J. Austral. Math. Soc. (*Series A*) **68** (2000), 55–67

AN L^p VERSION OF THE HARDY THEOREM FOR MOTION GROUPS

MASAAKI EGUCHI*, SHIN KOIZUMI* and KEISAKU KUMAHARA

(Received 3 May 1998; revised 26 April 1999)

Communicated by A. H. Dooley

Abstract

We describe a generalization of the Hardy theorem on the motion group. We prove that for some weight functions v, w growing very rapidly and a measurable function f, the finiteness of the L^p-norm of vf and the L^q-norm of $w\hat{f}$ implies $f = 0$ (almost everywhere).

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 43A30.
Keywords and phrases: uncertainty principle, Hardy theorem, motion group.

1. Introduction

The classical uncertainty principle asserts that a function and its Fourier transform cannot both be concentrated on intervals of small measure. In the case of the Euclidean space, various forms of the uncertainty principle are known. For example, it is known that if $f \in L^2(\mathbb{R}^n)$ and the supports of f and its Fourier transform $\hat{f}$ are bounded then $f = 0$ (almost everywhere). Benedicks generalized this result to the case when both $f \in L^p(\mathbb{R}^n)$ and $\hat{f}$ vanish outside sets of finite Lebesgue measure [1, Proposition 8]. For another example, the Hardy theorem [3, pp. 155–158] yields that if a measurable function f on $\mathbb{R}$ satisfies $|f| \leq C\exp(-ax^2)$ and $|\hat{f}| \leq C\exp(-by^2)$ and $ab > 1/4$ then $f = 0$ (almost everywhere). Here we take $\hat{f}(y) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)\exp(\sqrt{-1}xy)\,dx$ as the definition of the Fourier transform of f. More generally, Cowling and Price [2] obtained the following L^p version of the Hardy theorem: Suppose that $1 \leq p, q \leq \infty$ and one of them is finite. If a measurable function f on $\mathbb{R}$ satisfies $\|\exp(ax^2)f\|_{L^p(\mathbb{R})} < \infty$ and $\|\exp(by^2)\hat{f}\|_{L^q(\mathbb{R})} < \infty$ and $ab \geq 1/4$ then $f = 0$ (almost everywhere).

*Partly supported by the Grants-in-Aid for Scientific Research (No. 09640190).

Recently, Sitaram and Sundari [7] and Sundari [8] obtained generalizations of the Hardy theorem to some types of Lie groups. In [4], by applying similar arguments to [7], we also got the Hardy theorem for the Cartan motion group. In this note, we shall prove an L^p version of the Hardy theorem for the motion group, which is similar to that of Cowling and Price. In accordance with [5], by the motion group is meant the semidirect product $K \ltimes V$ of a real vector space V and a connected compact Lie group K acting orthogonally on V. In [6], Kumahara defined the Fourier transform on the motion group by using the representations induced from the characters of V and gave the characterization of the image of the Schwartz space under the Fourier transform. By the way, L^p functions on the motion group can be regarded as tempered distributions in the same fashion as in the case of the Euclidean space. These facts allow us to compute the Fourier transforms of L^p functions in such a way as tempered distributions. With the help of these and similar arguments to [4], we can get an L^p version of the Hardy theorem for the motion group.

2. Notation and preliminaries

The standard symbols $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ shall be used for the integers, the real numbers and the complex numbers. If U is a vector space over $\mathbb{R}$, U_c, U^* and U_c^* denote its complexification, its real dual and its complex dual, respectively. For $v \in U_c$, $\Re v$, $\Im v$ and $\bar{v}$ denote its real part, its imaginary part and its complex conjugate, respectively. For a Lie group L, $\hat{L}$ denotes the set of equivalence classes of irreducible unitary representations of L. As usual, we use lower case German letters to denote the corresponding Lie algebras and upper case German letters to denote their universal enveloping algebras.

If $\mathscr{H}$ is a complex separable Hilbert space, $\mathbf{B}(\mathscr{H})$ denotes the Banach space comprised of all bounded operators on $\mathscr{H}$ with operator norm $\|\cdot\|_\infty$. For $T \in \mathbf{B}(\mathscr{H})$ and $1 \le p < \infty$, we indicate its Schatten p-norm by $\|T\|_p$, that is, $\|T\|_p = \left(\operatorname{tr}(T^*T)^{p/2}\right)^{1/p}$, T^* being the adjoint operator of T. For a complex separable Hilbert space $\mathscr{H}$ and a σ-finite measure space (X, μ), we denote by $L^p(X, \mathbf{B}(\mathscr{H}))$ the Banach space comprised of all $\mathbf{B}(\mathscr{H})$-valued L^p functions on X. Here the L^p-norm $\|F\|_{L^p(X,\mathbf{B}(\mathscr{H}))}$ of $F \in L^p(X, \mathbf{B}(\mathscr{H}))$ is given by the following:

$$(2.1)\qquad \|F\|_{L^p(X,\mathbf{B}(\mathscr{H}))} = \left(\int_X \|F(x)\|_p^p \, d\mu(x)\right)^{1/p}, \quad (1 \le p < \infty),$$

$$\|F\|_{L^\infty(X,\mathbf{B}(\mathscr{H}))} = \operatorname*{ess\,sup}_{x \in X} \|F(x)\|_\infty.$$

As is well-known, $L^2(X, \mathbf{B}(\mathscr{H}))$ becomes a Hilbert space with inner product

$$(2.2)\qquad \langle F, G\rangle_{L^2(X,\mathbf{B}(\mathscr{H}))} = \int_X \operatorname{tr}\left(G(x)^* F(x)\right) d\mu(x), \quad (F, G \in L^2(X, \mathbf{B}(\mathscr{H}))).$$

Let V be a finite-dimensional vector space over $\mathbb{R}$ with inner product $(\cdot,\cdot)$ and its corresponding norm $|\cdot|$. We use the same symbols $(\cdot,\cdot)$ and $|\cdot|$ for the bilinear form and the norm on V^* induced by those on V. If $\xi \in V^*$, we define $X_\xi \in V$ by $\xi(X) = (X_\xi, X)$ for all $X \in V$. Let K be a connected compact Lie group acting orthogonally on V. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the canonical inner product and the corresponding norm on $L^2(K)$, respectively. In the situation described above, we call the semidirect product $G = K \ltimes V$ the motion group. For any $g \in G$, we write $g = (k(g), X(g))$, where $k(g) \in K$ and $X(g) \in V$. If $k \in K$ and $X \in V$, we simply write k for $(k, 0)$ and X for (e, X), e being the unit element in K. With these notations, we have

$$k(g^{-1}) = k(g)^{-1}, \qquad X(g^{-1}) = -k(g)^{-1}X(g),$$
$$k(g_1g_2) = k(g_1)k(g_2), \qquad X(g_1g_2) = k(g_1)X(g_2) + X(g_1), \tag{2.3}$$

for $g, g_1, g_2 \in G$. If $g \in G$ and $\xi \in V^*$, we define $g\xi \in V^*$ by $g\xi(X) = \xi(k(g)^{-1}X)$ for all $X \in V$.

Let $n = \dim V$, let $\{X_1, \dots, X_n\}$ be an orthonormal basis for V and let $\{\xi_1, \dots, \xi_n\}$ be its dual basis for V^*. Using these bases, we identify V and V^* with $\mathbb{R}^n$ and look upon K as a connected subgroup of $SO(n)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we define differential operators D^α_X on V and D^α_ξ on V^* by

$$D^\alpha_X = \frac{\partial^{|\alpha|}}{\partial X_1^{\alpha_1} \cdots \partial X_n^{\alpha_n}}, \qquad D^\alpha_\xi = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}, \tag{2.4}$$

where $|\alpha| = \sum_{j=1}^n \alpha_j$.

Finally, let dk be the Haar measure on K normalized as $\int_K dk = 1$. We normalize Euclidean measures on V and V^* by multiplying $(2\pi)^{-n/2}$ and denote them by dX and $d\xi$, respectively. Then $dg = dk\,dX$ is a Haar measure on G. For $\tau \in \hat{K}$, we denote by $d(\tau)$ and χ_τ its degree and its character, respectively and put $\xi_\tau = d(\tau)\bar{\chi}_\tau$.

3. The Fourier transform on G

In this note, according to [4–6], we shall define the Fourier transform on the motion group $G = K \ltimes V$ by using the representations induced from the characters of V instead of using the irreducible unitary representations for G. Let $\xi \in V^*$. Define the action π_ξ of G on $L^2(K)$ by

$$(\pi_\xi(g)\varphi)(k) = e^{\sqrt{-1}\xi(k^{-1}X(g))}\varphi(k(g)^{-1}k), \quad (\varphi \in L^2(K)). \tag{3.1}$$

Then $(\pi_\xi, L^2(K))$ is a (reducible) unitary representation for G. It is to be noted that any irreducible unitary representation for G is contained in π_ξ for some $\xi \in V^*$ as an

https://doi.org/10.1017/S1446788700001579 Published online by Cambridge University Press

irreducible component. For $f \in L^1(G)$, we define its Fourier transform by

$$\mathscr{F}f(\xi) = \hat{f}(\xi) = \int_G f(g)\pi_\xi(g)\,dg. \tag{3.2}$$

If $f \in C_c^\infty(G)$, the set of all smooth functions on G with compact support, the following inversion formula holds (compare with [6]).

$$f(g) = \int_{V^*} \operatorname{tr}\left(\hat{f}(\xi)\pi_\xi(g)^{-1}\right) d\xi. \tag{3.3}$$

In the following, for all $\tau \in \hat{K}$, we fix a representative of τ and by abuse of notation, write τ for it again. Since $\pi_\xi|_K$ is the left regular representation of K on $L^2(K)$, it follows from the Peter–Weyl theorem that

$$\pi_\xi|_K = \sum_{\tau\in\hat{K}} d(\tau)\tau, \qquad L^2(K) = \sum_{\tau\in\hat{K}} V_\tau^* \otimes V_\tau. \tag{3.4}$$

Here V_τ denotes the representation space of τ. If $T \otimes v \in V_\tau^* \otimes V_\tau$, we set $\varphi_{T\otimes v}(k) = T(\tau(k)^{-1}v) \in C^\infty(K)$. Let $g \in G$ and define

$$\sigma(g) = |X(g)|, \qquad \Phi_\xi(g) = \int_K e^{\sqrt{-1}\xi(k^{-1}X(g))}\,dk. \tag{3.5}$$

Let $\tau_1, \tau_2 \in \hat{K}$ and put $\mathscr{V} = \operatorname{Hom}_{\mathbb{C}}(V_{\tau_2}, V_{\tau_1})$. We denote by $\langle\cdot,\cdot\rangle_{\mathscr{V}}$ and $\|\cdot\|_{\mathscr{V}}$ the canonical inner product and the corresponding norm on $\mathscr{V}$ respectively, that is, for $S, T \in \mathscr{V}$, $\langle S, T\rangle_{\mathscr{V}} = \operatorname{tr}(T^*S)$ and $\|T\|_{\mathscr{V}}^2 = \langle T, T\rangle_{\mathscr{V}}$. For $T \in \mathscr{V}$, we set

$$E(T, \xi, g) = \int_K \tau_1(k)T\tau_2(k^{-1}k(g))e^{\sqrt{-1}\xi(k^{-1}X(g))}\,dk. \tag{3.6}$$

Then it is known, [4, Lemma 3.1], that the function $\xi \mapsto E(T, \xi, g)$ can be extended to a holomorphic function on V_c^*. We list here the essential properties of the functions σ, Φ and E: For $\alpha \in \mathbb{Z}_{\geq 0}^n$, $\xi \in V_c^*$, $k, k_1, k_2 \in K$ and $g \in G$,

$$\begin{gathered}\sigma(k_1gk_2) = \sigma(g), \qquad \sigma(g^{-1}) = \sigma(g),\\ \Phi_\xi(k_1gk_2) = \Phi_\xi(g), \qquad \Phi_\xi(g^{-1}) = \Phi_\xi(g),\\ E(T,\xi,k_1gk_2) = \tau_1(k_1)E(T,\xi,g)\tau_2(k_2), \quad E(T,\xi,g)^* = E(T^*,\bar{\xi},g^{-1}),\\ E(T,k\xi,g) = E(\tau_1(k)^{-1}T\tau_2(k),\xi,g),\\ \|D_\xi^\alpha E(T,\xi,g)\|_{\mathscr{V}} \leq \|T\|_{\mathscr{V}}(1+\sigma(g)^2)^{|\alpha|}\Phi_{\sqrt{-1}\Im\xi}(g),\\ \langle \pi_\xi(g)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\rangle = \langle E(T_1^*T_2,\xi,g)v_2, v_1\rangle_{\tau_1}.\end{gathered} \tag{3.7}$$

Here $\langle\cdot,\cdot\rangle_{\tau_1}$ is an abbreviation of $\langle\cdot,\cdot\rangle_{V_{\tau_1}}$ and when there is no possibility of confusion, we shall use similar abbreviations.

We shall now summarize some known facts concerning the Schwartz spaces $\mathscr{C}(G)$ on G and $\mathscr{Z}(V^*)$ on V^*. Let L and R be the left and right regular representation of K on $L^2(K)$ and use the same symbols their differentials. As usual, we look upon any element $\mathfrak{K}$ as a right invariant differential operator on K. Then $\mathfrak{K}$ operates on $L^2(K)$ in the sense of distribution. Let $\mathscr{C}(G)$ be the set of all smooth functions f on G satisfying the following condition: For any $\alpha \in \mathbb{Z}^n_{\geq 0}$, $r \in \mathbb{R}_{\geq 0}$ and $y, y' \in \mathfrak{K}$,

$$\sup_{g\in G}(1+\sigma(g)^2)^r\left|D^\alpha_X L(y)R(y')f(g)\right| < \infty. \tag{3.8}$$

As is well-known, the system of seminorms given by the left-hand side of (3.8) makes $\mathscr{C}(G)$ into a Fréchet space. Let $\mathscr{Z}(V^*)$ be the set of all $\mathbf{B}(L^2(K))$-valued smooth functions F on V^* satisfying the following conditions:

(i) For any $\alpha \in \mathbb{Z}^n_{\geq 0}$, $r \in \mathbb{R}_{\geq 0}$ and $y, y' \in \mathfrak{K}$, $D^\alpha_\xi F(\xi)$ leaves $C^\infty(K)$ stable and it is satisfied

$$\sup_{\xi\in V^*}(1+|\xi|^2)^r\left\|yD^\alpha_\xi F(\xi)y'\right\|_\infty < \infty; \tag{3.9}$$

(ii) for any $k \in K$ and $\xi \in V^*$,

$$R(k)F(\xi) = F(k\xi)R(k). \tag{3.10}$$

The system of seminorms given by the left-hand side of (3.9) also makes $\mathscr{Z}(V^*)$ into a Fréchet space. With these notation, the following proposition holds.

PROPOSITION 3.1 ([6, Theorem 3]). *The Fourier transform is a topological isomorphism of $\mathscr{C}(G)$ onto $\mathscr{Z}(V^*)$. The inverse Fourier transform is given by the formula in* (3.3).

Let $\mathscr{C}'(G)$ and $\mathscr{Z}'(V^*)$ denote the sets of all continuous linear functionals on $\mathscr{C}(G)$ and $\mathscr{Z}(V^*)$, respectively. Following [5], we define the Fourier transform on $\mathscr{C}'(G)$ by the transposed inverse of the Fourier transform on $\mathscr{C}(G)$ and denote it also by $\mathscr{F}$.

COROLLARY 3.2 ([5, Proposition 4]). *The Fourier transform is a topological isomorphism of $\mathscr{C}'(G)$ onto $\mathscr{Z}'(V^*)$.*

Let $T \in \mathscr{C}'(G)$ and $\tau_1, \tau_2 \in \hat{K}$. We define $T_{\tau_1,\tau_2} \in \mathscr{C}'(G)$ by $T_{\tau_1,\tau_2}[\phi] = T[\xi_{\tau_1} * \phi * \xi_{\tau_2}]$ for $\phi \in \mathscr{C}(G)$. Since $\sum_{\tau_1,\tau_2\in\hat{K}} \xi_{\tau_1} * \phi * \xi_{\tau_2}$ converges absolutely to ϕ in $\mathscr{C}(G)$ [9, Theorem 4.2.2.1], the following proposition is valid.

PROPOSITION 3.3. *Retain the above notation. Then we have for $T \in \mathscr{C}'(G)$ that*

$$T = \sum_{\tau_1,\tau_2\in\hat{K}} T_{\tau_1,\tau_2},$$

and the above series converges absolutely to T in the weak topology of $\mathscr{C}'(G)$.

In what follows we simply write $\mathscr{L}^p(V^*) = L^p(V^*, \mathbf{B}(L^2(K)))$. We first remark that $\mathscr{C}(G) \subseteq L^p(G)$ and $\mathscr{Z}(V^*) \subseteq \mathscr{L}^p(V^*)$ for all $1 \le p \le \infty$. And for $F \in \mathscr{L}^p(V^*)$ and $\varphi_1, \varphi_2 \in L^2(K)$, we have

$$(3.11) \qquad \begin{aligned} \left|\langle F(\xi)\varphi_1, \varphi_2\rangle\right|^p &\le \|F(\xi)\varphi_1\|^p \|\varphi_2\|^p \\ &= \langle F(\xi)\varphi_1, F(\xi)\varphi_1\rangle^{p/2} \|\varphi_2\|^p \\ &\le \operatorname{tr}(F(\xi)^* F(\xi))^{p/2} \|\varphi_1\|^p \|\varphi_2\|^p \\ &= \|F(\xi)\|_p^p \|\varphi_1\|^p \|\varphi_2\|^p, \end{aligned}$$

and whence

$$(3.12) \qquad \|\langle F(\xi)\varphi_1, \varphi_2\rangle\|_{L^p(V^*)} \le \|F\|_{\mathscr{L}^p(V^*)} \|\varphi_1\| \|\varphi_2\|.$$

For $f \in L^p(G)$ and $F \in \mathscr{L}^p(V^*)$, we define $T_f \in \mathscr{C}'(G)$ and $T_F \in \mathscr{Z}'(V^*)$ by setting

$$(3.13) \qquad \begin{aligned} T_f[\phi] &= \int_G f(g)\phi(g)\,dg, \quad (\phi \in \mathscr{C}(G)), \\ T_F[\Phi] &= \int_{V^*} \operatorname{tr}\left(F(\xi)\Phi(\xi)\right) d\xi, \quad (\Phi \in \mathscr{Z}(V^*)). \end{aligned}$$

The following is an easy consequence of Corollary 3.2.

COROLLARY 3.4. *Retain the above notation. Let $f \in L^1(G)$ and suppose $\hat{f} \in \mathscr{L}^1(V^*)$. Then $f(g) = (\mathscr{F}^{-1}\hat{f})(g)$ (almost everywhere).*

PROOF. Noting $f(g)\Phi(\xi)\pi_\xi(g)^{-1} \in L^1(G \times V^*, \mathbf{B}(L^2(K)))$, we have for any $\Phi \in \mathscr{Z}(V^*)$ that

$$(3.14) \qquad \begin{aligned} \mathscr{F}T_f[\Phi] = T_f[\mathscr{F}^{-1}\Phi] &= \int_G f(g) \left(\int_{V^*} \operatorname{tr}\left(\Phi(\xi)\pi_\xi(g)^{-1}\right) d\xi \right) dg \\ &= \int_{V^*} \operatorname{tr}\left(\int_G f(g)\Phi(\xi)\pi_\xi(g)^{-1}\,dg \right) d\xi \\ &= \int_{V^*} \operatorname{tr}\left((\check{f})\hat{}(\xi)\Phi(\xi) \right) d\xi = T_{(\check{f})\hat{}}[\Phi], \end{aligned}$$

where $\check{f}(g) = f(g^{-1})$. By the same computation as above, we also obtain that $\mathscr{F}^{-1}T_F = T_{(\mathscr{F}^{-1}F)^\vee}$. Accordingly we have from Corollary 3.2 that

$$(3.15) \qquad T_{\check{f}} = \mathscr{F}^{-1}(\mathscr{F}T_{\check{f}}) = \mathscr{F}^{-1}T_{\hat{f}} = T_{(\mathscr{F}^{-1}\hat{f})^\vee},$$

and this proves the assertion. □

4. The main theorem

Let $1 \le p \le \infty$ and p' denote its conjugate exponent, that is, $1/p + 1/p' = 1$. Let $a > 0$ and let f be a measurable function on G such that $\left\| e^{a\sigma(g)^2} f(g) \right\|_{L^p(G)}$ $(= C_f$, say $< \infty)$. Then $f \in L^1(G)$ follows from the Hölder inequality. Let $\tau_1, \tau_2 \in \hat{K}$ and put $\mathscr{V} = \mathrm{Hom}_{\mathbb{C}}(V_{\tau_2}, V_{\tau_1})$. For $\alpha \in \mathbb{Z}^n_{\ge 0}$ and $T \in \mathscr{V}$, it follows from (3.7) and the Hölder inequality that

$$
\begin{aligned}
(4.1)\quad \left\| \int_G f(g) D_\xi^\alpha E(T, \xi, g)\, dg \right\|_{\mathscr{V}} &\le C_f \left(\int_G e^{-ap'\sigma(g)^2} \| D_\xi^\alpha E(T, \xi, g) \|_{\mathscr{V}}^{p'}\, dg \right)^{1/p'} \\
&\le C_f \|T\|_{\mathscr{V}} \left(\int_G (1 + \sigma(g)^2)^{p'|\alpha|} \int_K e^{-ap'\sigma(g)^2 - p'\Im\xi(k^{-1}X(g))}\, dk\, dg \right)^{1/p'} \\
&\le C_f \|T\|_{\mathscr{V}} \left(\int_V (1 + |X|^2)^{p'|\alpha|} \int_K e^{-ap'|X|^2 - p'\Im\xi(k^{-1}X)}\, dk\, dX \right)^{1/p'} \\
&\le C_f \|T\|_{\mathscr{V}} \left(\int_K \int_V (1 + |X|^2)^{p'|\alpha|} e^{-ap'|X|^2 - p'\Im\xi(k^{-1}X)}\, dX\, dk \right)^{1/p'} \\
&\le C_f \|T\|_{\mathscr{V}} \left(\int_V (1 + |X|^2)^{p'|\alpha|} e^{-ap'|X|^2 - p'\Im\xi(X)}\, dX \right)^{1/p'} < \infty.
\end{aligned}
$$

Thus $D_\xi^\alpha \int_G f(g) E(T, \xi, g)\, dg = \int_G f(g) D_\xi^\alpha E(T, \xi, g)\, dg$ and, since $E(T, \xi, g)$ is holomorphic on V_c^*, so is $\int_G f(g) E(T, \xi, g)\, dg$. By using similar arguments to the proof of Lemma 2.1 in [7], we can prove the following lemma.

LEMMA 4.1. *Let $1 \le p \le \infty$. Let h be an entire function on $\mathbb{C}^n$ such that*

$$
|h(z)| \le C \prod_{j=1}^n e^{a(\Re z_j)^2}, \qquad \|h(x)\|_{L^p(\mathbb{R}^n)} \le C,
$$

for some $a > 0$ and $C > 0$. Then $h(z)$ is a constant function on $\mathbb{C}^n$. Moreover, if $p < \infty$ then $h(z) = 0$.

PROOF. By dilating $z \mapsto (\sqrt{\pi/a})z$, we may assume $a = \pi$. In case $n = 1$ and $p = \infty$, the assertion is an easy consequence of the Phragmén–Lindelöf theorem and the Liouville theorem. In case $n > 1$ and $p = \infty$, the assertion is obtained by the same arguments as the proof of Lemma 2.1 in [7]. In case $n = 1$ and $p < \infty$, the assertion was given by Cowling and Price [2]. Thus it remains only to prove the assertion for the case when $n > 1$ and $p < \infty$. The Fubini theorem yields that for almost all $(t_2, \ldots, t_n) \in \mathbb{R}^{n-1}$, the function $x \mapsto h(x, t_2, \ldots, t_n)$ belongs to $L^p(\mathbb{R})$.

Noting

$$|h(z_1, t_2, \ldots, t_n)| \le C \prod_{j=2}^{n} e^{\pi t_j^2} e^{\pi (\Re z_1)^2}, \tag{4.2}$$

for $z_1 \in \mathbb{C}$ and applying the same discussion as in [2] to the function $z_1 \mapsto h(z_1, t_2, \ldots, t_n)$, we obtain $h(z_1, t_2, \ldots, t_n) = 0$ for all $z_1 \in \mathbb{C}$. The continuity of h implies $h(z_1, x_2, \ldots, x_n) = 0$ on $\mathbb{C} \times \mathbb{R}^{n-1}$ and the theorem of identity implies $h(z) = 0$ on $\mathbb{C}^n$. □

We shall first show the following proposition.

PROPOSITION 4.2. *Let $1 \le p, q \le \infty$. Let f be a measurable function on G such that*

$$\left\| e^{a\sigma(g)^2} f(g) \right\|_{L^p(G)} \le C, \qquad \left\| e^{b|\xi|^2} \hat{f}(\xi) \right\|_{\mathscr{L}^q(V^*)} \le C,$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ (almost everywhere). Moreover, if $q < \infty$ and $ab \ge 1/4$ then $f = 0$ (almost everywhere).

PROOF. We first assume $ab \ge 1/4$. Let $\tau_1, \tau_2 \in \hat{K}$ and put $\mathscr{V} = \mathrm{Hom}_{\mathbb{C}}(V_{\tau_2}, V_{\tau_1})$. Let $T_i \in V_{\tau_i}^*$ and $v_i \in V_{\tau_i}$ $(i = 1, 2)$ be such that their norms are equal to 1. We have from (4.1) that

$$\begin{aligned} |\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\rangle| &= \left| \left\langle \left(\int_G f(g) E(T_1^* T_2, \xi, g)\, dg \right) v_2, v_1 \right\rangle_{\tau_1} \right| \\ &= \left\| \int_G f(g) E(T_1^* T_2, \xi, g)\, dg \right\|_{\mathscr{V}} \|v_1\|_{\tau_1} \|v_2\|_{\tau_2} \\ &\le C_f \|T_1^* T_2\|_{\mathscr{V}} \left(\int_V e^{-ap'|X|^2 - p'\Im\xi(X)}\, dX \right)^{1/p'} \\ &\le C_f \left(\int_V e^{-ap'|X|^2 - p'\Im\xi(X)}\, dX \right)^{1/p'}, \end{aligned} \tag{4.3}$$

where $C_f = \left\| e^{a\sigma(g)^2} f(g) \right\|_{L^p(G)}$. Taking into account $|X_{\Im\xi}| = |\Im\xi|$, we see that

$$\begin{aligned} |\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\rangle| &\le C_f \left(\int_V e^{-ap'|X|^2 - p'(X, X_{\Im\xi})}\, dX \right)^{1/p'} \\ &= C_f e^{|X_{\Im\xi}|^2/4a} \left(\int_V e^{-ap'(X + X_{\Im\xi}/2a, X + X_{\Im\xi}/2a)}\, dX \right)^{1/p'} \\ &\le C_f e^{|\Im\xi|^2/4a} \left(\int_V e^{-ap'|X|^2}\, dX \right)^{1/p'}. \end{aligned} \tag{4.4}$$

https://doi.org/10.1017/S1446788700001579 Published online by Cambridge University Press

Since the integral appeared in the last expression in (4.4) has to be bounded, we can find a constant $C_1 > 0$ such that

$$(4.5)\qquad \left|\left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle\right| \le C_1 e^{|\Im\xi|^2/4a} = C_1 \prod_{j=1}^{n} e^{(\Im\xi_j)^2/4a},$$

for all $\xi = (\xi_1, \dots, \xi_n) \in V_c^*$. Let $h(\xi)$ be a holomorphic function on V_c^* defined by

$$(4.6)\qquad h(\xi) = h(\xi_1, \dots, \xi_n) = \left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle \prod_{j=1}^{n} e^{\xi_j^2/4a}.$$

Then (4.5) is written as

$$(4.7)\qquad |h(\xi)| \le C_1 \prod_{j=1}^{n} e^{(\Re\xi_j)^2/4a}.$$

On the other hand, it follows from the assumption of $\hat{f}$ and (3.12) that for $\xi \in V^*$,

$$(4.8)\qquad \left\| e^{b|\xi|^2}\left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle\right\|_{L^q(V^*)} \le C\sqrt{d(\tau_1)d(\tau_2)}\left\| e^{b|\xi|^2}\hat{f}(\xi)\right\|_{\mathcal{L}^q(V^*)}.$$

In case $ab > 1/4$, the Hölder inequality implies

$$(4.9)\qquad \|h(\xi)\|_{L^q(V^*)} \le \left\| e^{-(b-1/4a)|\xi|^2}\right\|_{L^{q'}(V^*)} \left\| e^{b|\xi|^2}\left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle\right\|_{L^q(V^*)},$$

and hence we can find a constant $C_2 > 0$ such that if $ab \ge 1/4$ then

$$(4.10)\qquad \|h(\xi)\|_{L^q(V^*)} \le C_2.$$

Applying Lemma 4.1 together with (4.7) and (4.10), we can find a constant C_3 such that $h(\xi) = C_3$ and moreover if $q < \infty$ then $C_3 = 0$. Assume now that $q = \infty$ and $ab > 1/4$. We have from the definition of $h(\xi)$ that if $\xi = (\xi_1, \dots, \xi_n) \in V^*$ then

$$(4.11)\qquad \left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle = C_3 \prod_{j=1}^{n} e^{-\xi_j^2/4a}.$$

In case $q = \infty$, the assumption of $\hat{f}$ is expressed as

$$(4.12)\qquad \left|\left\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\right\rangle\right| \le Ce^{-b|\xi|^2} = C\prod_{j=1}^{n} e^{-b\xi_j^2} \quad \text{(almost everywhere)},$$

and the continuity of the left-hand side in (4.12) shows that (4.12) holds for all $\xi \in V^*$. Accordingly we obtain

$$(4.13)\qquad |C_3| \prod_{j=1}^{n} e^{(b-1/4a)\xi_j^2} \le C,$$

https://doi.org/10.1017/S1446788700001579 Published online by Cambridge University Press

and, since $b - 1/(4a) > 0$, this is impossible unless $C_3 = 0$.

Summarizing these, we see that if $ab \geq 1/4$ and $q < \infty$, or if $ab > 1/4$ and $q = \infty$ then

$$\langle \hat{f}(\xi)\varphi_{T_2\otimes v_2}, \varphi_{T_1\otimes v_1}\rangle = 0. \tag{4.14}$$

Interchanging f and $\check{f}$ and applying the same discussion as above, we have from (3.14) and (4.14) that

$$\mathscr{F}T_f[\Phi] = T_{(\check{f})\hat{}}[\Phi] = \int_{V^*} \operatorname{tr}\left((\check{f})\hat{}(\xi)\Phi(\xi)\right) d\xi = 0, \tag{4.15}$$

for all $\Phi \in \mathscr{Z}(V^*)$. From Corollary 3.2, we have $T_f[\phi] = 0$ for all $\phi \in \mathscr{C}(G)$ and hence $f = 0$ (almost everywhere). □

If $p = q = \infty$, Proposition 4.2 takes on the following form.

COROLLARY 4.3 ([4, Theorem 4.1]). *Let f be a measurable function on G such that*

$$|f(g)| \leq Ce^{-a\sigma(g)^2} \text{ (almost everywhere)},$$
$$\|\hat{f}(\xi)\|_\infty \leq Ce^{-b|\xi|^2} \text{ (almost everywhere)},$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab > 1/4$ then $f = 0$ (almost everywhere).

Let p, q, a, b and f be as in Proposition 4.2. In case $p < \infty$, $q = \infty$ and $ab = 1/4$, Proposition 4.2 dose not yield $f = 0$ (almost everywhere). However in this case, with the help of Proposition 3.3 and Corollary 3.4, we can prove $f = 0$ (almost everywhere). We first consider the case when $\hat{f}$ is of trace class.

LEMMA 4.4. *Let $1 \leq p < \infty$ and let f be a measurable function on G such that*

$$\left\|e^{a\sigma(g)^2} f(g)\right\|_{L^p(G)} \leq C, \qquad \|\hat{f}(\xi)\|_\infty \leq Ce^{-b|\xi|^2},$$

for $C > 0$, $a > 0$ and $b > 0$. If $\hat{f}$ is of trace class and $ab = 1/4$ then $f = 0$ (almost everywhere).

PROOF. Under these assumptions, in view of (4.11), we can write $\hat{f}(\xi) = e^{-b|\xi|^2}T$ for some $T \in \mathbf{B}(L^2(K))$ of trace class. For each $\varphi \in L^2(K)$, $k \in K$ and $g \in G$, we have

$$\int_{V^*} e^{-b|\xi|^2}(\pi_\xi(g)^{-1}T\varphi)(k)\, d\xi = \int_{V^*} e^{-b|\xi|^2} e^{-\sqrt{-1}\xi(k^{-1}k(g)^{-1}X(g))}(T\varphi)(k(g)k)\, d\xi \tag{4.16}$$

https://doi.org/10.1017/S1446788700001579 Published online by Cambridge University Press

$$= e^{-|k^{-1}k(g)^{-1}X(g)|^2/4b}(T\varphi)(k(g)k)$$
$$= e^{-a\sigma(g)^2}(T\varphi)(k(g)k),$$

and thus by the Fubini theorem,

$$\begin{aligned}(4.17)\qquad \int_{V^*}\int_K & e^{-b|\xi|^2}(\pi_\xi(g)^{-1}T\varphi)(k)\overline{\varphi(k)}\,dk\,d\xi \\ &= \int_K\int_{V^*} e^{-b|\xi|^2}(\pi_\xi(g)^{-1}T\varphi)(k)\overline{\varphi(k)}\,d\xi\,dk \\ &= \int_K e^{-a\sigma(g)^2}(T\varphi)(k(g)k)\overline{\varphi(k)}\,dk.\end{aligned}$$

Take $\{\varphi_j : j \in \mathbb{Z}_{\geq 0}\}$ to be an orthonormal basis for $L^2(K)$. Since $e^{-b|\xi|^2}T\pi_\xi(g)^{-1} \in \mathscr{L}^1(V^*)$, we have

$$\begin{aligned}(4.18)\qquad \int_{V^*}\operatorname{tr}\left(e^{-b|\xi|^2}T\pi_\xi(g)^{-1}\right)d\xi &= \sum_{j=0}^{\infty}\int_{V^*} e^{-b|\xi|^2}\langle\pi_\xi(g)^{-1}T\varphi_j, \varphi_j\rangle\,d\xi \\ &= \sum_{j=0}^{\infty} e^{-a\sigma(g)^2}\int_K (T\varphi_j)(k(g)k)\overline{\varphi_j(k)}\,dk \\ &= e^{-a\sigma(g)^2}\operatorname{tr}\left(L(k(g))^{-1}T\right).\end{aligned}$$

Therefore, we obtain from Corollary 3.3 that $f(g) = e^{-a\sigma(g)^2}\operatorname{tr}\left(L(k(g))^{-1}T\right)$ (almost everywhere). From the assumption of f in Proposition 4.2, we have

$$(4.19)\qquad \left\|e^{a\sigma(g)^2}f(g)\right\|_{L^p(G)} = \left\|\operatorname{tr}\left(L(k(g))^{-1}T\right)\right\|_{L^p(G)} < C,$$

and, since $p < \infty$, this is impossible unless $T = 0$. Thus we conclude $f = 0$ (almost everywhere). □

Combining Proposition 3.3, Proposition 4.2 and Lemma 4.4, we finally have the following theorem.

THEOREM 4.5 (L^p version of the Hardy theorem). *Suppose* $1 \leq p, q \leq \infty$ *and one of* p, q *is finite. Let* f *be a measurable function on* G *such that*

$$\left\|e^{a\sigma(g)^2}f(g)\right\|_{L^p(G)} \leq C, \qquad \left\|e^{b|\xi|^2}\hat{f}(\xi)\right\|_{\mathscr{L}^q(V^*)} \leq C,$$

for $C > 0$, $a > 0$ *and* $b > 0$. *If* $ab \geq 1/4$ *then* $f = 0$ (*almost everywhere*).

PROOF. We use $\check{\tau}$ to denote the contragradient representation of $\tau \in \hat{K}$. It remains only to prove the assertion for the case when $p < \infty$, $q = \infty$ and $\hat{f}$ is not always of

https://doi.org/10.1017/S1446788700001579 Published online by Cambridge University Press

trace class. However, replacing f by $\xi_{\tau_1} * f * \xi_{\tau_2}$ if necessary, we can get the desired result. Suppose that $p < \infty$, $q = \infty$ and $\hat{f}$ is not of trace class. We can easily check that $\xi_{\tau_1} * f * \xi_{\tau_2}$ $(\tau_1, \tau_2 \in \hat{K})$ satisfies the same assumptions as in Lemma 4.4. Thus taking into account $T_{f,\tau_1,\tau_2} = T_{\xi_{\check{\tau}_1} * f * \xi_{\check{\tau}_2}}$, we deduce from Proposition 3.3 and Lemma 4.4 that $f = 0$ (almost everywhere). □

In case $K = \{e\}$ and $V = \mathbb{R}^n$, the motion group $G = K \ltimes V$ coincides with $\mathbb{R}^n$ and the Fourier transform given by the formula in (3.2) coincides with the Euclidean Fourier transform on $\mathbb{R}^n$.

COROLLARY 4.6. *Suppose that $1 \leq p, q \leq \infty$ and one of p, q is finite. Let f be a measurable function on $\mathbb{R}^n$ such that*

$$\left\| e^{a|x|^2} f(x) \right\|_{L^p(\mathbb{R}^n)} \leq C, \quad \left\| e^{b|\xi|^2} \hat{f}(\xi) \right\|_{L^q(\mathbb{R}^n)} \leq C,$$

for $C > 0$, $a > 0$ and $b > 0$. If $ab \geq 1/4$ then $f = 0$ (almost everywhere).

Acknowledgement

The second author wants to thank Dr Makoto Abe for useful suggestions on Lemma 4.1.

References

[1] W. O. Amrein and A. Berthier, 'On support properties of L^p-functions and their Fourier transforms', *J. Funct. Anal.* **24** (1977), 258–267.
[2] M. G. Cowling and J. F. Price, *Generalizations of Heisenberg's inequality*, Lecture Notes in Math. 992 (Springer, Berlin, 1983) pp. 443–449.
[3] H. Dym and H. P. McKean, *Fourier series and integral* (Academic Press, New York, 1972).
[4] M. Eguchi, S. Koizumi and K. Kumahara, 'An analogue of the Hardy theorem for the Cartan motion group', *Proc. Japan Acad.* **74** (1998), 149–151.
[5] M. Eguchi, K. Kumahara and Y. Muta, 'A subspace of Schwartz space on motion group', *Hiroshima Math. J.* **10** (1980), 691–698.
[6] K. Kumahara, 'Fourier transforms on the motion groups', *J. Math. Soc. Japan.* **26** (1976), 18–32.
[7] A. Sitaram and M. Sundari, 'An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups', *Pacific J. Math.* **177** (1997), 187–200.
[8] M. Sundari, 'Hardy's theorem for the n-dimensional Euclidean motion group', *Proc. Amer. Math. Soc.* **126** (1998), 1199–1204.
[9] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I (Springer, New York, 1972).

Faculty of Integrated Arts and Sciences
Hiroshima University
Kagamiyama 1-7-1
Higashi-Hiroshima, 739-8521
Japan
e-mail: eguchi@humpty.mis.hiroshima-u.ac.jp

Onomichi Junior College
Hisayamada 1600
Onomichi, 722-8506
Japan

The University of the Air
2-11 Wakaba, Mihama-ku
Chiba, 261-8586
Japan
e-mail: kumahara@u-air.ac.jp