BULL. AUSTRAL. MATH. SOC. 17B75, 17B35
VOL. 71 (2005) [327-335]

GENERIC LIE COLOUR ALGEBRAS

KENNETH L. PRICE

We describe a type of Lie colour algebra, which we call generic, whose universal enveloping algebra is a domain with finite global dimension. Moreover, it is an iterated Ore extension. We provide an application and show Gröbner basis methods can be used to study universal enveloping algebras of factors of generic Lie colour algebras.

1. INTRODUCTION

Throughout k denotes a field of characteristic different from 2. Iterated Ore extensions play an important role in this work. All of the Ore extensions we introduce are of the form $A[\theta;\sigma,\delta]$ where A is a k-algebra, σ is an automorphism, and δ is a σ-derivation. In this case if A is Noetherian, a domain, or has finite right global dimension, then $A[\theta;\sigma,\delta]$ has that corresponding property as well. These facts are well known (see [**11**, Theorem I.2.9 and Theorem VII.5.3]) and will be invoked as needed without further comment.

We follow the notation on Lie colour algebras in [**13**, Sections 1, 2, and 6]. Let $k^\times = k\backslash\{0\}$ denote the group of units of k.

DEFINITION 1.1: Let G be an Abelian group. A map $\varepsilon : G\times G \to k^\times$ is called a *skew-symmetric bicharacter* on G if it satisfies (1) and (2) below, for any $f,g,h\in G$.

1. $\varepsilon(f,g+h)=\varepsilon(f,g)\varepsilon(f,h)$ and $\varepsilon(g+h,f)=\varepsilon(g,f)\varepsilon(h,f)$
2. $\varepsilon(g,h)\varepsilon(h,g)=1$

All gradings are with respect to an additively written Abelian group. We denote the degree of a nonzero homogeneous element x by ∂x. We want to avoid statements like "$\partial x = g$ or $x = 0$" so we often write $\partial x = g$ to handle both cases. If x and y are homogeneous and ε is a skew-symmetric bicharacter, then we shorten our notation by writing $\varepsilon(x,y)$ instead of $\varepsilon(\partial x,\partial y)$.

DEFINITION 1.2: A (G,ε)-*Lie colour algebra* is a G-graded vector space $\mathcal{L}$ equipped with a graded bilinear map $\langle,\rangle : \mathcal{L}\times\mathcal{L}\to\mathcal{L}$, called the *bracket* of $\mathcal{L}$, which satisfies the following for any homogeneous $x,y,z\in\mathcal{L}$.

$$\langle x,y\rangle = -\varepsilon(x,y)\langle y,x\rangle \qquad \varepsilon\text{-skew-symmetry}$$
$$\varepsilon(z,x)\langle x,\langle y,z\rangle\rangle+\varepsilon(y,z)\langle z,\langle x,y\rangle\rangle+\varepsilon(x,y)\langle y,\langle z,x\rangle\rangle=0 \qquad \varepsilon\text{-Jacobi identity}$$

Received 17th November, 2004

A skew-symmetric bicharacter ε satisfies $\varepsilon(g,g) = \pm 1$ for each $g \in G$. Note that

$$G_+ = \{g \in G : \varepsilon(g,g) = 1\}$$

is a subgroup such that $[G : G_+] \leqslant 2$. Set

$$G_- = \{g \in G : \varepsilon(g,g) = -1\} = G \backslash G_+.$$

For any G-graded vector space V we set $V_\pm = \bigoplus_{g \in G_\pm} V_g$.

We may view Lie superalgebras and graded Lie algebras, which are graded over $\mathbb{Z}_2$ and $\mathbb{Z}$, respectively, as special types of Lie colour algebras. The 'absolutely torsion free' condition was introduced by R. Bøgvad to find graded Lie algebras with finite global dimension (see [**5**, Theorem 1]). It is well-known that the universal enveloping algebra of a finite dimensional Lie superalgebra may have infinite global dimension (see [**3**, Proposition 5]). M. Aubry and J.-M. Lemaire have shown that the universal enveloping algebra of an absolutely torsion free graded Lie algebra (or Lie superalgebra) is a domain (see [**1**]).

Now consider a finite dimensional Lie colour algebra $(\mathcal{L}, \langle , \rangle)$. Its universal enveloping algebra $U(\mathcal{L})$ need not even be semiprime (see [**13**, Example 2.9]). Conditions for $U(\mathcal{L})$ to be semiprime or prime are provided in [**13**, Theorem 2.5], which extends the analogous result, [**4**, Theorem 1.5], for Lie superalgebras. When $U(\mathcal{L})$ has finite global dimension, then it must equal $\dim \mathcal{L}_+$ by [**12**, Theorem 3.1], which extends the analogous result, [**9**, Proposition 2.3], for Lie superalgebras.

We do not know how to extend Aubry and Lemaire's or Bøgvad's theorems to Lie colour algebras (see [**1**] and [**5**]). However we believe it may be achieved through the use of generic Lie colour algebras, which are defined in Section 2.

We handle the case with $\dim \mathcal{L}_- \leqslant 2$ in Section 3. In this case Theorem 3.5 provides conditions for $U(\mathcal{L})$ to be a domain with finite global dimension. Example 3.2 shows this is possible even when there exists $x \in \mathcal{L}_-$ such that $\langle x, x \rangle = 0$. Results of Behr, Bøgvad, and Aubry and Lemaire (discussed above) imply that this is not possible for Lie superalgebras or graded Lie algebras. Our proof uses generic Lie colour algebras.

In Section 4 we explain how to find the Gröbner basis of an ideal generated by positive elements of a generic Lie colour algebra. We refer the reader to [**7**] for background on Gröbner bases, to [**8**] for a Gröbner basis test to determine if an ideal of an iterated Ore extension is completely prime, and to [**6**] for a Gröbner basis method to calculate projective dimension.

2. Generic Lie colour Algebras

For a Lie colour algebra $(\mathcal{L}, \langle , \rangle)$ and a linear subspace V of $\mathcal{L}_-$ we let $\langle V, V \rangle$ denote the linear subspace of $\mathcal{L}_+$ generated by brackets between elements of V. If $n = \dim V$ then it is easy to see that $\dim \langle V, V \rangle \leqslant n(n+1)/2$.

https://doi.org/10.1017/S0004972700038284 Published online by Cambridge University Press

DEFINITION 2.1: A Lie colour algebra $(\mathcal{X}, \langle , \rangle)$ is called *generic* if $\dim \mathcal{X}_+ = m(m+1)/2$, where $m = \dim \mathcal{X}_- < \infty$, and $\mathcal{X}_+ = \langle \mathcal{X}_-, \mathcal{X}_- \rangle$ is colour central, that is, $\langle x, y \rangle = 0$ for all $x \in \mathcal{X}_+$ and $y \in \mathcal{X}$.

REMARK 2.2. Let V be a graded subspace of $\mathcal{X}_-$. We set $\mathcal{L}_- = V$ and $\mathcal{L}_+ = \langle V, V \rangle$. Then it is easy to show the sub Lie colour algebra $\mathcal{L}$ of $\mathcal{X}$ is also a generic Lie colour algebra.

LEMMA 2.3. *Let $(\mathcal{L}, \langle , \rangle)$ be a finite dimensional Lie colour algebra such that $\mathcal{L}_+ = \langle \mathcal{L}_-, \mathcal{L}_- \rangle$ is colour central. Then there is a generic Lie colour algebra $\mathcal{X}$ with $\dim \mathcal{X}_- = \dim \mathcal{L}_-$ and a surjective Lie colour algebra homomorphism $\psi : \mathcal{X} \to \mathcal{L}$.*

The proof of Lemma 2.3 is straightforward.

Given a G-graded algebra A we write $A^{\varepsilon}[t_1, t_2, \dots, t_n]$ to denote the colour polynomial algebra over A in n homogeneous variables. It is G-graded and isomorphic to an iterated Ore extension of A (see [**2**] for details).

THEOREM 2.4. *Suppose $\mathcal{X}$ is generic and $x_1, \dots, x_m$ form a homogeneous basis of $\mathcal{X}_-$. Set $g_i = \partial x_i$ and $t_{i,j} = \langle x_i, x_j \rangle / 2$ for $1 \leqslant i < j \leqslant n$. Then $U(\mathcal{X})$ is isomorphic to the iterated Ore extension*

$$(1) \qquad U(\mathcal{X}) \cong S[x_1; \alpha_1][x_2; \alpha_2, \delta_2] \dots [x_m; \alpha_m, \delta_m]$$

with $S = k^{\varepsilon}[t_{i,j} : 1 \leqslant i < j \leqslant m]$ such that for all $a \leqslant m$ and $1 \leqslant b < c \leqslant m$

1. $\alpha_a(t_{b,c}) = \varepsilon(g_a, g_b + g_c) t_{b,c}$,
2. $\delta_a(t_{b,c}) = 0$,
3. $\alpha_c(x_b) = \varepsilon(g_c, g_b) x_b$,
4. *and* $\delta_c(x_b) = -2\varepsilon(g_c, g_b) t_{b,c} = \langle x_c, x_b \rangle$.

Theorem 2.4 can be proved in the same way as [**10**, Theorem II.3.1]. In fact the definition of generic Lie colour algebra was motivated by the treatment of generic Clifford algebras in [**10**, Chapter 2].

3. AN APPLICATION

We assume k is algebraically closed throughout this section.

LEMMA 3.1. *Let $(\mathcal{L}, \langle , \rangle)$ be a Lie colour algebra such that $\mathcal{L}_+$ is colour central and $\dim \mathcal{L}_- = \dim \mathcal{L}_+ = 2$. Suppose there does not exist homogeneous $x \in \mathcal{L}_-$ such that $\langle x, x \rangle = 0$. Then $U(\mathcal{L}) \cong k[\theta_1][\theta_2; \sigma_q]$ where q is a nonzero scalar, $\sigma_q(\theta_1) = q\theta_1$, and either θ_1, θ_2 are homogeneous or $\theta_1 + \theta_2, \theta_1 - \theta_2$ are homogeneous and $q = -1$.*

PROOF: In view of Lemma 2.3 we may pass to the case that $\mathcal{L} \cong \mathcal{X}/K$ where $K = \ker \psi$ is a homogeneous linear subspace of $\mathcal{X}_+$ with $\dim K = 1$.

STEP 1. Choose nonzero homogeneous $t \in K$. There is a homogeneous basis $\{x_1, x_2\}$ of $\mathcal{X}_-$ such that either

$$t = \lambda_1\langle x_1, x_1\rangle + \lambda_2\langle x_2, x_2\rangle \quad \text{or} \quad t = \langle x_1, x_2\rangle.$$

Set $g_1 = \partial x_1$ and $g_2 = \partial x_2$.

Let y_1, y_2 form a homogeneous basis of of $\mathcal{X}_-$. Then we may write t as in equation 2 for some $\mu_1, \mu_2, \mu_3 \in k$.

$$t = \mu_1\langle y_1, y_1\rangle + \mu_2\langle y_2, y_2\rangle + \mu_3\langle y_1, y_2\rangle \tag{2}$$

If $\mu_3 = 0$ or $\mu_1 = \mu_2 = 0$ then step 1 follows immediately. We pass to the case $\mu_2 \neq 0$ and $\mu_3 \neq 0$ by relabeling y_1 and y_2 if necessary. This implies

$$\partial t = 2\partial y_2 = \partial y_1 + \partial y_2,$$

which yields $\partial y_1 = \partial y_2$. Set

$$\lambda_1 = \mu_1 - (2^{-1}\mu_3)^2(\mu_2)^{-1}, \quad x_1 = y_1, \quad \lambda_2 = \mu_2,$$

and

$$x_2 = y_2 + \mu_3(2\mu_2)^{-1}y_1.$$

A straightforward calculation shows

$$t = \lambda_1\langle x_1, x_1\rangle + \lambda_2\langle x_2, x_2\rangle.$$

STEP 2. Set $T = U(\mathcal{X})$. Then $T \cong k[t_{12}][x_1; \sigma_1][x_2; \sigma_2, \delta_2]$, defined as in Theorem 2.4, and $U(\mathcal{L}) \cong T/(t)$.

The last statement follows from universal properties of enveloping algebras and Ore extensions.

STEP 3. If $t = \langle x_1, x_2\rangle$ the lemma holds with $q = \varepsilon(g_1, g_2)$. In this case θ_1 and θ_2 are homogeneous.

This is easy since $t = 2t_{12}$ so $U(\mathcal{L}) \cong T/(t_{12})$.

STEP 4. Suppose $t = \lambda_1\langle x_1, x_1\rangle + \lambda_2\langle x_2, x_2\rangle$. The lemma holds with $q = -1$. If $\partial x_1 = \partial x_2$ then θ_1 and θ_2 are homogeneous. Otherwise $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are homogeneous.

If $\lambda_i = 0$ for $i = 1$ or $i = 2$ then $\langle x, x\rangle = 0$ with homogeneous $x = \psi(x_i) \in \mathcal{L}_-$. We assumed this could not happen. Thus $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ which implies $\partial t = 2g_1 = 2g_2$ and $\varepsilon(g_1, g_2)^2 = 1$. By replacing x_1 and x_2 by appropriate scalar multiples, if necessary, we may assume $\lambda_1 = 1$ and $\lambda_2 = -1$.

Set $S = k[u, z_1][z_2; \sigma, \delta]$, where $\sigma(z_1) = -z_1$, $\sigma(u) = u$, $\delta(z_1) = 2u$ and $\delta(u) = 0$. There is an isomorphism $\phi : T \to S$ determined by $\phi(x_1) = z_1 + z_2$, $\phi(x_2) = z_1 - z_2$, and $\phi(t_{12})$ is given by equation 3.

$$\phi(t_{12}) = \frac{1}{2}\big(1 - \varepsilon(g_1, g_2)\big)\big((z_1)^2 - (z_2)^2\big) + \big(1 + \varepsilon(g_1, g_2)\big)(u - z_1z_2) \tag{3}$$

https://doi.org/10.1017/S0004972700038284 Published online by Cambridge University Press

Then $\phi(t) = u$ so $U(\mathcal{L}) \cong S/(u)$. ∎

EXAMPLE 3.2. Suppose ε is a skew-symmetric bicharacter on G and there exist $g_1, g_2 \in G_-$ such that $2g_1 = 2g_2$, and $\varepsilon(g_1, g_2) = 1$ (bicharacters with this property are described in [**13**, Lemma 2.7]). Let $\mathcal{L}$ be the Lie colour algebra with homogeneous basis $\{u_1, u_2, x_1, x_2\}$ such that $\partial u_1 = 2g_1 = 2g_2$, $\partial u_2 = g_1 + g_2$, $\partial x_1 = g_1$, $\partial x_2 = g_2$, and the brackets between basis elements are all zero except for the ones listed below.

$$\langle x_1, x_1\rangle = 2u_1 \quad \langle x_2, x_2\rangle = 2u_1 \quad \langle x_1, x_2\rangle = u_2 \quad \langle x_2, x_1\rangle = -u_2$$

Choose $\zeta \in k$ which satisfies $\zeta^2 = -1$, then $x = x_1 + \zeta x_2$ satisfies $\langle x, x\rangle = 0$. However $U(\mathcal{L}) \cong k[\theta_1][\theta_2; \sigma_{-1}]$ by the proof of Lemma 3.1.

REMARK 3.3. The product of linearly independent elements $x_1 - x_2, x_1 + x_2 \in \mathcal{L}_-$ is $u_2 \in \mathcal{L}_+$. At first glance this may appear to violate the PBW theorem (see [**2**, Theorem 3.2.2]). It does not since the elements $x_1 - x_2$ and $x_1 + x_2$ are not homogeneous.

REMARK 3.4. In example 3.2, $\mathcal{L} \cong L^\gamma$ for some torsion free Lie superalgebra L with appropriate G-grading and γ a two-cocycle on G (see [**13**, Corollary 6.1] for notation). There is an algebra isomorphism $U(\mathcal{L}) \cong U(L)^\gamma \cong U(L)$.

THEOREM 3.5. *Let $(\mathcal{L}, \langle,\rangle)$ be a finite dimensional Lie colour algebra. Suppose* $\dim \mathcal{L}_- \leqslant 2$ *and* $\dim V \leqslant \dim\langle V, V\rangle$ *for each graded subspace V of $\mathcal{L}_-$. Then $U(\mathcal{L})$ is a domain with global dimension equal to* $\dim \mathcal{L}_+$.

PROOF: We may reduce to the case $\mathcal{L}_+$ is colour central by [**13**, Lemma 6.2] and [**11**, Corollary 6.18]. Let $\mathcal{L}'$ be the smallest sub Lie colour algebra which contains $\mathcal{L}_-$. Then $\mathcal{L}'_+ = \langle \mathcal{L}_-, \mathcal{L}_-\rangle$, $\mathcal{L}'_- = \mathcal{L}_-$, and $U(\mathcal{L})$ is a colour polynomial algebra over $U(\mathcal{L}')$. Thus we may pass to the case $\mathcal{L} = \mathcal{L}'$. If $\dim \mathcal{L}'_- = \dim \mathcal{L}'_+ = 2$ then we may apply Lemma 3.1. Otherwise apply Lemma 2.4. ∎

4. GRÖBNER BASIS METHODS

Throughout this section $\mathcal{X}$ denotes a generic Lie colour algebra and $x_1, x_2, \dots, x_m$ form a homogeneous basis of $\mathcal{X}_-$. We define a chain of generic sub Lie colour algebras which are used to express $U(\mathcal{X})$ as an iterated Ore extension. Then we explain how to find the Gröbner basis of an ideal generated by homogeneous elements of $\mathcal{X}_+$.

Set $l = (m-1)/2m$ and $p = l + m$. We specify a homogeneous subset of $\mathcal{X}$ and define a function $\phi : \{1, 2, \dots, p\} \to \{1, 2\}$. Fix the following notation for $1 \leqslant i \leqslant j \leqslant m$.

1. $s(i,j) = j(j-1)/2 + i$
2. $t_{s(j,j)} = x_j$
3. $\phi(s(j,j)) = 2$
4. If $i < j$ then $t_{s(i,j)} = \langle x_i, x_j\rangle$ and $\phi(s(i,j)) = 1$.

For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \mathbb{N}^p$ we write $\mathbf{t}^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_p^{\alpha_p} \in U(\mathcal{X})$. Set $e_i = \big(0, \dots, \underset{(i)}{1}, \dots, 0\big) \in \mathbb{N}^p$ for $1 \leqslant i \leqslant p$. Then a homogeneous basis for $\mathcal{X}_+$ is

$$\big\{(t_i)^{\phi(i)} : 1 \leqslant i \leqslant p\big\} = \big\{\mathbf{t}^{\phi(i)e_i} : 1 \leqslant i \leqslant p\big\}.$$

Proceeding as in Remark 2.2, we let $\mathcal{X}_i$ be the generic sub Lie colour algebra generated by $\{x_1, x_2, \dots, x_i\}$.

Lemma 4.1. *Set $U_1 = k[t_1]$, and for $j = 2, \dots, m$, define U_j recursively by equation 4.*

$$U_j = U_{j-1}\big[t_{s(1,j)}; \sigma_{s(1,j)}\big] \cdots \big[t_{s(j-1,j)-1}; \sigma_{s(j-1,j)}\big]\big[t_{s(j,j)}; \sigma_{s(j,j)}, \delta_{s(j,j)}\big] \tag{4}$$

1. *Then $\mathcal{B} = \{\mathbf{t}^\alpha : \alpha \in \mathbb{N}^p\}$ is a basis for $U(\mathcal{X})$,*
2. *$U(\mathcal{X}_i) \cong U_i$ for $1 \leqslant i \leqslant m$, and*
3. *there is a $\mathcal{B}$-admissible ordering $\preceq$ on $U(\mathcal{X})$ such that $t_1 \prec t_2 \prec \cdots \prec t_p$.*

Proof: Part 1 follows from the PBW Theorem. Part 2 can be proved in the same way as [**10**, Theorem 3.1]. Part 3 follows from part 2 and [**7**, Theorem 1.10]. The maps $\sigma_1, \sigma_2, \dots, \sigma_p$ and $\delta_3, \delta_6, \dots, \delta_p$ are determined by the rules below.

(i) $\sigma_j(t_i) = \varepsilon(t_j, t_i)t_i$ for $1 \leqslant i < j \leqslant p$

(ii) $\delta_{s(j,j)}(t_i) = \langle t_{s(j,j)}, t_i\rangle$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i < s(j,j)$ ∎

We adopt notation and terminology from [**7**].

Definition 4.2: Choose nonzero $u \in U(\mathcal{X})$. By part 1 of Lemma 4.1 the expression in equation 5 is unique with $c_\alpha \in k$ for each $\alpha \in \mathbb{N}^p$.

$$u = \sum_{\alpha \in \mathbb{N}^p} c_\alpha \mathbf{t}^\alpha \tag{5}$$

1. Equation 5 is called the *standard representation* of u.
2. The *Newton diagram* of u is $\mathcal{N}(u) = \{\alpha \in \mathbb{N}^p : c_\alpha \neq 0\}$.
3. The *exponent* of u is $\exp(u) = \max_{\preceq} \mathcal{N}(u)$.
4. The *leading coefficient* of u is $lc(u) = c_{\exp(u)}$.

In particular, if $u \in \mathcal{X}_+$ then $\mathcal{N}(u) \subseteq \big\{\phi(i)e_i : 1 \leqslant i \leqslant p\big\}$.

Definition 4.3: Let $u_1, u_2 \in U(\mathcal{X})$ be given and set

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) = \exp(u_1)$$

and

$$\beta = (\beta_1, \beta_2, \dots, \beta_p) = \exp(u_2).$$

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_p)$ be such that $\gamma_i = \max\{\alpha_i, \beta_i\}$ for each $i = 1, 2, \dots, p$. The *left S-polynomial* of u_1 and u_2, denoted $S^\ell(u_1, u_2)$, is shown in equation 6 where $\lambda = lc(u_2)\big(lc(\mathbf{t}^\alpha \mathbf{t}^{\gamma-\alpha})\big)^{-1}$ and $\mu = lc(u_1)\big(lc(\mathbf{t}^\beta \mathbf{t}^{\gamma-\beta})\big)^{-1}$.

$$S^\ell(u_1, u_2) = \lambda \mathbf{t}^{\gamma-\alpha} u_1 - \mu \mathbf{t}^{\gamma-\beta} u_2 \tag{6}$$

Lemma 4.4. *If K is a homogeneous linear subspace of $\mathcal{X}_+$ then there is a homogeneous basis $\mathcal{G} = \{u_1, u_2, \dots, u_n\}$ of K such that $u_1 \prec u_2 \prec \cdots \prec u_n$ and $\exp(u_i) \notin \mathcal{N}(u_j)$ for all i, j with $1 \leqslant i < j \leqslant n$.*

Proof: We prove such a basis $\mathcal{G}$ exists by induction on n, with the case $n = 1$ being trivial.

Step 1. If K' is a homogeneous linear subspace of K with $\dim K' = n-1$ then there is a basis $\mathcal{G}' = \{u_1, u_2, \dots, u_{n-1}\}$ of K' such that $u_1 \prec u_2 \prec \cdots \prec u_{n-1}$ and $\exp(u_i) \notin \mathcal{N}(u_j)$ for all i, j with $1 \leqslant i < j \leqslant n-1$.

This follows from the inductive hypothesis.

Step 2. If K' and G' are as in Step 1 then there exists homogeneous $u \in K \backslash K'$ such that $\exp(u) \neq \exp(u_{n-1})$.

Suppose $v \in K \backslash K'$ is homogeneous and $\exp(v) = \exp(u_{n-1})$. Then $\partial v = \partial u_{n-1}$ so $u = v - lc(v)\big(lc(u_{n-1})\big)^{-1} u_{n-1} \in K \backslash K'$ is homogeneous with $\exp(u) \prec \exp(u_{n-1})$.

Step 3. There exist K' and $\mathcal{G}'$ as in Step 1 such that some homogeneous $u \in K \backslash K'$ satisfies $\exp(u_{n-1}) \prec \exp(u)$.

Let K' and G' be as in Step 1. By Step 2 there exists homogeneous $w \in K \backslash K'$ such that $\exp(w) \neq \exp(u_{n-1})$. If $\exp(w) \prec \exp(u_{n-1})$ set $u = u_{n-1}$ and let K'' be the linear subspace spanned by $\{u_1, u_2, \dots, u_{n-2}, w\}$. It is easy to see $\exp(w') \prec \exp(u_{n-1})$ for all $w' \in K''$. Choose a basis $\mathcal{G}''$ as in Step 1 and replace K' with K'' and $\mathcal{G}'$ with $\mathcal{G}''$.

Step 4. There is a homogeneous basis $\mathcal{G} = \{u_1, u_2, \dots, u_n\}$ of K such that $u_1 \prec u_2 \prec \cdots \prec u_n$ and $\exp(u_i) \notin \mathcal{N}(u_j)$ for all i, j with $1 \leqslant i < j \leqslant n$.

Let K', $\mathcal{G}'$, and u be as in Step 3 and write u as in equation 5. Set

$$\mathcal{M} = \mathbb{N}^p \backslash \{\exp(u_i) : i = 1, 2, \dots, n-1\}$$

and set

$$u_n = u - \sum_{i=1}^{n-1} c_{\exp(u_i)} \big(lc(u_i)\big)^{-1} u_i.$$

Clearly $\exp(u_{n-1}) \prec \exp(u_n)$ so we only need to show $\mathcal{N}(u_n) \subseteq \mathcal{M}$. Set

$$v = u - \sum_{i=1}^{n-1} c_{\exp(u_i)} \mathbf{t}^{\exp(u_i)}$$

and set $v_i = u_i - lc(u_i)\mathbf{t}^{\exp(u_i)}$ for $i = 1, 2, \dots, n-1$. Then $\mathcal{N}(v) \subseteq \mathcal{M}$ by construction and $\mathcal{N}(v_i) \subseteq \mathcal{M}$ for $i = 1, 2, \dots, n-1$ by our choice of $\mathcal{G}'$. Moreover

$$u_n = v - \sum_{i=1}^{n-1} c_{\exp(u_i)} lc(u_i)^{-1} v_i$$

so

$$\mathcal{N}(u_n) \subseteq \mathcal{N}(v) \cup \left(\bigcup_{i=1}^{n-1} \mathcal{N}(v_i) \right) \subseteq \mathcal{M}$$

as desired. ∎

THEOREM 4.5. *Let K be a homogeneous linear subspace of $\mathcal{X}_+$ and let I be the ideal generated by K. Then $I = U(\mathcal{X})K$ and $U(\mathcal{X}/K) \cong U(\mathcal{X})/I$. Moreover the basis $\mathcal{G}$ of K described in Lemma 4.4 is a Gröbner basis of I.*

PROOF: It is easy to see $\mathcal{X}_+$, and hence K, contains only normal elements of $U(\mathcal{X})$. The relations $I = U(\mathcal{X})K$ and $U(\mathcal{X}/K) \cong U(\mathcal{X})/I$ can be proved using the PBW Theorem (see [**2**, Thoerem 3.2.2]) and the universal property of the enveloping algebra.

To prove $\mathcal{G}$ is a Gröbner basis of I we show the remainder is 0 when the left division algorithm by $\mathcal{G}$ is applied to $S^\ell(u_i, u_j)$ (see [**7**, Theorem 2.1]). Then $\mathcal{G}$ satisfies the so-called "Buchberger S-pair criterion", which is [**7**, Theorem 3.2].

For each i, j with $1 \leqslant i < j \leqslant n$ it is enough to show that $S^\ell(u_i, u_j) = v_1 u_i + v_2 u_j$ where

$$\exp(v_1 u_i) \preccurlyeq \exp\big(S^\ell(u_i, u_j)\big), \quad \exp(v_2 u_j) \preccurlyeq \exp\big(S^\ell(u_i, u_j)\big)$$

and for all $\alpha \in \mathcal{N}(v_i)$ there does not exist $\beta \in \mathbb{N}^p$ such that $\exp(u_j) + \alpha = \exp(u_i) + \beta$. We may pass to the case $i = 1$ and $j = 2$ by [**7**, Proposition 2.11].

Since $u_1, u_2 \in \mathcal{X}_+$ and $u_1 \prec u_2$ there exist $p_1, p_2 \in \{1, 2, \ldots, p\}$ with $p_1 < p_2 \leqslant p$ such that $\exp(u_1) = \phi(p_1)e_{p_1}$ and $\exp(u_2) = \phi(p_2)e_{p_2}$. Thus $\alpha = \phi(p_1)e_{p_1}$, $\beta = \phi(p_2)e_{p_2}$, and $\gamma = \alpha + \beta$, which implies $\lambda = lc(u_2)$ and $\mu = lc(u_1)\varepsilon(u_2, u_1)$. Set $v_1 = \varepsilon(u_2, u_1)\big(u_2 - lc(u_2)t_{p_2}^{\phi(p_2)}\big)$ and $v_2 = \varepsilon(u_2, u_1)\big(u_1 - lc(u_1)t_{p_1}^{\phi(p_1)}\big)$. Then

$$\begin{aligned} S^\ell(u_1, u_2) &= \lambda \mathbf{t}^{\gamma - \alpha} u_1 - \mu \mathbf{t}^{\gamma - \beta} u_2 \\ &= lc(u_2) t_{p_2}^{\phi(p_2)} u_1 - lc(u_1)\varepsilon(u_2, u_1) t_{p_1}^{\phi(p_1)} u_2 \\ &= (u_2 + v_1)u_1 - \big(\varepsilon(u_2, u_1)u_1 - v_2\big)u_2 \\ &= v_1 u_1 + v_2 u_2 \end{aligned}$$

with $\exp(v_1) \prec \exp(u_2)$ and $\exp(v_2) \prec \exp(u_1)$.

Let $\alpha \in \mathcal{N}(u_2)$ be arbitrarily chosen. Then $\alpha = \phi(p_3)e_{p_3}$ for some $p_3 \in \{1, 2, \ldots, p\}$ with $p_3 \leqslant p_2$ since $u_2 \in \mathcal{X}_+$. If there exists $\beta \in \mathbb{N}^p$ such that $\exp(u_j) + \alpha = \exp(u_i) + \beta$ then $\phi(p_2)e_{p_2} + \phi(p_3)e_{p_3} = \phi(p_1)e_{p_1} + \beta$. We must have $\phi(p_3)e_{p_3} = \phi(p_1)e_{p_1}$ and $\beta = \phi(p_2)e_{p_2}$ since $p_1 < p_2$. But this implies $\exp(u_1) = \alpha \in \mathcal{N}(u_2)$, which contradicts our choice of basis.

It follows from what we just proved that $\exp(v_1 u_1) \neq \exp(v_2 u_2)$. This implies $\exp(v_1 u_1) \preccurlyeq \exp\big(S^\ell(u_1, u_2)\big)$ and $\exp(v_2 u_2) \preccurlyeq \exp\big(S^\ell(u_1, u_2)\big)$ as desired. ∎

References

[1] M. Aubry and J.-M. Lemaire, 'Zero divisors in enveloping algebras of graded Lie algebras', *J. Pure Appl. Algebra* **38** (1985), 159–166.

[2] Yu. A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky and M.V. Zaicev, *Infinite dimensional Lie superalgebras* (DeGruyter, Berlin, 1992).

[3] E.J. Behr, 'Enveloping algebras of Lie superalgebras', *Pacific J. Math.* **130** (1987), 9–25.

[4] A.D. Bell, 'A criterion for primeness of enveloping algebras of Lie superalgebras', *J. Pure Appl. Algebra* **69** (1990), 111–120.

[5] R. Bøgvad, 'Some elementary results on the cohomology of graded Lie algebras', in *Homotopie Algébrique et Algèbre Locale (Luminy, 1982)*, Astérisque **113–114** (Soc. Math. France, Paris, 1984), pp. 156–166.

[6] J.L. Bueso, F.J. Castro-Jiménez, J. Gómez-Torrecillas and F. J. Lobillo, 'Homological computations in PBW modules', *Algebr. Represent. Theory* **4** (2001), 201–218.

[7] J.L. Bueso, F.J. Castro-Jiménez, J. Gómez-Torrecillas and F. J. Lobillo, 'An introduction to effective calculus in quantum groups', in *Rings, Hopf algebras and Brauer Groups*, (S. Caenepell and A. Verschoren, Editors), Lecture Notes in Pure and Applied Mathematics **197** (Marcel Dekker, New York, 1998), pp. 55–83.

[8] J.L. Bueso, F.J. Castro-Jiménez, J. Gómez-Torrecillas and F.J. Lobillo, 'Primality test in iterated ore extensions', *Comm. Algebra* **29** (2001), 1357–1371.

[9] E. Kirkman, J. Kuzmanovich and L. Small, 'Finitistic dimensions of Noetherian rings', *J. Algebra* **147** (1992), 350–364.

[10] L. Le Bruyn, 'Trace rings of generic 2 by 2 matrices', *Mem. Amer. Math. Soc.* **66** (1987).

[11] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian rings* (Wiley, Chichester, England, 1987).

[12] K.L. Price, 'Homological properties of color Lie super algebras', in *Advances in Ring Theory (Granville, OH, 1996)*, (S.K. Jain and S. Tariq Rizvi, Editors) (Birkhäuser, 1997), pp. 287–293.

[13] K.L. Price, 'Primeness criterion for universal enveloping algebras of Lie color algebras', *J. Algebra* **235** (2001), 589–607.

Department of Mathematics
University of Wisconsin Oshkosh
Oshkosh, WI 54901-8631
United States of America
e-mail: pricek@uwosh.edu