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A NOTE ON ABSOLUTELY PURE MODULES

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Fieldhouse observed that any finitely presented left R-module P is projective with respect to pure exact sequences, i.e.

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \swarrow & \downarrow & & \\ O & \to & M' & \to & M & \to & M'' & \to & O \end{array}$$

can always be completed to a commutative diagram when the sequence is pure exact. A left R-module A is absolutely pure if it is a pure submodule of every module which contains it. Hence $\mathrm{Ext}^1_R(P, A)=0$ if P is finitely presented and A is absolutely pure.

The object of this note is to show that finitely presented modules P are characterized among the finitely generated modules by the property that $\mathrm{Ext}^1_R(P, A)=0$ for all absolutely pure modules A. If $0\to S\to F\to P\to 0$ is exact where F is free on a finite number of generators then $\mathrm{Ext}^1_R(P, A)=0$ if and only if

$$\begin{array}{ccc} S & \to & F \\ \downarrow & \swarrow & \\ A & & \end{array}$$

can always be completed to a commutative diagram. P is finitely presented if and only if S is finitely generated. Hence it suffices to prove that if the diagram can always be completed to a commutative diagram then S must be finitely generated. This turns out to be true even if we drop the hypothesis that F be free and only require that it be finitely generated.

PROPOSITION. *If $S\subset M$ is a submodule of a finitely generated module M, then if for every absolutely pure module A and any linear map $S\to A$*

$$\begin{array}{ccc} S & \to & M \\ \downarrow & \swarrow & \\ A & & \end{array}$$

can be completed to a commutative diagram, S is finitely generated.

Proof. Suppose S is not finitely generated. Let $X\subset S$ be a set of generators of least possible cardinality. Well order X so that X has no largest element and so that every section has cardinality smaller than the cardinality of X. For each

$x \in X$ consider the module. $\sum_{y \le x} Ry / \sum_{y \le x} Ry$. Imbed this module in an injective module I_x. Extend the obvious linear map $\sum_{y \le x} Ry \to I_x$ to a linear map $S \to I_x$. Let $G \subset \prod I_x$ consist of all families $(i_x) x \in X$ such that for some $z \in X$, $i_x = 0$ if $x \ge z$. Then G is the union of a collection of direct summands of $\prod_{x \in X} I_x$ which is filtering to the right; and hence G is a pure submodule. Since $\prod I_x$ is injective, G is absolutely pure. Also the image of S is contained in G.

By hypothesis the linear map $S \dashrightarrow G$ can be extended to a linear map $M \leftarrow G$. M is finitely generated and G is the union of the submodules $\prod_{y \le x} I_y$ for $x \in X$ so that the image of M would be contained in one of the submodules. But if the image of S were in $\prod_{y \le x} I_y$, S would be generated by the section of X determined by x. This contradicts the choice of X and the order imposed on it.

References

1. B. H. Maddox, *Absolutely Pure Modules*, Proc. Amer. Math. Soc. **18** (1967) 155–158.
2. C. Megibben, *Absolutely Pure Modules*, Proc. Amer. Math. Soc. **26** (1970) 561–566.

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