16

The theory of strong interactions: quantum chromodynamics

The basic features of the quark model of hadrons were set out in Chapter 1. Quarks carry a colour index, and interact with the gluon fields which mediate the strong interaction.

We have seen that in the Standard Model the electromagnetic interaction and the weak interaction are well described by gauge theories. In the Standard Model the strong interaction also is described by a gauge theory. In this chapter we show how this is done. The theory is known as *quantum chromodynamics* (QCD) and has the remarkable property that in the theory quarks are confined, as appears to be the case experimentally (Section 1.4). In this chapter we concentrate exclusively on the strong interaction. The electromagnetic and weak interactions of quarks are neglected.

16.1 A local $SU(3)$ gauge theory

In QCD, we have three fields for each flavour of quark. These are put into so-called *colour triplets.* For example the u quark is associated with the triplet

$$\mathbf{u} = \begin{pmatrix} \mathrm{u_r} \\ \mathrm{u_g} \\ \mathrm{u_b} \end{pmatrix},$$

where $\mathrm{u_r}$, $\mathrm{u_g}$, $\mathrm{u_b}$ are four-component Dirac spinors, and the subscripts r, g, b label the colour states (red, green, blue, say).

We then postulate that the theory is invariant under a local $SU(3)$ transformation

$$\mathbf{q} \to \mathbf{q}' = \mathbf{U}\mathbf{q} \tag{16.1a}$$

where $\mathbf{q}$ is any quark triplet, and $\mathbf{U}$ is any space- and time-dependent element of the group $SU(3)$. The mathematical steps follow those of the $SU(2)$ theory of the weak interaction of leptons. We introduce a 3×3 matrix gauge field $\mathbf{G}_\mu$, which

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is the analogue of the matrix field $\mathbf{W}_\mu$ of the electroweak theory. Under an $SU(3)$ transformation,

$$\mathbf{G}_\mu \to \mathbf{G}_\mu{}' = \mathbf{U}\mathbf{G}_\mu\mathbf{U}^\dagger + (\mathrm{i}/g)(\partial_\mu\mathbf{U})\mathbf{U}^\dagger. \tag{16.1b}$$

We define

$$D_\mu\mathbf{q} = (\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{q}. \tag{16.2}$$

It follows that under a local $SU(3)$ transformation

$$D_\mu{}'\mathbf{q}' = \mathbf{U}D_\mu\mathbf{q} \tag{16.3}$$

where $D_\mu{}'\mathbf{q}' = (\partial_\mu + ig\mathbf{G}_\mu{}')\mathbf{q}'$. The parameter g that appears in these equations is the *strong coupling constant.*

$\mathbf{G}_\mu$ is taken to be Hermitian and traceless, like $\mathbf{W}_\mu$ in the electroweak theory, and hence it can be expressed in terms of the eight matrices λ_a set out in Appendix B, Section B.7:

$$\mathbf{G}_\mu = \frac{1}{2}\sum_{a=1}^{8} G_\mu^a\lambda_a \tag{16.4}$$

where the coefficients $G_\mu^a(x)$ are eight real independent gluon gauge fields. (The factor $\frac{1}{2}$ is conventional.)

The Yang–Mills construction (cf. Section 11.2),

$$\mathbf{G}_{\mu\nu} = \partial_\mu\mathbf{G}_\nu - \partial_\nu\mathbf{G}_\mu + \mathrm{i}g(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu), \tag{16.5}$$

leads to the result that, under $SU(3)$ transformations of the form (16.1b),

$$\mathbf{G}'_{\mu\nu} = \mathbf{U}\mathbf{G}_{\mu\nu}\mathbf{U}^+. \tag{16.6}$$

The gluon Lagrangian density is taken to be

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{2}\mathrm{Tr}[\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}]. \tag{16.7}$$

It follows from (16.16) and the cyclic invariance of the trace that $\mathcal{L}_{\text{gluon}}$ is gauge invariant.

We can expand $\mathbf{G}_{\mu\nu}$ in terms of its ‘components’,

$$\mathbf{G}_{\mu\nu} = \frac{1}{2}\sum_{a=1}^{8} G_{\mu\nu}^a\lambda_a, \tag{16.8}$$

using equation (B.27) of Appendix B. Hence, using also the property (B.28), that

$$\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab},$$

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the gluon Lagrangian density becomes

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{4}\sum_{a=1}^{8} G^a_{\mu\nu} G^{a\mu\nu}. \tag{16.9}$$

The quark Lagrangian density is taken to be of the standard Dirac form (equation (7.7)):

$$\mathcal{L}_{\text{quark}} = \sum_{f=1}^{6} [\bar{\mathbf{q}}_f \mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{q}_f - m_f \bar{\mathbf{q}}_f \mathbf{q}_f], \tag{16.10}$$

where the sum is over all flavours of quark and m_f are the ‘true’ quark masses defined in Section 14.2. $\mathcal{L}_{\text{quark}}$ is evidently invariant under an $SU(3)$ transformation (using (16.3)). The reader should note here the very compact notation that has been developed: as well as the explicit sum over flavours, there are sums over colour indices and sums over the indices of the four-component Dirac spinor and γ matrices. It is perhaps instructive for the reader to write out the expression in full.

The total strong interaction Lagrangian density is

$$\mathcal{L}_{\text{strong}} = \mathcal{L}_{\text{gluon}} + \mathcal{L}_{\text{quark}}. \tag{16.11}$$

The eight gluon gauge fields have no mass terms. There is no direct coupling of the gluon fields to the Higgs field. The Higgs field is relevant in that it gives mass to the quarks. The field equations follow from Hamilton’s principle of stationary action. For the six quark triplets we easily obtain (cf. Section 5.5)

$$(\mathrm{i}\gamma^\mu D_\mu - m_f)\mathbf{q}_f = 0. \tag{16.12}$$

For the eight gluon fields, variation of the Lagrangian density with respect to the field G^a_ν gives (cf. Section 4.2)

$$\partial_\mu G^{a\mu\nu} = j^{a\nu} \tag{16.13}$$

where

$$j^{a\nu} = g[f_{abc} G^b_\mu G^{c\mu\nu} + \sum_f \bar{\mathbf{q}}_f \gamma^\nu (\lambda_a/2)\mathbf{q}_f]. \tag{16.14}$$

Here f_{abc} are the $SU(3)$ structure constants, defined by

$$[\lambda_a, \lambda_b] = \lambda_a\lambda_b - \lambda_b\lambda_a = 2i\sum_{c=1}^{8} f_{abc}\lambda_c. \tag{16.15}$$

(See Appendix B, Section B.7.) Their appearance here stems from the definition (16.5) of $G_{\mu\nu}$.

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Since $\mathbf{G}^{\mu\nu} = -\mathbf{G}^{\nu\mu}$ it follows that

$$\partial_\nu j^{a\nu} = 0, \tag{16.16}$$

and we have eight conserved currents. These are the Noether currents, which are a consequence of the $SU(3)$ symmetry taken as a global symmetry. We therefore have eight constants of the motion, associated with the time-independent operators

$$Q^a = \int j^{a0} \mathrm{d}^3\mathbf{x}. \tag{16.17}$$

The field equations, and in particular the gluon field equations, are non-linear, like the equations of the electroweak theory. It is clear from (16.14) that both the quarks and the gluon fields themselves contribute to the currents $j^{a\nu}$ which are the sources of the gluon fields. The quarks interact through the mediation of the gluon fields; the gluon fields are also self-interacting.

Since the gluon fields are massless we might anticipate colour forces to be long range, which appears inconsistent with the short range of the strong interaction. However, the fields are known to be *confining* on a length scale greater than about 10^{-15} m = 1 fm: neither free quarks nor free 'gluons' have ever been observed.

In the electroweak theory, the 'free field' approximation in which all coupling constants are set to zero is the basis for the successful perturbation calculations we have seen in the preceding chapters. The free field approximation for quarks and gluons is not a good starting point for calculations in QCD, except on the scale of very small distances (≤ 0.1 fm) or very high energies (> 10 GeV). For low energy physics, the equations of the theory are analytically highly intractable. Even the vacuum state is characterised by complicated field configurations that have so far defied analysis. There is no analytical proof of confinement. Confinement is not displayed in perturbation theory, but numerical simulations demonstrate convincingly that QCD has this necessary property for an acceptable theory.

16.2 Colour gauge transformations on baryons and mesons

Since colour symmetry plays such an important part in the theory of strong interactions, it is natural to ask why it is not readily apparent in the particles, baryons and mesons, formed from quarks by the strong interaction. Here we attempt to answer that question.

In Section 1.4 we asserted that baryons are essentially made up of three quarks, and mesons are essentially quark–antiquark pairs. We shall denote a three-quark state in which quark 1 is in colour state i, quark 2 is in colour state j, and quark 3 is in colour state k by $|i, j, k\rangle$, and take the colour indices to be the numbers 1, 2, 3. We have suppressed all other aspects (position, spin, flavour) of the quarks. In

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Section 1.7 we saw that the Pauli principle required baryon states to be antisymmetric in the interchange of colour indices. The only antisymmetric combination of colour states we can construct is

$$|\text{state}\rangle = (1/\sqrt{6})\varepsilon_{ijk}|i, j, k\rangle, \tag{16.18}$$

where ε_{ijk} is defined by:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{321} = -\varepsilon_{213} = 1,$$

and $\varepsilon_{ijk} = 0$ if any two of *i, j, k* are the same. $(1/\sqrt{6})$ is a normalisation factor.

How does this state transform under a colour $SU(3)$ transformation? We restrict the discussion to a global (space- and time-independent) transformation, since a baryon is an object extended in space. We consider the quark fields to be transformed by $\mathbf{q} \to \mathbf{q}' = \mathbf{U}\mathbf{q}$. In quantum field theory, these fields destroy quarks and create antiquarks. It follows that under the transformation the baryon state (16.18) will transform as $|\,\text{state}\rangle \to |\,\text{state}\rangle' = (1/\sqrt{6})|a,b,c\rangle U^*_{ai}U^*_{bj}U^*_{ck}\varepsilon_{ijk}$. But $\varepsilon_{ijk}U^*_{ai}U^*_{bj}U^*_{ck} = \varepsilon_{abc}\det\mathbf{U}^* = \varepsilon_{abc}$, since the determinant of an $SU(3)$ matrix is 1. Thus we have the important result that under an $SU(3)$ transformation, $|\text{state}\rangle' = |\text{state}\rangle$. The transformation of the state is a trivial multiplication by unity. The state is said to be a *colour singlet.*

Turning now to the mesons, we denote a state of a quark, colour i, and an antiquark of colour j by $|i, \bar{j}\rangle$. Again, we have suppressed all other aspects of the quarks. Meson states are linear combinations

$$|\text{mesons}\rangle = (1/\sqrt{3})(|1, \bar{1}\rangle + |2, \bar{2}\rangle + |3, \bar{3}\rangle). \tag{16.19}$$

Under an $SU(3)$ transformation,

$$|\text{meson}\rangle \to |\text{meson}\rangle' = (1/\sqrt{3})|a, \bar{b}\rangle U^*_{ai}U_{bi}.$$

But $U^*_{ai}U_{bi} = U_{bi}U^\dagger_{ia} = \delta_{ab}$, so that

$$|\text{meson}\rangle' = |\text{meson}\rangle.$$

The meson states, like the baryon states, are colour singlets.

In the quark model, we see that colour transformations have no effect on the observed particles. It can also be shown that the eight gluon colour operators Q^a, defined by (16.17), give zero when they act on these states. Thus the $SU(3)$ symmetry is well hidden by Nature: the particles are blind to the transformation of colour symmetry. These observations can be related to lattice QCD, in which calculations indicate that all the allowed states of the theory have this property.

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16.3 Lattice QCD and asymptotic freedom

Numerical simulations of QCD replace continuous space-time by a finite but large four-dimensional space and time lattice of points. The quark and gluon fields are only defined at these points. Sophisticated computer programs have been written that are capable of handling the lattice. Gluon fields are commuting boson fields. The quark fields are anticommuting fermion fields and pose a technically much more difficult numerical problem. In fact the first lattice calculations were done neglecting all quark fields, even those of the light u and d quarks, and thus excluding all effects of virtual quark pair creation and annihilation. In this so-called *quenched approximation* the Lagrangian density it taken to be the $\mathcal{L}_{\text{gluon}}$ of (16.9). $\mathcal{L}_{\text{gluon}}$ displays confinement at distances greater than about a fermi.

At shorter distances, less than about 0.2 fermi, both $\mathcal{L}_{\text{gluon}}$ and the full QCD Lagrangian density display another important property, known as *asymptotic freedom.* The *effective* strong interaction coupling constant becomes so small at short distances that quarks and gluons can be considered as approximately free, and their interactions can be treated in perturbation theory.

To set the scene for the discussion of the effective 'running' strong interaction coupling constant, we first discuss the case of electromagnetism.

At atomic distances $\sim 10^{-10}$ m, the electrostatic interaction between an electron and a positron is given by the Coulomb energy $V(r) = -e^2/4\pi r$. In the lowest order of perturbation theory, the amplitude for electron–positron Coulomb scattering is proportional to the Fourier transform $V(Q^2)$ of $V(r)$,

$$V(Q^2) = \int V(r) e^{i\mathbf{Q}\cdot\mathbf{r}} d^3 r = -e^2/Q^2, \qquad (16.20)$$

where Q is the momentum transfer in the centre of mass system.

In QED, this result is modified by quantum corrections: virtual e^+e^- pairs created from the vacuum are polarised by the electric field of a charge, so that its measured charge at atomic distances is a 'bare' charge screened by virtual e^+e^- pairs. At short distances the screening is reduced, so that the effective charge is greater. Perturbation calculations in QED that include vacuum polarisation effects (Fig. 16.1) show that at large Q^2, (16.20) is modified to

$$V(Q^2) = -\frac{e^2}{Q^2} \frac{1}{1 - (e^2/12\pi^2)\ln(Q^2/4m^2)} \qquad (16.21)$$

where m is the electron mass. This result holds for large $Q^2 \gg 4m^2$ (but not so large Q^2 that the denominator vanishes!). Thus at large Q^2 we have an effective coupling constant

$$\alpha(Q^2) = \frac{e^2(Q^2)}{4\pi} = \frac{(e^2/4\pi)}{1 - (e^2/12\pi^2)\ln(Q^2/4m^2)}, \qquad (16.22)$$

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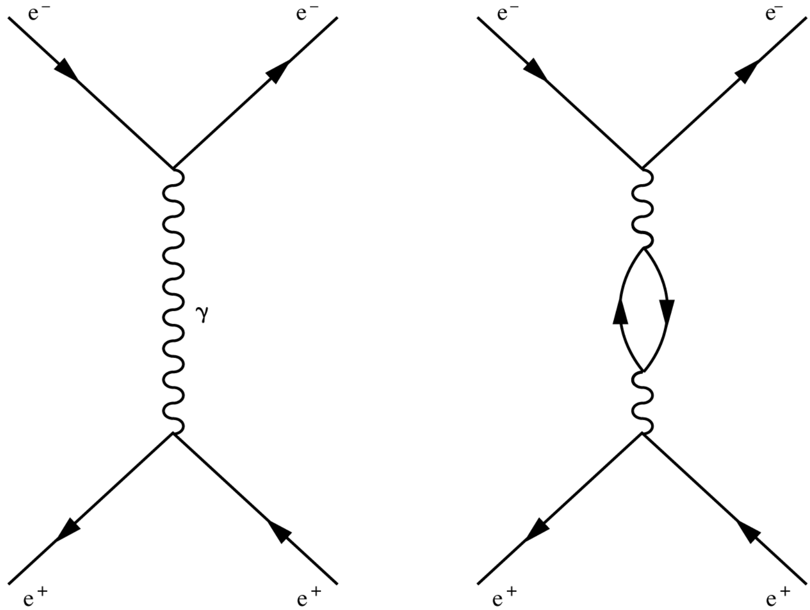

(a) (b)

Figure 16.1 (a) The lowest order Feynman diagram representing single photon exchange. The corresponding perturbation calculation reproduces the result of (16.20). (b) The lowest order modification due to vacuum polarisation. Including this effect gives, at large Q^2/m^2, the result of (16.21).

which increases as Q^2 increases (or, equivalently, as we probe shorter distances). Because $e^2/12\pi^2 \approx 10^{-3}$ the effects of vacuum polarisation are small, but in atomic physics they have been calculated and measured with high precision.

Similar vacuum polarisation effects occur in QCD, but the coupling is much larger and the consequences are more dramatic. If the scattering of a quark and an antiquark is calculated to the same order of perturbation theory as that used to obtain (16.22), then at large Q^2 the effective strong coupling constant $\alpha_s(Q^2)$ is (see Close, 1979, p. 217)

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{g^2/4\pi}{1 + (g^2/16\pi^2)[11 - (2/3)n_{\mathrm{f}}]\ln(Q^2/\lambda^2)}. \tag{16.23}$$

In this expression λ is a parameter with the dimensions of energy that replaces the electron mass appearing in QED. It is a necessary parameter associated with the renormalisation scheme. n_{f} is the effective number of quark flavours. For very large $Q^2 >$ (mass of the top quark)^2, $n_{\mathrm{f}} = 6$, but n_{f} is smaller at smaller Q^2. The important point to note is that $(11 - (2/3)n_{\mathrm{f}})$ is a positive number. Thus, in contrast to what happens in QED, $g(Q^2)$ decreases as Q^2 increases, and this is the basis of

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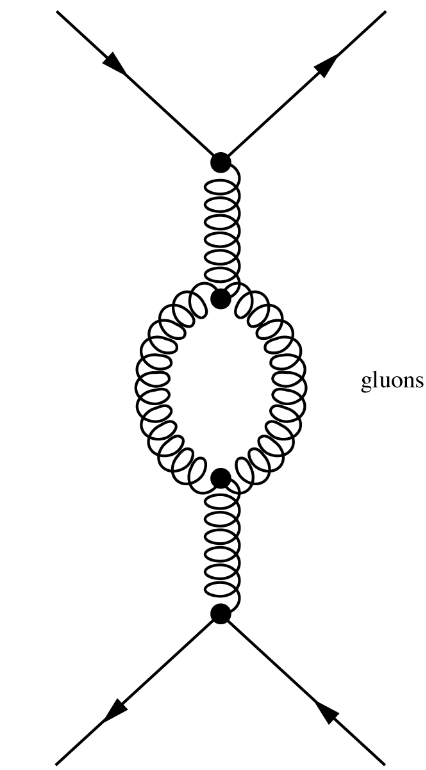

Figure 16.2 There are Feynman graphs similar to those of Fig. 16.1 but for gluon exchange between quarks and antiquarks. An additional lowest order contribution to vacuum polarisation is associated with this Feynman graph coming from the gluon self-coupling.

asymptotic freedom. As with QED the fermions contribute with a negative sign, but their contribution is outweighed by the virtual gluons that contribute the number 11. The difference is due to the presence of gluon loops in QCD (Fig. 16.2). This property of QCD was discovered by Gross and Wilczek (1973) and Politzer (1973).

Although renormalisation seems to necessitate the introduction of a second, dimensioned, parameter λ, the effective coupling constant is in fact dependent on only one parameter. We can set

$$\frac{1}{g^2} - \frac{1}{16\pi^2}[11 - (2/3)n_f]\ln\lambda^2 = -\frac{1}{16\pi^2}[11 - (2/3)n_\mathrm{f}]\ln\Lambda^2, \qquad (16.24)$$

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thus defining Λ, and then

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{[11 - (2/3)n_{\mathrm{f}}]\ln(Q^2/\Lambda^2)}. \tag{16.25}$$

This remarkable feature survives in all orders of perturbation theory. Higher terms in the expansion of $\alpha_s(Q^2)$ are given in, for example, Particle Data Group (2005).

Λ is well defined in the limit of large Q^2, and it is standard practice to regard the one parameter Λ, rather than the two parameters g and λ, as the fundamental constant of QCD, which must be determined from experiment. It is also interesting to note that we have replaced a dimensionless parameter g by a dimensioned one, Λ. Asymptotic freedom is displayed since $\alpha_s(Q^2) \to 0$ as $Q^2 \to \infty$. It is clear from (16.25) that perturbation theory breaks down at $Q^2 = \Lambda^2$, when the effective coupling constant becomes infinite. Small values of Q^2 are associated with large distances, and the length scale Λ^{-1} is called the *confinement length.*

16.4 The quark–antiquark interaction at short distances

In QED, single photon exchange between an electron and a positron gives the Coulomb potential

$$V(r) = \frac{1}{(2\pi)^3}\int V(Q^2)\mathrm{e}^{-\mathrm{i}Q\cdot r}\mathrm{d}^3Q = \frac{e^2}{4\pi r} = -\frac{\alpha}{r},$$

where $V(Q^2) = -e^2/Q^2$ and α is the fine-structure constant. In QCD perturbation theory, single photon exchange is replaced by the sum of eight single gluon exchanges. To lowest order, the Coulomb-like potential between a quark and an antiquark in a colour singlet state and at a distance r apart may be shown to be (see Leader and Predazzi, 1982, p. 175)

$$V_{\mathrm{QCD}}(r) = -\sum_a \frac{g^2}{4\pi r}\frac{1}{3}\frac{\lambda_{aij}}{2}\frac{\lambda_{aji}}{2} = -\sum_a \frac{g^2}{4\pi r}\frac{1}{12}\mathrm{Tr}(\lambda_a\lambda_a) = -\frac{4}{3}\frac{g^2}{4\pi r}. \tag{16.26}$$

The factor (1/3) is from the normalisation of the colour singlet state (see (16.19)). With quantum corrections, the effective potential at short distances becomes

$$V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s(r)}{r},$$

where

$$\frac{\alpha_s(r)}{r} = \frac{4\pi}{(2\pi)^3}\int\frac{\alpha_s(Q^2)}{Q^2}\mathrm{e}^{-\mathrm{i}Q\cdot r}\mathrm{d}^3Q. \tag{16.27}$$

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This is a significant result for the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ systems, in which the heavy quark and antiquark are slowly moving. In these systems the colour Coulomb energy is the main contribution to the potential energy: colour magnetic effects are of relative order v/c. The behaviour of $\alpha_s(Q^2)$ at large Q^2 gives the dominant contribution to $V_{\mathrm{QCD}}(r)$ at small r (Problem 16.5). We shall return to charmonium and bottomonium in Chapter 17.

16.5 The conservation of quarks

In addition to the $SU(3)$ local colour symmetry, the Lagrangian density (16.11) has six global $U(1)$ symmetries:

$$\mathbf{q}_{\mathrm{f}} \to \mathbf{q}_{\mathrm{f}}{}' = \exp(\mathrm{i}\alpha_{\mathrm{f}})\mathbf{q}_{\mathrm{f}}. \tag{16.28}$$

In the Standard Model these remain global and are not elevated into local gauge symmetries. They imply conservation of quark number for each flavour of quark. Thus the strong interaction does not change quark flavour. Regarding mesons and baryons, the K^+, for example, which can be denoted $K(u\bar{s})$ has u quark number 1 and s quark number -1, the proton P (uud) has u quark number 2 and d quark number 1. Only the weak interaction, as exemplified in weak decays, can change quark flavour. Including the weak interaction, and in particular that part involving the Kobayashi–Maskawa mixing matrix, the six $U(1)$ symmetries reduce to one. Individual quark flavour numbers are not conserved, and only the overall quark number remains constant.

16.6 Isospin symmetry

The estimated masses of the u quark ($1.5\,\mathrm{MeV} < m_{\mathrm{u}} < 4\,\mathrm{MeV}$) and d quark ($4\,\mathrm{MeV} < m_{\mathrm{d}} < 8\,\mathrm{MeV}$) are small compared with those of the s quark ($100\,\mathrm{MeV} < m_{\mathrm{s}} < 300\,\mathrm{MeV}$) and the heavy c, b and t quarks. The masses of the u and d quarks are also small compared with those of the lightest hadrons: the π^0 has a mass $\sim 135\,\mathrm{MeV}$ and the proton has a mass $\sim 938\,\mathrm{MeV}$. At low energies we may therefore neglect all but the u and d quarks, and consider the Lagrangian density to be, as a first approximation,

$$\mathcal{L}_{\mathrm{ud}} = \bar{\mathbf{u}}\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{u} + \bar{\mathbf{d}}\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{d} - m_{\mathrm{u}}\bar{\mathbf{u}}\mathbf{u} - m_{\mathrm{d}}\bar{\mathbf{d}}\mathbf{d} \tag{16.29}$$

where here $\mathbf{G}_\mu$ is the gluon field matrix, evaluated from the field equations (16.13) with all but the $\mathbf{u}$ and $\mathbf{d}$ quark fields neglected. The fields $\mathbf{u}$ and $\mathbf{d}$ in (16.29) are triplets of Dirac fermion fields; colour indices and Dirac indices have been suppressed.

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We now combine the **u** and **d** fields into an *isospin doublet*,

$$\mathbf{D}(x) = \begin{pmatrix} \mathbf{u}(x) \\ \mathbf{d}(x) \end{pmatrix} \tag{16.30}$$

and we can write

$$\mathcal{L}_{\mathrm{ud}} = \bar{\mathbf{D}}\mathrm{i}\gamma^{\mu}(\partial_{\mu} + \mathrm{i}g\mathbf{G}_{\mu})\mathbf{D} - (1/2)(m_{\mathrm{u}} + m_{\mathrm{d}})\bar{\mathbf{D}}\mathbf{D} - (1/2)(m_{\mathrm{u}} - m_{\mathrm{d}})\bar{\mathbf{D}}\tau_3\mathbf{D} \tag{16.31}$$

where

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{D}} = (\mathrm{u}^{+}\gamma^{0}, \mathrm{d}^{\dagger}\gamma^{0}).$$

$\mathcal{L}_{\mathrm{ud}}$ is invariant under a global $U(1)$ transformation

$$\mathbf{D} \to \mathbf{D}' = \exp(-\mathrm{i}\alpha^{0})\mathbf{D}, \tag{16.32}$$

which leads (cf. Section 4.1) to the conserved quark current

$$J^{\mu} = \bar{\mathbf{D}}\gamma^{\mu}\mathbf{D} = \bar{\mathbf{u}}\gamma^{\mu}\mathbf{u} + \bar{\mathbf{d}}\gamma^{\mu}\mathbf{d}. \tag{16.33}$$

It is also invariant under a global $U(1)$ transformation

$$\mathbf{D} \to \mathbf{D}' = \exp(-\mathrm{i}\alpha^{3}\tau^{3})\mathbf{D} \tag{16.34}$$

which leads to the conserved current

$$J_3{}^{\mu} = \bar{\mathbf{D}}\gamma^{\mu}\tau^{3}\mathbf{D} = \bar{\mathbf{u}}\gamma^{\mu}\mathbf{u} - \mathbf{d}\gamma^{\mu}\mathbf{d}. \tag{16.35}$$

(16.33) and (16.35) show that this Lagrangian density (16.31) conserves both u and d quark numbers separately.

So-called *isospin symmetry* appears if we neglect the mass difference $(m_{\mathrm{u}} - m_{\mathrm{d}})$. The resulting, simplified, Lagrangian density is invariant under the global $SU(2)$ transformation

$$\mathbf{D} \to \mathbf{D}' = \exp(-\mathrm{i}\alpha^{k}\tau^{k})\mathbf{D} \tag{16.36}$$

where the τ^k are the generators of the group $SU(2)$ (Appendix B, Section B.3). In addition to the conserved current (16.35) we now have also the conserved currents

$$J_1^{\mu} = \bar{\mathbf{D}}\gamma^{\mu}\tau^{1}\mathbf{D}, \quad J_2^{\mu} = \bar{\mathbf{D}}\gamma^{\mu}\tau^{2}\mathbf{D} \tag{16.37}$$

and the corresponding time-independent quantities

$$\int \mathbf{D}^{\dagger}\tau^{k}\mathbf{D}\,\mathrm{d}^{3}x, \quad k = 1,2,3. \tag{16.38}$$

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$SU(2)$ transformations are equivalent to rotations in a three-dimensional 'isospin space'. In analogy with the intrinsic angular momentum operator $\mathbf{S} = (1/2)\boldsymbol{\sigma}$, we define the isospin operator $\mathbf{I} = (1/2)\boldsymbol{\tau}$; then

$$\mathbf{I}^2 = {I_1}^2 + {I_2}^2 + {I_3}^2 = (3/4)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A u quark state is an eigenstate of $\mathbf{I}^2$ and I_3 with $I = 1/2$, $I_3 = 1/2$, and a d quark state is an eigenstate with $I = 1/2$, $I_3 = -1/2$. The mathematics of isospin is identical to the mathematics of angular momentum, and the formalism of isospin is very useful in understanding and classifying hadron states, as indicated in Chapter 1. We see here its origin in QCD, with the neglect of the u $-$ d mass difference and the electromagnetic and weak interactions.

16.7 Chiral symmetry

If we neglect entirely the quark masses, further approximate symmetries arise. These are of interest in particle physics. The Lagrangian density (16.31) may be written in terms of the left-handed and right-handed isospin doublets $\mathbf{L} = (1/2)(1-\gamma^5)\mathbf{D}$ and $\mathbf{R} = (1/2)(1+\gamma^5)\mathbf{D}$. Neglecting the mass terms it becomes

$$\mathcal{L} = \mathbf{L}^\dagger \mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{L} + \mathbf{R}^\dagger \mathrm{i}\sigma^\mu(\partial_\mu + \mathrm{i}g\mathbf{G}_\mu)\mathbf{R}. \tag{16.39}$$

$\mathbf{L}$ and $\mathbf{R}$ are now doublets of two-component spinors, and there are eight conserved currents:

$$\mathbf{L}^\dagger\tilde{\sigma}^\mu\mathbf{L}, \quad \mathbf{L}^\dagger\tilde{\sigma}^\mu\tau^k\mathbf{L}, \quad \mathbf{R}^\dagger\sigma^\mu\mathbf{R}, \quad \mathbf{R}^\dagger\sigma^\mu\tau^k\mathbf{R}, \quad k = 1, 2, 3.$$

An important observation is that the currents $\mathbf{L}^\dagger\tilde{\sigma}^\mu\tau^1\mathbf{L}$ and $\mathbf{L}^\dagger\tilde{\sigma}^\mu\tau^2\mathbf{L}$ couple to the $W^\pm$ boson fields in the Lagrangian density (14.15), and appear in the effective Lagrangian density (14.22). The relevant quark factor in (14.15) is $\mathbf{u}_\mathrm{L}^\dagger\tilde{\sigma}^\mu\mathbf{d}_\mathrm{L}V_{\mathrm{ud}}$, and we may write

$$\begin{aligned} \mathbf{u}_\mathrm{L}^\dagger\tilde{\sigma}^\mu\mathbf{d}_\mathrm{L} &= \mathbf{L}^\dagger\tilde{\sigma}^\mu(1/2)(\tau^1 + \mathrm{i}\tau^2)\mathbf{L}, \\ \mathbf{d}_\mathrm{L}^\dagger\tilde{\sigma}^\mu\mathbf{u}_\mathrm{L} &= \mathbf{L}^\dagger\tilde{\sigma}^\mu(1/2)(\tau^1 - \mathrm{i}\tau^2)\mathbf{L}. \end{aligned} \tag{16.40}$$

This observation gives insight into the nature of the effective Lagrangian for β decay, as we shall see in Chapter 18.

The independent symmetry transformations

$$\mathbf{L} \to \mathbf{L}' = \exp[-\mathrm{i}(\alpha^0 + \alpha^k\tau^k)]\mathbf{L}, \quad \mathbf{R} \to \mathbf{R}$$

and

$$\mathbf{R} \to \mathbf{R}' = \exp[-\mathrm{i}(\beta^0 + \beta^k\tau^k)]\mathbf{R}, \quad \mathbf{L} \to \mathbf{L}$$

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may be written in terms of Dirac spinors as

$$\mathbf{D} \to \mathbf{D}' = \exp[-\mathrm{i}(\alpha^0 + \alpha^k \tau^k)(1/2)(1-\gamma^5)]\mathbf{D}, \tag{16.41}$$
$$\mathbf{D} \to \mathbf{D}' = \exp[-\mathrm{i}(\beta^0 + \beta^k \tau^k)(1/2)(1+\gamma^5)]\mathbf{D}, \tag{16.42}$$

respectively.

The eight independent symmetry operations can also be taken as

$$\mathbf{D} \to \mathbf{D}' = \exp[-\mathrm{i}(\alpha'^0 + \alpha'^k \tau^k)]\mathbf{D} \tag{16.43}$$

which give conservation of quark number and isospin, and

$$\mathbf{D} \to \mathbf{D}' = \mathbf{exp}[-\mathrm{i}(\beta'^0 + \beta'^k \tau^k)\gamma^5]\mathbf{D} \tag{16.44}$$

The last four are known as the *chiral symmetries.*

Problems

16.1 Show that

$$G^a_{\mu\nu} = (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) - g \sum_{b,c} f_{abc} G^b_\mu G^c_\nu.$$

16.2 Using Problem 16.1, show that the gluon self-coupling terms in the Lagrangian density (16.9) are

$$\mathcal{L}_{\text{int}} = g(\partial_\mu G^a_\nu f_{abc} G^{b\mu} G^{c\nu}) - (g^2/4) f_{abc} f_{ade} G^b_\mu G^c_\nu G^{d\mu} G^{e\nu}.$$

16.3 Verify the expression (16.14) for the current $j^{a\nu}$.

16.4 Estimate the value of Q for which $V(Q^2)$ of equation (16.21) becomes infinite.

16.5 From (16.27) show that

$$\alpha_s(r) = \frac{2}{\pi} \int_0^\infty \alpha_s(x^2/r^2) \frac{\sin x}{x}\, \mathrm{d}x.$$

(Note that the expression (16.25) for $\alpha_s(x^2/r^2)$ is only valid for $x > \Lambda r$, but for small r this range may be anticipated to give the main contribution to the integral.)

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