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BICYCLIC UNITS IN SOME INTEGRAL GROUP RINGS

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ABSTRACT. A description is given of the unit group $\mathcal{U}(\mathbf{Z}G)$ for the two groups $G = D_{12}$ and $G = D_8 \times C_2$. In particular, it is shown that in both cases the bicyclic units generate a torsion-free normal complement. It follows that the Bass-cyclic units together with the bicyclic units generate a subgroup of finite index in $\mathcal{U}(\mathbf{Z}D_{2n})$, for all $n \geq 3$.

In this paper we compute the group of units of the integral group ring of the dihedral group of order 12, denoted D_{12}, and the direct product of the dihedral group of order 8 with the cyclic group of order 2, $D_8 \times C_2$. In both cases it is shown that there exists a torsion-free normal complement which is a free rank 3 extension of a free normal subgroup. Furthermore, in each case the normal complement is generated by the bicyclic units. Hence, because of the results in [7], it follows that the Bass-cylic units together with the bicyclic units generate a subgroup of finite index in the unit group of the integral group ring of an arbitrary dihedral group. Furthermore, because of the results in [4] it follows that the only groups of order 16 for which the Bass cyclic units together with the bicyclic units generate a subgroup of finite index in the unit group of their respective integral group ring are D_{16} and $D_8 \times C_2$.

Throughout we follow the notation of [6]. For a group G the group of units of augmentation 1 in $\mathbf{Z}G$ is denoted $\mathcal{U}_1(\mathbf{Z}G)$. The dihedral group of order $2n$ is denoted D_{2n}. We use the following presentations: $D_6 = \langle a, b \mid a^3 = b^2 = 1, ba = a^2 b\rangle$ and clearly $D_{12} = D_6 \times C_2$ where $C_2 = \langle c \mid c^2 = 1\rangle$, the cyclic group of order 2.

The ring of two-by-two matrices over $\mathbf{Q}$ is denoted $M_2(\mathbf{Q})$. For subsets A, B, C, D of $\mathbf{Q}$ we write $\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\det=1}$ for the set

$$\left\{ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a \in A, b \in B, c \in C, d \in D, \det(M) = 1 \right\}.$$

The congruence group $\Gamma(n)$ of level $n > 0$ is $\begin{bmatrix} 1 + n\mathbf{Z} & n\mathbf{Z} \\ n\mathbf{Z} & 1 + n\mathbf{Z} \end{bmatrix}_{\det=1}$.

For our computations in $\mathbf{Z}D_{12}$ we need the following result of Jespers and Parmenter on $\mathbf{Z}D_6$.

PROPOSITION 1 ([3]). *The following statements hold in* $\mathbf{Z}D_6$*:*

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(1)

$$\mathbf{Z}D_6(1-a) \cong \begin{bmatrix} 3\mathbf{Z} & 2\mathbf{Z} \\ \frac{3}{2}\mathbf{Z} & 3\mathbf{Z} \end{bmatrix}$$

where the isomorphism maps $(\alpha_0 + \alpha_1 a + \beta_0 b + \beta_1 ba)(1-a)$, $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbf{Z}$, *to*

$$\begin{bmatrix} 3(\alpha_1 + \beta_0 - \beta_1) & 2(-\alpha_0 + 2\alpha_1 + 2\beta_0 - \beta_1) \\ \frac{3}{2}(\alpha_0 - 2\alpha_1 - \beta_0 + 2\beta_1) & 3(\alpha_0 - \alpha_1 - \beta_0 + \beta_1); \end{bmatrix}$$

(2)

$$\Delta_{\mathbf{Z}}(D_6)(1-a) \cong \begin{bmatrix} 3\mathbf{Z} & 3\mathbf{Z} \\ 3\mathbf{Z} & 3\mathbf{Z} \end{bmatrix},$$

this isomorphism follows from the above and conjugating by $\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$;

(3) D_6 *has a torsionfree normal complement* $W = 1 + \{u = 1 + \alpha(1-a) \mid \alpha \in \Delta_{\mathbf{Z}}(D_6), u$ *a unit*$\} \cong \Gamma(3)$. *Further* W *is a free group of rank 3 generated by the 3 distinct (up to inverses) bicyclic units.*

THEOREM 2. *In* $\mathcal{U}_1(\mathbf{Z}D_{12})$, D_{12} *has a torsion-free normal complement* V *which is a semi-direct product of a free group of rank* 5 *by a free group of rank* 3. *Furthermore,* V *is generated by the bicyclic units.*

PROOF. As said above, we write $D_{12} = D_6 \times C_2$.

Let $V = \{u = 1 + \alpha(1-a) \mid \alpha \in \Delta_{\mathbf{Z}}(D_{12}), u \text{ a unit}\}$. It is well-known [1] that V is a torsion-free normal complement for D_{12} in $\mathcal{U}_1(\mathbf{Z}D_{12})$.

Let $\pi\colon \mathbf{Z}(D_6 \times C_2) \longrightarrow \mathbf{Z}D_6$ be the natural epimorphism (mapping c to 1). Because of Proposition 1, $\pi(V) = W$ is a free group of rank 3. It follows that V is a semi-direct product of $K = \{u = 1 + \alpha(1-c)(1-a) \mid \alpha \in \mathbf{Z}D_6, u \text{ a unit}\}$ by the free group of rank 3 generated by the following bicyclic units in $\mathbf{Z}D_6 \subseteq \mathbf{Z}D_{12}$;

$$\begin{aligned} x_1 &= 1 + (1 - ba^2)a(1 + ba^2), \\ x_2 &= 1 + (1 - ba)a(1 + ba), \\ x_3 &= 1 + (1 - b)a(1 + b). \end{aligned}$$

Clearly the natural mapping $K \longrightarrow K\frac{1}{2}(1-c)$ is an isomorphism. Hence

$$K \cong K\frac{1}{2}(1-c) \cong W_2 = \{u = 1 + 2\alpha(1-a) \mid \alpha \in \mathbf{Z}D_6, u \text{ a unit}\}.$$

Proposition 1 yields

$$W_2 \cong \begin{bmatrix} 1 + 6\mathbf{Z} & 4\mathbf{Z} \\ 3\mathbf{Z} & 1 + 6\mathbf{Z} \end{bmatrix}_{\det=1}.$$

Hence

$$\begin{aligned} W_2 &\cong \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 + 6\mathbf{Z} & 4\mathbf{Z} \\ 3\mathbf{Z} & 1 + 6\mathbf{Z} \end{bmatrix}_{\det=1} \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \\ &\cong \begin{bmatrix} 1 + 6\mathbf{Z} & 2\mathbf{Z} \\ 6\mathbf{Z} & 1 + 6\mathbf{Z} \end{bmatrix}_{\det=1} \end{aligned}$$

The latter group contains as a subgroup $\Gamma(6)$, a free group of rank 13 (*cf.* [5]). Let L be the subgroup of K isomorphic (under the above mentioned isomorphism) to $\Gamma(6)$. We first show that L is generated by products of bicyclic units. For this we first need a set of generators for $\Gamma(6)$. We will obtain these, using the Reidemeister-Schreier method, from a set of generators of $\Gamma(3)$.

Again from Proposition 1 we know that $\Gamma(3)$ is freely generated by the matrix representations of the bicyclic units x_1, x_2 and x_3 (abusing notation we will use the same notation for these elements):

$$x_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -5 & 3 \\ -12 & 7 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}.$$

Since $\Gamma(6)$ is a free group of rank 13 and of finite index in the free group $\Gamma(3)$ of rank 3, the quotient group $\Gamma(3)/\Gamma(6)$ is a group of of order 6. Using the matrix representations it is easily verified that

$$\Gamma(3)/\Gamma(6) = \langle \overline{x_1}, \overline{x_1x_2} \mid \overline{x_1}\overline{x_1x_2} = \overline{x_1x_2}^{-1}\overline{x_1} \rangle \cong D_6.$$

Here we denote by $\bar{x}$ the coset $x\Gamma(6)$. So

$$T = \{1, x_1, x_1(x_1x_2), x_1x_2, x_1(x_1x_2)^2, (x_1x_2)^2\}$$

is a transversal for $\Gamma(6)$ in $\Gamma(3)$. Hence by the Reidemeister-Schreier method the elements of the type

$$s^{-1}x_kt \in \Gamma(6), \quad (s, t \in T, 1 \leq k \leq 3)$$

form a set of generators for $\Gamma(6)$. Using the above presentation for $\Gamma(3)/\Gamma(6)$, and $\overline{x_2} = \overline{x_1}(\overline{x_1x_2})$ and $\overline{x_3} = \overline{x_1x_2x_1}$ one verifies that these generators are:

(i)
$$x_1^2,$$
$$(x_1x_2)^{-1}x_1^2(x_1x_2),$$
$$(x_1x_2)^{-2}x_1^2(x_1x_2)^2,$$

(ii)
$$x_2x_1^2x_2,$$
$$(x_1x_2)^{-1}x_2x_1x_2(x_1x_2),$$

(iii)
$$(x_1x_2)^{-3}x_1x_2^2x_1,$$

(iv)
$$x_1^{-2}(x_1x_2)^3,$$
$$x_2^{-1}x_1^{-2}x_2,$$
$$(x_1x_2)^{-1}x_2^{-1}x_1^{-2}x_2(x_1x_2),$$

(v)
$$(x_1x_2)^{-1}x_3x_1,$$
$$(x_1x_2)^{-1}[(x_1x_2)^{-1}x_3x_1](x_1x_2),$$
$$(x_1x_2)^3\{(x_1x_2)^{-2}[(x_1x_2)^{-1}x_3x_1](x_1x_2)^2\},$$

(vi)
$$x_1^{-1}x_3^2x_1\left((x_1x_2)^{-1}x_3x_1\right)^{-1},$$
$$(x_1x_2)^{-1}\left[x_1^{-1}x_3^2x_1\left((x_1x_2)^{-1}x_3x_1\right)^{-1}\right](x_1x_2),$$
$$(x_1x_2)^{-2}\left[x_1^{-1}x_3^2x_1\left((x_1x_2)^{-1}x_3x_1\right)^{-1}\right](x_1x_2)^2(x_1x_2)^{-3}.$$

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So to prove that L is generated by products of bicyclics it is sufficient to show that for each of the above listed generators α one can find a product of bicyclics, say b, such that $b\frac{1}{2}(1+c)=\frac{1}{2}(1+c)$ and $b\frac{1}{2}(1-c)=\alpha$ (here we identify the elements of $K\frac{1}{2}(1-c)$ with those of W_2). Notice that this property is preserved under conjugation by products of bicyclics. Hence, we only have to deal with the generators (i)–(vi). Note that for (iii) and (iv) it is sufficient to deal with the case (iii′): $(x_1x_2)^3$. Furthermore (vi) may be replaced by (vi′): x_3^2. We need the following bicyclic units:

$$x_k(c)=1+(1-ba^{3-k}c)a(1+ba^{3-k}c),$$
$$y_k=1+(1-ba^{3-k})ac(1+ba^{3-k}),$$
$$y_k(c)=1+(1-ba^{3-k}c)ac(1+ba^{3-k}c),$$

where $1\leq k\leq 3$. In the next table we list for each of the five generators α the corresponding product b of bicyclic units.

(i)	$x_1y_1^{-1}$,
(ii)	$y_2^{-1}x_1y_1^{-1}x_2$,
(iii′)	$x_2^{-1}x_3(c)^{-1}x_3x_2(c)^{-1}x_2x_1(c)^{-1}x_1x_2$,
(v)	$y_2x_2^{-1}y_3(c)^{-1}x_3^{-1}x_1(c)x_1y_1(c)^{-1}x_1(c)^{-1}x_3(c)y_3(c)x_1^{-1}x_3y_3^{-1}x_1$,
(vi′)	$x_3y_3^{-1}$.

We have therefore shown that L is indeed generated by products of bicyclics. Finally we show that the same holds for K.

Notice that the quotient group

$$K/L\cong\begin{bmatrix}1+6\mathbf{Z} & 2\mathbf{Z}\\ 6\mathbf{Z} & 1+6\mathbf{Z}\end{bmatrix}_{\det=1}/\Gamma(6)$$

is a cyclic group of order 3, generated by the coset of the matrix $\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Hence

$$K/L\cong\begin{bmatrix}1+6\mathbf{Z} & 2\mathbf{Z}\\ 6\mathbf{Z} & 1+6\mathbf{Z}\end{bmatrix}_{\det=1}/\Gamma(6).$$

Since $x_3(c)^{-1}x_3\in K$ corresponds with the matrix

$$\begin{bmatrix}-5 & 8\\ -12 & 19\end{bmatrix}\notin\Gamma(6)$$

it follows that K is generated by L and the element $x_3(c)^{-1}x_3$. Therefore the result follows. ■

In the remainder we study the unit group of $\mathbf{Z}(D_8\times C_2)$. The first part of the statement in the next theorem is proved, by Jespers and Parmenter, in [4].

THEOREM 3. *In $\mathcal{U}_1\big(\mathbf{Z}(D_8\times C_2)\big)$, $D_8\times C_2$ has a torsion-free normal complement V which is a semi-direct product of a free group of rank* 9 *by a free group of rank* 3. *Furthermore, V is generated by the bicyclic units.*

PROOF. Write $G = D_8 \times C_2$, where $D_8 = \langle a, b \mid a^4 = b^2 = 1, ba = a^3 b\rangle$. Let $V = \{u = 1 + \alpha(1 - a^2) \mid \alpha \in \Delta_{\mathbf{Z}}(G), u \text{ a unit}\}$. Because of [1], V is a torsion-free normal complement for G in $\mathcal{U}_1(\mathbf{Z}G)$. It follows from [4] that V is the semidirect product of the normal subgroup

$$K = \{u = 1 + \alpha(1 - a^2)(1 - c) \mid \alpha \in \mathbf{Z}D_8, u \text{ a unit}\}$$

by the group

$$W = \{u = 1 + \alpha(1 - a^2) \mid \alpha \in \Delta_{\mathbf{Z}}(D_8), u \text{ a unit}\}.$$

Furthermore W is a free group with basis any three of the following bicyclic units:

$$\begin{aligned} x_1 &= 1 + (1 - b)a(1 + b), \\ x_2 &= 1 + (1 - ab)a(1 + ab), \\ x_3 &= 1 + (1 - a^2b)a(1 + a^2b), \\ x_4 &= 1 + (1 - a^3b)a(1 + a^3b). \end{aligned}$$

Note that $x_4 = x_3^{-1}x_2^{-1}x_1^{-1}$. Also, from [4] one obtains that

$$W \cong \overline{\begin{bmatrix} 1 + 2\mathbf{Z} & 4\mathbf{Z} \\ 2\mathbf{Z} & 1 + 2\mathbf{Z} \end{bmatrix}_{\det=1}}.$$

The latter group is by definition the group $\begin{bmatrix} 1 + 2\mathbf{Z} & 4\mathbf{Z} \\ 2\mathbf{Z} & 1 + 2\mathbf{Z} \end{bmatrix}_{\det=1}$ modulo its center. In the proof we will often identify x_i with its matrix representation under the above isomorphism. Hence

$$x_4 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$x_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix}.$$

Clearly,

$$K \cong K\frac{1}{2}(1 - c) \cong W_2 = \{u = 1 + 2\alpha(1 - a^2) \mid \alpha \in \mathbf{Z}D_8, u \text{ a unit}\};$$

and

$$W_2 \cong \begin{bmatrix} 1 + 4\mathbf{Z} & 8\mathbf{Z} \\ 4\mathbf{Z} & 1 + 4\mathbf{Z} \end{bmatrix}_{\det=1},$$

a free group of rank 9 (*cf.* [4]). It is easily verified that $\{1, x_4^{-1}, x_3^{-1}, x_3^{-1}x_4^{-1}\}$ is a transversal for the subgroup $\begin{bmatrix} 1 + 4\mathbf{Z} & 8\mathbf{Z} \\ 4\mathbf{Z} & 1 + 4\mathbf{Z} \end{bmatrix}_{\det=1}$ of $\overline{\begin{bmatrix} 1 + 2\mathbf{Z} & 4\mathbf{Z} \\ 2\mathbf{Z} & 1 + 2\mathbf{Z} \end{bmatrix}_{\det=1}}$. Hence applying the

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Reidemeister-Schreier method to this transversal and the generators x_4^{-1}, x_3^{-1}, x_2^{-1} one obtains (*cf.* [2], Proof of Theorem 5.1, p. 1824):

$$W_2 = \langle w_1 = x_4^{-2}, w_2 = x_4^{-1}x_2^{-1}, w_3 = x_2^{-1}x_4, w_4 = x_3^{-2}, \\ w_5 = x_3^{-1}x_4^{-1}x_3^{-1}x_4, w_6 = x_4^{-1}x_3^{-1}x_4x_3, w_7 = x_3^{-1}x_4^{-2}x_3, \\ w_8 = x_3^{-1}x_4^{-1}x_2^{-1}x_3, w_9 = x_3^{-1}x_2^{-1}x_4x_3\rangle.$$

So to prove the result it is now sufficient to show that for each of the generators w_i of W_2 there exists a product b_i of bicyclic units in K such that $\frac{1}{2}(1-c)b_i = w_i$ (here again we identify the elements of $K\frac{1}{2}(1-c)$ with those of W_2). For this we need more bicyclic units, $1 \le i \le 4$:

$$x_i(c) = 1 + (1 - a^{i-1}bc)a(1 + a^{i-1}bc), \\ y_i = 1 + (1 - a^{i-1}b)ac(1 + a^{i-1}b), \\ y_i(c) = 1 + (1 - a^{i-1}bc)ac(1 + a^{i-1}bc).$$

Further we note that

$$x_1 = 1 + (1 + a^2b)a(1 - a^2b) \\ x_2 = 1 + (1 + a^3b)a(1 - a^3b) \\ x_3 = 1 + (1 + b)a(1 - b) \\ x_4 = 1 + (1 + ab)a(1 - ab).$$

Using these identities one can verify the following table which list for each w_i its corresponding b_i.

$$w_1 : x_2(c)^{-1}y_2(c) \\ w_2 : x_2(c)^{-1}y_2 \\ w_3 : x_4(c)^{-1}x_4 \\ w_4 : x_1(c)^{-1}y_1x_1x_1(c)^{-1} \\ w_5 : x_3^{-1}x_1x_2y_2(c)^{-1}x_1^{-1}y_3(c)^{-1}y_3x_1(c)x_1^{-1}y_1^{-1}x_1(c)x_3x_1(c)^{-1}y_1x_1x_1(c)^{-1} \\ w_6 : (x_3w_5x_3^{-1})x_1(c)x_1^{-1}y_1^{-1}x_1(c) \\ w_7 : x_1(c)^{-1}x_2(c)^{-1}y_2(c)x_1(c) \\ w_8 : x_3^{-1}x_2(c)^{-1}y_2x_3 \\ w_9 : x_3^{-1}x_4(c)^{-1}x_4x_3.$$

Hence the result follows. ■

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