\sum_{m=-1}^{\infty}\frac{(2m+4)!}{(m+2)!\,(m+3)!}(1+\delta)^m Y^{m+1}.$$

Consider the special case of $m = 0$. We employ the generating function

$$h(x) = \sum_{n=0}^{\infty} h_n x^n,$$

where h_n is the number of simple triangular maps of type $[n, 0]$. By (3.5)

$$h(X) - 1 = 1 - \frac{(1+\delta)^3}{(1+2\delta)^2}.$$

By Lagrange's theorem

$$h(X) = 1 + \sum_{n=1}^{\infty}\frac{X^n}{n!}D^{n-1}\left[(1+\delta)^{3n+2}\frac{(1-2\delta)}{(1+2\delta)^3}\right]_{\delta=0},$$

from which

$$h_n = \frac{1}{n}\sum_{r=0}^{n-1}\binom{3n+2}{n-1-r}(-2)^r(r+1)^2,$$

a simplified version of a result due to Tutte (**6**, p. 33). It is shown in (**6**, p. 37) that as $n \to \infty$

$$h_n \sim \frac{1}{128}\sqrt{\frac{3}{\pi}}\,n^{-5/2}\left(\frac{27}{4}\right)^{n+1}.$$

For $m > 0$, we apply Lagrange's expansion to (3.5) to obtain

$$k_{n,m} = \frac{(2m+2)!\,(3n+m-1)!}{n!\,(m-1)!\,(m+2)!\,(2n+m+1)!}. \tag{3.6}$$

As $n \to \infty$ for fixed m,

$$k_{n,m} \sim \frac{(2m+2)!}{(m-1)!\,(m+2)!}\frac{1}{6}\left(\frac{9}{4}\right)^n\left(\frac{3}{2}\right)^m n^{-5/2}\sqrt{\frac{3}{2\pi}}. \tag{3.7}$$

4. A coincidence? It was pointed out by Tutte that by comparing the formula for the number of rooted simple triangulations of type $[n, 1]$ in the preceding section with the formula for the number of rooted non-separable maps obtained by Tutte (**7**, p. 257) one might observe that the number of rooted simple triangulations of a quadrilateral with n internal vertices is equal to the number of rooted non-separable maps with $n + 1$ edges. This author knows of no direct correspondence which can be established between the two classes of maps. It has been observed also by Tutte and Brown that the number of rooted quadrangulations (**5**) of a quadrilateral with n internal vertices is equal to the number of rooted non-separable maps with $n + 2$ edges, with the exception that there are two non-separable maps with one edge and no quadrangulations with -1 internal vertices. A direct correspondence between the latter pair of systems is shown in (**5**). Perhaps it might be more natural to attempt to establish a direct relation between simple triangulations and quadrangulations of a quadrilateral.

5. Polygons in triangular maps on the sphere. One may consider triangular maps *of type* $[n, 0]$ as being drawn on the surface of a sphere. In this event it is convenient to consider the map as rooted by distinguishing three mutually incident elements of dimension 0, 1, and 2 as root vertex, edge, and face respectively. We shall call such an entity a *rooted global map* of $n + 3$ vertices and shall determine the average number of polygons of $k > 3$ edges passing through the root edge. Such a polygon divides the sphere into two regions, one of which is distinguished by the fact that it contains the root face. If we consider the boundary as being contained in both regions, we may consider each as the rooted triangular map by retaining the root edge and vertex as distinguished elements. Therefore the number of rooted global maps of n vertices which may be derived from a polygon with $k \geqslant 3$ edges is the coefficient of x^{n-k} in the expansion of $T_k^2(x)$, where

$$T_k(x) = \sum_{n=0}^{\infty} t_{n,k-3}\, x^n$$

where $t_{n,m}$ is defined as in Section 1. By equation (2.5),

$$T_k(x) = \frac{(2k-4)!}{k!\,(k-2)!} w^{2k-3}[k - 2(2k-3)u],$$

where $x = uw^{-3}$ and $w = 1 + 2u$. Therefore

$$(5.1) \qquad T_k^{\,2}(x) = \frac{(2k-4)!^2}{k!^2\,(k-2)!^2}\left\{k^2 \sum_{n=0}^{\infty} \frac{(4k-6)2^n(3n+4k-7)!}{n!\,(2n+4k-6)!} x^n \right.$$

$$- 4k(2k-3) \sum_{n=0}^{\infty} \frac{(4k-3)2^n(3n+4k-4)!}{n!\,(2n+4k-3)!} x^{n+1}$$

$$\left. + 4(2k-3)^2 \sum_{n=0}^{\infty} \frac{4k2^n(3n+4k-1)!}{n!\,(2n+4k)!} x^{n+2} \right\}.$$

Hence the coefficient of x^n in (5.1) is

$$\frac{(2k-3)!\,(2k-4)!\,2^{n+2}(3n+4k-7)!}{k!\,(k-1)!\,(k-2)!^2\,(n-2)!\,(2n+4k-6)!} \times \frac{[8k^3-18k^2+10k+3nk-3n]}{n(n-1)(2n+4k-5)(2n+4k-4)}$$

and therefore the number of triangular maps derivable from the preceding construction is

$$\frac{(2k-3)!\,(2k-4)!\,2^{n-k+2}(3n+k-7)!\,F(n,k)}{k!\,(k-2)!^3\,(n-k)!\,(2n+2k-4)!},$$

where $F(n, k) = 8k^2 - 13k + 3n$. The number of rooted global maps of n vertices is, of course, $t_{n-3,0}$ (where $n \geqslant 3$) or

$$\frac{3.2^{n-1}(3n-7)!}{(n-3)!\,(2n-2)!}.$$

Hence, the average number of polygons of k edges passing through the root of a global map of n vertices is

$$\frac{(n-3)!\,(2n-2)!\,(2k-3)!\,(2k-4)!\,(3n+k-7)!\,F(n,k)}{3.2^{k-3}k!\,(3n-7)!\,(k-2)!^3\,(n-k)!\,(2n+2k-4)!}. \tag{5.2}$$

For fixed k, as $n \to \infty$ (5.2) is asymptotically

$$\frac{4}{n}\left(\frac{3}{8}\right)^{k-1}\frac{(2k-3)!\,(2k-4)!\,(8k^2-13k+3n)}{k!\,(k-2)!^3},$$

which is asymptotically

$$32\left(\frac{3}{8}\right)^{k}\frac{(2k-3)!\,(2k-4)!}{k!\,(k-2)!^3},$$

an expression independent of n, which is not too surprising.

To find the average number of digons through the root of a global map we note that $xT_3(x) = T_2(x)$; hence the average number of digons through the root of a global map $n + 2$ vertices is the same as the number of triangles through the root of a global map of n vertices, since to each pair of digons in the construction of this section there is a corresponding pair of triangles with a total of two fewer vertices, and the correspondence is one to one.

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