Proceedings of the Edinburgh Mathematical Society (2024) **66**, 1110–1118
doi:10.1017/S0013091523000615

A WEAKENED MARKUS–YAMABE CONDITION FOR PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE $(1, n)$

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(Received 9 May 2023)

Abstract For a general autonomous planar polynomial differential system, it is difficult to find conditions that are easy to verify and which guarantee global asymptotic stability, weakening the Markus–Yamabe condition. In this paper, we provide three conditions that guarantee the global asymptotic stability for polynomial differential systems of the form $x' = f_1(x, y)$, $y' = f_2(x, y)$, where f_1 has degree one, f_2 has degree $n \geq 1$ and has degree one in the variable y. As a consequence, we provide sufficient conditions, weaker than the Markus–Yamabe conditions that guarantee the global asymptotic stability for any generalized Liénard polynomial differential system of the form $x' = y$, $y' = g_1(x) + yg_2(x)$ with g_1 and g_2 polynomials of degrees n and m, respectively.

Keywords: global asymptotic stability; Markus–Yamabe conjecture; planar polynomial vector fields

2010 Mathematics subject classification: Primary 34C05

1. Introduction and statement of the main results

Since the time of Liapunov, it has become evident that finding conditions that guarantee global asymptotic stability of an equilibrium point in a differential system, even in two dimensions, is a difficult problem. Liapunov's approach is probably the most widespread general method used, though constructing a Liapunov function usually requires ingenuity, experience and some luck. For the two-dimensional autonomous system

$$x' = f_1(x, y), \qquad y' = f_2(x, y), \tag{1}$$

with $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$, we seek for a set of easily verifiable conditions on the function f which can give global asymptotic stability. A result to this end was proven in

1993, the so-called Markus–Yamabe conjecture in two dimensions (see [8, 10, 11]). This result shows that the global asymptotic stability is obtained if the eigenvalues of the Jacobian matrix $Df(x, y)$ have negative real part for all $(x, y) \in \mathbb{R}^2$. We remark that the Markus–Yamabe conjecture holds in the positive sense in $\mathbb{R}^2$ (see [8, 10, 11]), but it does not hold in $\mathbb{R}^n$ with $n > 2$, see [1, 5].

The aim of this paper is to weaken the Markus–Yamabe condition and still obtain global asymptotic stability for some classes of differential systems (1). The Markus–Yamabe condition ensures the global asymptotic stability, provided that the differential system has a unique equilibrium point, the trace of Df is negative ($\mathrm{Tr}Df < 0$) and the determinant of Df is positive ($\det Df > 0$) for all $(x, y) \in \mathbb{R}^2$. The trace condition guarantees that each region of finite area shrinks under the flow, while the determinant has a priori no known geometric interpretation. Several results (see [2, 9, 12, 13]) weaken the Markus–Yamabe condition by replacing the determinant condition by other conditions. These new requirements on the equilibrium point seem unremovable because they are necessary for the global asymptotic stability and they are easy to verify. Therefore, guided by the results in [4] for polynomial differential systems of degrees 2 and 7 in the plane (we recall that the degree of a polynomial map f is n if the components of f are polynomials of degree at most n), we consider the following open problem.

Open problem. *Assume that* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a polynomial map of degree n and satisfies the following conditions:*

- (c1) Tr $(Df) < 0$ *for all* $(x, y) \in \mathbb{R}^2$;
- (c2) *The differential equation (1) has a unique equilibrium point* $p \in \mathbb{R}^2$.
- (c3) *The equilibrium point p is locally asymptotically stable.*

Which is the largest family of planar differentiable systems (1) for which the assumptions (c1)–(c3) *imply that p is globally asymptotically stable.*

In view of [4], any planar differential system $(x, y)' = f(x, y)$ with f being a polynomial map of degree two satisfying conditions (c1)–(c3) imply globally asymptotical stability. On the other hand, in view of [3], there are polynomial differential systems $(x, y)' = f(x, y)$ of degree seven (i.e. f has degree seven) for which conditions (c1)–(c3) do not imply globally asymptotical stability. In this paper, we will consider the polynomial differential systems

$$x' = a_{1,0}x + a_{0,1}y, \qquad y' = \sum_{j=1}^{n} b_{j,0}x^j + y\sum_{i=0}^{m} b_{i,1}x^i, \tag{2}$$

with $m, n \geq 1$.

The following is our main result.

Theorem 1. *Every planar polynomial differential system (2) satisfying conditions* (c1)–(c3) *is globally asymptotically stable.*

As a corollary, we obtain the following result.

Corollary 2. *Any generalized Liénard polynomial differential system*

$$x' = y, \qquad y' = \sum_{j=1}^{n} b_{j,0} x^j + y \sum_{i=0}^{m} b_{i,1} x^i,$$

with n, m integers satisfying conditions (c1)–(c3) *is globally asymptotically stable.*

Theorem 1 is proven in § 2. By taking $a_{1,0}$ and $a_{0,1} = 1$, Corollary 2 follows.

2. Proof of Theorem 1

The case $n = 1$ is trivial, and the case $n = 2$ was proved in [4]. So in this paper, we consider the case $n \geq 3$.

The proof of Theorem 1 is divided into different cases. We first need Theorem 3 (see below) that provides the local phase portraits of semi-hyperbolic equilibrium points for planar polynomial differential equations. See for instance [7, Theorem 2.19] for a proof of Theorem 1.

Theorem 3. *Let* $(0,0)$ *be an isolated equilibrium point of the planar polynomial differential system*

$$x' = F(x,y), \qquad y' = y + G(x,y),$$

with F and G being polynomials without constant and linear terms in the variables x and y. Let $y = g(x)$ be the solution of $y' = y + G(x,y) = 0$ and assume that $F(x, g(x)) = a_m x^m + \cdots$, where $m \geq 2$ and $a_m \neq 0$. Then,

(i) If m is odd and $a_m > 0$, then $(0,0)$ is an unstable node.
(ii) If m is odd and $a_m < 0$, then $(0,0)$ is a saddle.
(iii) If m is even, then $(0,0)$ is a saddle node.

2.1. Case 1: $a_{0,1} = 0$

In this case, system (2) becomes

$$x' = a_{1,0} x = f_1(x,y), \qquad y' = \sum_{j=1}^{n} b_{j,0} x^j + y \sum_{i=0}^{m} b_{i,1} x^i = f_2(x,y). \tag{3}$$

The divergence of this system is

$$\text{div} = \text{Tr} Df = a_{1,0} + \sum_{i=0}^{m} b_{i,1} x^i.$$

Imposing the condition (c1), i.e., $\text{Tr} Df < 0$ for all $(x,y) \in \mathbb{R}^2$, we must have that m is even and

$$a_{1,0} + b_{0,1} < 0, \qquad a_{1,0} + \sum_{i=0}^{m} b_{i,1} x^i = b_{m,1} \prod_{i=1}^{m/2} \left((x - \alpha_i)^2 + \beta_i^2 \right), \quad b_{m,1} < 0,$$

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with $\alpha_i, \beta_i \in \mathbb{R}$ and $\beta_i \neq 0$. So, system (3) becomes

$$x' = a_{1,0}x, \qquad y' = (b_{0,1} - a_{1,0})y + \sum_{j=1}^{n} b_{j,0}x^j + y b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2\right). \tag{4}$$

Note that if $b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}(\alpha_i^2 + \beta_i^2) = 0$, then the line $x = 0$ is filled by equilibria and so this is not possible. So $b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}(\alpha_i^2 + \beta_i^2) \neq 0$ and $(0,0)$ is the unique equilibrium point of system (4) yielding that condition (c2) is satisfied.

On the other hand, the matrix $Df(0,0)$ has eigenvalues $a_{1,0}$ and $b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}(\alpha_i^2 + \beta_i^2)$. Imposing condition (c3) and taking into account that $b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}(\alpha_i^2 + \beta_i^2) \neq 0$, we must have $a_{1,0} < 0$ and $b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}(\alpha_i^2 + \beta_i^2) < 0$, in which case the origin is a stable node.

Now, we shall prove that the origin is globally asymptotically stable. Any solution of Equation (4) with initial condition (x_0, y_0) is given by $(x(y), y(t))$ with $x(t) = x_0\,\mathrm{e}^{a_{1,0}t}$ and $y(t)$ can be computed, thanks to the variation of constant method, that is, setting

$$a(t) = b_{0,1} - a_{1,0} + b_{m,1}\prod_{i=1}^{m/2}\left((x_0\,\mathrm{e}^{a_{1,0}t} - \alpha_i)^2 + \beta_i^2\right),$$

we get,

$$y(t) = y_0\,\mathrm{e}^{\int_{t_0}^{t} a(s)\,\mathrm{d}s} + \sum_{i=0}^{n} b_{i,0}x_0^i \int_{t_0}^{t} \mathrm{e}^{a_{1,0}i\tau}\mathrm{e}^{\int_{\tau}^{t} a(s)\,\mathrm{d}s}\,\mathrm{d}\tau, \quad t \geq t_0.$$

Since $a_{1,0} < 0$ and $x_0\mathrm{e}^{a_{1,0}t} \to 0$ when $t \to +\infty$, there exists $T > 0$ such that for $t \geq T$, we get

$$a(t) < \frac{1}{2}\left(b_{0,1} - a_{1,0} + b_{m,1}\prod_{j=0}^{m/2}(\alpha_j^2 + \beta_j^2)\right) =: a^* < 0.$$

Then, for $t, \tau \geq T$, we have $0 \leq \mathrm{e}^{\int_{\tau}^{t} a(s)\,\mathrm{d}s} \leq \mathrm{e}^{a^*(t-\tau)}$ and $\mathrm{e}^{a_{1,0}i\tau}\mathrm{e}^{\int_{\tau}^{t} a(s)\,\mathrm{d}s} \leq \mathrm{e}^{a_{1,0}i\tau + a^*(t-\tau)}$. Therefore, for $t, t_0 \geq T$ and $t \geq t_0$,

$$0 \leq |y(t)| \leq |y_0|\mathrm{e}^{a^*(t-t_0)} + \sum_{i=0}^{n} \frac{|b_{i,0}x_0^i|}{|a_{1,0}i - a^*|}\left(\mathrm{e}^{a_{1,0}it} - \mathrm{e}^{a_{1,0}it_0 + a^*(t-t_0)}\right)$$

if $a^* \neq i a_{1,0}$ for any $i = 0, 1, \ldots, n$, and

$$\begin{aligned} 0 \leq & |y(t)| \leq |y_0|\mathrm{e}^{a^*(t-t_0)} + |b_{i^*,0}x_0^{i^*}|(t-t_0)\mathrm{e}^{a^*t} \\ & + \sum_{i=0, i\neq i*}^{n} \frac{|b_{i,0}x_0^i|}{|a_{1,0}i - a^*|}\left(\mathrm{e}^{a_{1,0}it} - \mathrm{e}^{a_{1,0}it_0 + a^*(t-t_0)}\right) \end{aligned}$$

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if there exists $i^* \in \{0, 1, \ldots, n\}$ such that $a^* = i^* a_{1,0}$.

In both cases using that $a_{1,0} < 0$ and $a^* < 0$, we get that any solution $(x(t), y(t))$ with initial condition (x_0, y_0) at time t_0 tends to the origin as $t \to +\infty$ and so the origin is globally asymptotically stable. The proof of the theorem is proved in this case.

2.2. Case 2: $a_{0,1} \neq 0$

In this case, introducing the change of variables:

$$X = x, \qquad Y = a_{1,0}x + a_{0,1}y,$$

Equation (2) writes

$$X' = Y, \qquad Y' = a_{1,0}Y + a_{0,1}\sum_{i=1}^{n} b_{i,0}X^i + (Y - a_{1,0}X)\sum_{i=0}^{m} b_{i,1}X^i,$$

which also writes

$$x' = y, \qquad y' = a_{1,0}y + \sum_{i=1}^{p} \tilde{b}_{i,0}x^i + y\sum_{i=0}^{m} b_{i,1}x^i, \tag{5}$$

where in order to avoid cumbersome notation, we have renamed the variables $(X,\ Y)$ again as $(x,\ y)$ and the new parameters $\tilde{b}_{i,0} = a_{0,1}b_{i,0} - a_{1,0}b_{i-1,1}$ for $i = 1, \ldots, p$, where

$$p = \begin{cases} n, & \text{if } n \geq m+1 \text{ or } a_{1,0} = 0, \\ m+1, & \text{if } n \leq m+1 \text{ and } a_{1,0} \neq 0. \end{cases}$$

Taking into account that

$$\mathrm{Tr}Df = a_{1,0} + \sum_{i=0}^{m} b_{i,1}x^i$$

and imposing condition (c1), i.e. $\mathrm{Tr}Df < 0$ for all $(x, y) \in \mathbb{R}^2$, we must have that m is even and

$$a_{1,0} + b_{0,1} < 0, \qquad a_{1,0} + \sum_{i=0}^{m} b_{i,1}x^i = b_{m,1}\prod_{i=1}^{m/2}\left((x-\alpha_i)^2 + \beta_i^2\right), \quad b_{m,1} < 0,$$

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with α_i, $\beta_i \in \mathbb{R}$ and $\beta_i \neq 0$. Hence, system (5) becomes

$$\begin{aligned} x' &= y, \\ y' &= \sum_{i=1}^{p} \tilde{b}_{i,0} x^i + y b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2 \right) \\ &= Ay + \sum_{i=1}^{p} \tilde{b}_{i,0} x^i + y \left(b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2 \right) - A \right), \end{aligned} \tag{6}$$

where $A = b_{m,1} \prod_{i=1}^{m/2} (\alpha_i^2 + \beta_i^2)$ and $b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2 \right) - A$ has no constant terms.

Case 2.1: p even

Note that if $\tilde{b}_{10} \neq 0$, then Equation (6) has always an equilibrium point besides the origin, taking into account that

$$y'|_{y=0} = x \sum_{i=0}^{p-1} \tilde{b}_{i+1,0} x^i,$$

with $\tilde{b}_{1,0} \tilde{b}_{p,0} \neq 0$ and $p-1$ being odd (equation $\sum_{i=0}^{p-1} \tilde{b}_{i+1,0} x^i = 0$ has always a real solution different from $x = 0$). Therefore, in order that condition (c2) is fulfilled, we must have $\tilde{b}_{1,0} = 0$. In this case, system (6) becomes

$$x' = y, \qquad y' = Ay + \sum_{i=2}^{p} \tilde{b}_{i,0} x^i + y \left(b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2 \right) - A \right). \tag{7}$$

In this case, the origin is semi-hyperbolic, and in order to apply Theorem 3, we must write Equation (7) in canonical Jordan form. For doing this, we apply the change of variables $X = x - Y/A$, $Y = y$, and system (7) becomes

$$\begin{aligned} X' &= -\frac{1}{A} \left(\sum_{i=2}^{p} \tilde{b}_{i,0} \left(X + \frac{Y}{A} \right)^i + Y b_{m,1} \prod_{i=1}^{m/2} \left(\left(X + \frac{Y}{A} - \alpha_i \right)^2 + \beta_i^2 \right) \right), \\ Y' &= AY + \sum_{i=2}^{p} \tilde{b}_{i,0} \left(X + \frac{Y}{A} \right)^i + Y \left(b_{m,1} \prod_{i=1}^{m/2} \left(\left(X + \frac{Y}{A} - \alpha_i \right)^2 + \beta_i^2 \right) - A \right). \end{aligned}$$

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Now, after rescaling by the time variable $\mathrm{d}s = A\,\mathrm{d}t$, and using Newton's binomial formula, we obtain the system in canonical normal form:

$$X' = -\sum_{i=2}^{p} \frac{\tilde{b}_{i,0}}{A^2} \sum_{j=0}^{i} \binom{i}{j} \frac{X^j Y^{i-j}}{A^{i-j}} - Y \left(\frac{b_{m,1}}{A^2} \prod_{i=1}^{m/2} \left(\left(X + \frac{Y}{A} - \alpha_i \right)^2 + \beta_i^2 \right) - \frac{1}{A} \right),$$
$$Y' = Y + \sum_{i=2}^{p} \frac{\tilde{b}_{i,0}}{A} \sum_{j=0}^{i} \binom{i}{j} \frac{X^j Y^{i-j}}{A^{i-j}} + \frac{Y}{A} \left(b_{m,1} \prod_{i=1}^{m/2} \left(\left(X + \frac{Y}{A} - \alpha_i \right)^2 + \beta_i^2 \right) - A \right), \tag{8}$$

where now the prime means derivative in the new time s (note that since $A < 0$, the original system in time t changes the direction of the orbits). Applying Theorem 3, we get that

$$Y = -\tilde{b}_{2,0} X^2/A + \cdots \text{ and then } F(X,Y) = -\tilde{b}_{2,0} X^2/A^2 + \cdots,$$

implying that the origin $(0,0)$ is a saddle-node, which is not possible. So in order that condition (c3) is fulfilled, we must have $\tilde{b}_{2,0} = 0$, but then system (7) would be of the form

$$x' = y, \qquad y' = Ay + x^3 \sum_{i=0}^{p-3} \tilde{b}_{i+3,0} x^i + y \left(b_{m,1} \prod_{i=1}^{m/2} \left((x - \alpha_i)^2 + \beta_i^2 \right) - A \right),$$

and the equation $\sum_{i=0}^{p-3} \tilde{b}_{i+3,0} x^i = 0$ has a real solution different from $x = 0$ unless $\tilde{b}_{3,0} = 0$, but then again applying Theorem 3, we get that

$$Y = -\tilde{b}_{4,0} X^4/A + \cdots, \text{ and then } F(X,Y) = -\tilde{b}_{4,0} X^4/A^2 + \cdots,$$

implying that the origin $(0,0)$ is a saddle-node, which is not possible. Proceeding inductively, we conclude that $\tilde{b}_{i,0} = 0$ for $i \geq 1$, and in this case, taking into account that p is even, we get that the origin is a saddle-node and condition (c3) is not satisfied. In short, in this case, no system satisfies conditions (c1)–(c3) and there is nothing to prove.

Case 2.2: p odd

Note that if $\tilde{b}_{10} \neq 0$, the matrix $Df(0,0)$ has eigenvalues:

$$\lambda_{\pm} = \frac{1}{2} \left(A \pm \sqrt{A^2 + 4\tilde{b}_{1,0}} \right),$$

and imposing condition (c3), we must have $\tilde{b}_{1,0} < 0$, and since $A < 0$, the origin is a stable node if $A^2 + 4\tilde{b}_{1,0} \geq 0$ and a stable focus if $A^2 + 4\tilde{b}_{1,0} < 0$.

If $\tilde{b}_{1,0} = 0$, then we get system (7) and the origin is semi-hyperbolic. Proceeding as for Case 2.1, we get that if $\tilde{b}_{2,0} \neq 0$, then the origin is a saddle-node, which is not possible.

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So, $\tilde{b}_{2,0} = 0$. Now applying Theorem 3 to system (8), we get that $Y = -\tilde{b}_{3,0}X^3/A + \cdots$ and then $F(X,Y) = -\tilde{b}_{3,0}X^3/A^2 + \cdots$, implying that, if $\tilde{b}_{3,0} \neq 0$, for the origin to be a node, we must have $\tilde{b}_{3,0} < 0$. If $\tilde{b}_{3,0} = 0$, then in order that condition (c3) is fulfilled, we must have $\tilde{b}_{4,0} = 0$ and so on. In short, in order that condition (c3) if fulfilled for system (7), there must exists an odd integer $i^* \in \{1, \ldots, n\}$ for which $\tilde{b}_{i^*,0} < 0$ and system (7) writes

$$x' = y, \qquad y' = Ay + x^{i^*} \sum_{i=0}^{p-i^*} \tilde{b}_{i+i^*,0}x^i + y \left(b_{m,1} \prod_{i=1}^{m/2} \left((x-\alpha_i)^2 + \beta_i^2\right) - A \right).$$

Note that $p - i^*$ is even and that equation $\sum_{i=0}^{p-i^*} \tilde{b}_{i+i^*,0}x^i = 0$ cannot have a real solution (otherwise condition (c2) is not satisfied). Taking this into account, we can write

$$\sum_{i=0}^{p-i^*} \tilde{b}_{i+i^*,0}x^i = \tilde{b}_{p,0} \prod_{k=1}^{(p-i^*)/2} (x^2 - 2\tilde{\alpha}_k x + (\tilde{\alpha}_k^2 + \tilde{\beta}_k^2)), \quad \text{for some } \tilde{\alpha}_k, \tilde{\beta}_k \in \mathbb{R} \text{ with } \tilde{\beta}_k \neq 0.$$

In this case, the origin of system (6) is the unique equilibrium point of system (6) and taking $\tilde{b}_{i^*,0} < 0$ and $A < 0$, all conditions (c1)–(c3) are satisfied. Note that

$$\tilde{b}_{i^*,0} = \tilde{b}_{p,0} \prod_{k=1}^{(p-i^*)/2} (\alpha_k^2 + \beta_k^2),$$

and since $\tilde{b}_{i^*,0} < 0$, this implies that $\tilde{b}_{p,0} < 0$.

We recall that p is odd, m is even and $b_{p,0} < 0$. A study (see [6]) of such system in a neighbourhood of the origin on the Poincaré sphere forms the backbone of the proof. In this case system, (6) is a generalized Liénard differential system with p odd, m even, $p > m+1$ and $\tilde{b}_{p,0} < 0$. We can indeed make the rescaling

$$x = \alpha X, \quad y = \beta Y, \quad t = \gamma s, \quad \alpha = \left(\frac{b_{m,1}^2}{\tilde{b}_{p,0}} \right)^{1/(p-1-2m)}, \quad \gamma = -\frac{1}{b_{m,1}\alpha^m}, \quad \beta = \frac{\alpha}{\gamma},$$

and system (6) becomes

$$X' = Y, \qquad Y' = AY + X^{i^*} \sum_{i=0}^{p-i^*-1} \hat{b}_{i+i^*,0}X^i - X^p + Y \left(\hat{b}_{m,1} \prod_{i=1}^{m/2} \left((X-\hat{\alpha}_i)^2 + \hat{\beta}_i^2\right) - A \right), \tag{9}$$

where $\hat{b}_{2i,1}$, $\hat{b}_{m,1}$, $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the new parameters, and the coefficient of YX^m is -1.

Note that now system (9) is a generalized Liénard differential system with p odd, m even, and the coefficient of X^p equal to -1. Using the differential system (3) of [6] with $\varepsilon = 1$, m odd and n even (in the notation of [6]), we get that the infinity of system (9)

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must be one of the following five infinities described in the phase portraits of Figure 5(3), Figure 6(1), Figure 7(3), Figure 8(6) and Figure 9(3).

From these previous figures, we get that either the infinity is a repeller (there are orbits which come from infinity, but there are no orbits going to infinity) in Figure 5(3), Figure 6(1) and Figure 7(3) or there are no orbits going or coming from infinity in Figure 8(6) and Figure 9(3).

Note that since the divergence of the system is negative (condition (c1)), thanks to Poincaré-Bendixson Theorem (see for instance [7, Theorem 7.10]), no periodic orbit exist and conditions (c1)–(c3) together with the behaviour at infinity previously described imply that the origin is globally asymptotically stable. This concludes the proof of Theorem 1.

Funding Statement. The first author is partially supported by the Agencia Estatal de Investigación grant no. PID2019-104658GB-I00, the H2020 European Research Council grant no. MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant no. 2022-SGR 00113, and the Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

Competing Interests. The authors have no competing interests.

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