A RELATION BETWEEN LAPLACE AND HANKEL TRANSFORMS

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The Laplace transform of a function $f(t) \in L(0, \infty)$ is defined by the equation

$$\mathscr{L}\{f;\ p\} = \int_0^\infty e^{-pt} f(t)\, dt \quad (\mathscr{R}(p) > 0), \tag{1}$$

and its Hankel transform of order ν is defined by the equation

$$\mathscr{H}_\nu\{f;\ \xi\} = \int_0^\infty t f(t) J_\nu(\xi t)\, dt \quad (\xi > 0). \tag{2}$$

The object of this note is to obtain a relation between the Laplace transform of $t^\mu f(t)$ and the Hankel transform of $f(t)$, when $\mathscr{R}(\mu) > -1$. The result is stated in the form of a theorem which is then illustrated by an example.

THEOREM. *If f and $\mathscr{H}_\nu\{f; \xi\}$ belong to $L(0, \infty)$ and if $\mathscr{R}(\mu) > -1$, $\mathscr{R}(\mu+\nu) > -1$, $\mathscr{R}(p) > 0$, then*

$$\mathscr{L}\{t^\mu f(t);\ p\} = \int_0^\infty k(p, \xi) \mathscr{H}_\nu\{f;\ \xi\}\, d\xi,$$

where

$$k(p, \xi) = \Gamma(\mu+\nu+1)\xi(p^2+\xi^2)^{-\frac{1}{2}(\mu+1)} P_\mu^{-\nu}\left(\frac{p}{\sqrt{(p^2+\xi^2)}}\right).$$

Proof. Since $f \in L(0, \infty)$ we have, by the Hankel inversion theorem [**2**, p. 52], that

$$f(t) = \int_0^\infty \xi \mathscr{H}_\nu\{f;\ \xi\} J_\nu(t\xi)\, d\xi.$$

Hence

$$\mathscr{L}\{t^\mu f(t)\} = \int_0^\infty \xi \mathscr{H}_\nu\{f;\ \xi\} \mathscr{L}\{t^\mu J_\nu(t\xi)\}\, d\xi;$$

the change of order of integration is justified because $e^{-pt} t^\mu \in L(0, \infty)$ if $\mathscr{R}(\mu) > -1$, $\mathscr{R}(p) > 0$ and $\mathscr{H}_\nu\{f, \xi\} \in L(0, \infty)$, $J_\nu(t\xi)$ being a bounded function of both the variables. The theorem then follows from the fact that

$$\mathscr{L}\{t^\mu J_\nu(t\xi);\ p\} = \Gamma(\mu+\nu+1)(p^2+\xi^2)^{-\frac{1}{2}(\mu+1)} P_\mu^{-\nu}\left(\frac{p}{\sqrt{(p^2+\xi^2)}}\right) \tag{3}$$

[**2**, p. 182].

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As an example of the use of this result, let $f(t) = t^{m-1}e^{-\alpha t}$. Then

$$\mathscr{L}\{t^{\mu}f(t); p\} = (p+\alpha)^{-m-\mu}\Gamma(m+\mu)$$

and

$$\mathscr{H}_{\nu}\{f; \xi\} = \int_0^{\infty} t^m e^{-\alpha t} J_{\nu}(t\xi)\, dt = \mathscr{L}\{t^m J_{\nu}(t\xi); \alpha\}.$$

This last integral can be evaluated by using (3). Substituting these expressions in the theorem we obtain the result

$$\int_0^{\infty} \xi(p^2+\xi^2)^{-\frac{1}{2}(\mu+1)}(\alpha^2+\xi^2)^{-\frac{1}{2}(m+1)} P_{\mu}^{-\nu}\left(\frac{p}{\sqrt{(p^2+\xi^2)}}\right) P_m^{-\nu}\left(\frac{\alpha}{\sqrt{(\alpha^2+\xi^2)}}\right) d\xi = \frac{\Gamma(m+\mu)(p+\alpha)^{-m-\mu}}{\Gamma(\mu+\nu+1)\Gamma(m+\nu+1)},$$

where $\alpha > 0$, $\mathscr{R}(p) > 0$, $\mathscr{R}(\mu) > -\frac{1}{2}$, $\mathscr{R}(m+\mu) > 0$, $\mathscr{R}(\mu+\nu) > -1$, $\mathscr{R}(m+\nu) > -1$.

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REFERENCES

1. I. N. Sneddon, *Fourier transforms* (McGraw-Hill, New York, 1951).
2. A. Erdélyi, editor, *Tables of integral transforms* (McGraw-Hill, New York, 1954), vol. 1.

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