Canad. Math. Bull. Vol. **17** (4), 1974

LYAPUNOV INEQUALITIES AND BOUNDS ON SOLUTIONS OF CERTAIN SECOND ORDER EQUATIONS*

BY
STANLEY B. ELIASON

1. **Introduction.** In this paper we consider the equation

$$(r(t)y'(t))'+p(t)f(y(t)) = 0 \tag{1.1}$$

under the conditions

$((H_O)$: the real valued functions r, r' and p are continuous on a non-trivial interval J of reals, and $r(t)>0$ for $t \in J$;

and

(H_1): $f: R \to R$ is continuously differentiable and odd with $f'(y)>0$ for all real y.

We also consider the equation

$$y''(t)+m(t)y'(t)+n(t)f(y(t)) = 0 \tag{1.2}$$

under the conditions (H_1) and

(H_2): the real valued functions m and n are continuous on a non-trivial interval J of reals.

Later in the paper, for $y \neq 0$, we let $f_1(y) \equiv f(y)/y$. All solutions considered are real valued.

By a well-known method, equation (1.2) may be expressed in the form of (1.1). By simply multiplying (1.2) by

$$r(t) \equiv \exp \int_\alpha^t m(s)\,ds, \quad \text{for} \quad t \in J. \tag{1.3}$$

where $\alpha \in J$ is fixed we obtain (1.1) and the relation

$$p(t) = r(t)n(t). \tag{1.4}$$

With $f(y) \equiv y$, hereafter called the linear case, A. M. Fink and D. F. St. Mary [3, (3)] establish that if $a<b$ in J are consecutive zeros of a non-trivial solution of (1.2) then

$$(b-a)\int_a^b n^+ - 4 \exp\left[-\left(\frac{1}{2}\right)\int_a^b |m|\right] > 0. \tag{1.5}$$

Received by the editors February 14, 1973.

* Based on research supported in part by the U.S. Army Research Office-Durham through Grant Number DA-ARO-D-31-124-72-G154 with the University of Oklahoma Research Institute.

As usual, for a real valued function g we let g^+ and g^- be defined by

$$g^+(t) = \max\{g(t), 0\} \quad \text{and} \quad g^-(t) = \max\{-g(t), 0\}.$$

Inequality (1.5) is also announced in a paper by Levin [5].

Also inequality (1.5) implies an inequality of Nehari [7], displayed as (4) in Fink and St. Mary [3].

For equation (1.1), related inequalities known as Lyapunov inequalities, have been established in several places, first of all in the linear case by A. Liapounoff [6]; and elsewhere, see Hartman [4] for work and references. In the case of certain non-linear functions f, such an inequality is established by the author [1]. In this situation the inequality involves a bound on the solution between its zeros, but nevertheless, for a large class of such functions f the inequality remains sharp.

One purpose of this paper is to establish inequalities which improve (1.5) in many cases when m and n are not of constant sign. They will follow by first establishing lower bounds on certain positive solutions of (1.1). The bounds are expressed in terms of a maximum value of the solution and integral functionals involving the coefficients, as defined below.

For reals $d<e$ we let

$$\begin{aligned} R(d, e; p) &= \sup_{d\le x\le e} \int_x^e p, \qquad L(d, e; p) = \sup_{d\le x\le e} \int_d^x p, \\ S(d, e; p) &= \sup_{d\le u\le v\le e} \int_u^v p, \quad \text{and} \quad I(d, e; p) = \inf_{d\le u\le v\le e} \int_u^v p. \end{aligned} \tag{1.6}$$

Clearly, we have

$$-\int_d^e p^- \le F(d, e; p) \le \int_d^e p^+ \tag{1.7}$$

holding for F denoting R, L, S, or I. Also for fixed e, R and S decrease monotonically as d increases. Other obvious monotoneity properties of L, S, and I will be used without explicitly stating them here. By studying the relationships (1.7) more closely one may also see how the inequalities become strict in certain cases when p is not of constant sign on $[d, e]$.

Two inequalities improving (1.5) in the linear case are

$$(b-a)\int_a^b n^+ - 4\exp\{(\tfrac{1}{2})[I(a, b; m) - S(a, b; m)]\} > 0; \tag{1.8}$$

and, when $m\equiv 0$ and the solution y is positive on (a, b) and for some $c \in (a, b)$ satisfies

$$(c-t)y'(t) \ge 0 \quad \text{for} \quad t \in [a, b], \tag{1.9}$$

$$(b-a)S(a, b; n) > 4. \tag{1.10}$$

https://doi.org/10.4153/CMB-1974-088-2 Published online by Cambridge University Press

By (1.7), the improvement of (1.8) follows from

$$I(a, b; m)-S(a, b; m) \geq \int_a^b -(m^-+m^+) = -\int_a^b |m|. \tag{1.11}$$

Strict inequality holds here, for example, when $a=0$, $b=4\pi$ and $m(t)= \sin kt$, where k is a positive integer. In fact we here have the rather interesting phenomena that $-\int_a^b |m|$ remains constant while $I(a, b; m)-S(a, b; m)\to 0$ as $k\to\infty$.

2. **Bounds on Solutions and Related Inequalities.** We first consider a solution y of (1.1) where $y'(c)=0$ for some $c \in J$. By integrating twice and applying an integration by parts, for $x \in J$ we have

$$\begin{aligned} y(c)-y(x) \\ = \int_x^c [r(t)]^{-1}\left\{\left(\int_t^c p(\tau)\, d\tau\right) f(y(t))+\int_t^c \left(\int_s^c p(\tau)\, d\tau\right) f'(y(s))y'(s)\, ds\right\} dt \end{aligned} \tag{2.1}$$

By the oddness of f, if $y(c)\neq 0$, we may assume $y(c)>0$; and throughout the paper, between consecutive zeros we will assume a solution is positive. Thus if $x<c$ and if y is positive and monotone increasing on $(x, c]$ we may conclude from (2.1) that

$$\begin{aligned} y(c)-y(x) &\leq \int_x^c [r(t)]^{-1}R(t, c; p)[f(y(t))+f(y(c))-f(y(t))]\, dt \\ &= f(y(c))\int_x^c [r(t)]^{-1}R(t, c; p)\, dt \\ &\leq f(y(c))R(x, c; p)\int_x^c (1/r). \end{aligned} \tag{2.2}$$

Furthermore, by (H_1) and (2.1), if $y(x)<y(c)$, then y' and p must both be positive on some subinterval of $[x, c]$. As a result it may be argued that the inequalities in (2.2) are strict in this case.

By a similar argument if $x>c$ and if y is positive and monotone decreasing on $[c, x]$ then

$$\begin{aligned} y(c)-y(x) &\leq f(y(c))\int_c^x [r(t)]^{-1}L(c, t; p)\, dt \\ &\leq f(y(c))L(c, x; p)\int_c^x (1/r), \end{aligned} \tag{2.3}$$

where the same conclusions on strictness apply here if $y(x)<y(c)$.

The inequalities in (2.2) and (2.3) clearly yield lower bounds on the solution y. They will be next used to place implicit lower bounds on the distance from c to the first possible zero of y lying to the left or right of c.

Suppose, then, that $a<b$ in J are two consecutive zeros of a solution y and suppose $c \in (a, b)$ satisfies (1.9), where, as is understood, y is positive on (a, b).

With $f_1(y)=f(y)/y$ for $y\neq 0$, (2.2) and (2.3) respectively yield

$$1 < f_1(y(c))\int_a^c [r(t)]^{-1}R(t, c; p)\,dt < f_1(y(c))R(a, c; p)\int_a^c (1/r) \tag{2.4}$$

and

$$1 < f_1(y(c))\int_c^b [r(t)]^{-1}L(c, t; p)\,dt < f_1(y(c))L(c, b; p)\int_c^b (1/r). \tag{2.5}$$

The inequalities provided by the extremes of (2.4) and (2.5) improve those of D. F. St.Mary [8, Theorem 7] when

$$R(a, c; p) < \int_a^c p^+ \quad \text{or} \quad L(c, b; p) < \int_c^b p^+ \tag{2.6}$$

respectively and, of course, (1.9) hold.

When $f(y)=y^{2k+1}$, k being a positive integer, related inequalities are provided by the author [2] in Corollary 2 of Theorem 1, where non-linear eigenvalue problems are studied.

We now attack a "distance between zeros" problem. By using different variables of integration and then multiplying, from (2.4) and (2.5) we obtain the Lyapunov inequalities

$$\begin{aligned} 1 &< f_1^2(y(c))\int_a^c\int_c^b [r(u)r(v)]^{-1}R(u, c; p)L(c, v; p)\,dv\,du \\ &\leq f_1^2(y(c))4^{-1}\int_a^c\int_c^b [r(u)r(v)]^{-1}[S(u, v; p)]^2\,dv\,du \\ &< f_1^2(y(c))4^{-2}[S(a, b; p)]^2\left(\int_a^b (1/r)\right)^2. \end{aligned} \tag{2.7}$$

The second inequality above follows from $\alpha\beta\leq 4^{-1}(\alpha+\beta)^2$ and

$$0 \leq R(u, c; p)+L(c, v; p) \leq S(u, v; p).$$

The third inequality follows from monotoneity properties of S and

$$\int_a^c (1/r)\int_c^b (1/r) \leq 4^{-1}\left(\int_a^b (1/r)\right)^2.$$

Inequality (1.10) is now a special case of (2.7), by simply taking square roots in (2.7), where, of course, $f_1(v)\equiv 1$.

In order to obtain (1.8) we consider $a<b$ to be two consecutive zeros of a solution y of (1.2), where y is positive on (a, b). Then for some $a<c_1\leq c_2<b$ we have $y'(c_1)=y'(c_2)=0$ and y is monotone on $(a, c_1]$ and on $[c_2, b)$.

https://doi.org/10.4153/CMB-1974-088-2 Published online by Cambridge University Press

Using (1.3) and (1.4), the first inequality of (2.4) yields

$$\begin{aligned}
1 &< f_1(y(c_1))\int_a^{c_1}\left[\exp-\int_a^t m(w)\,dw\right]\max_{t\le s\le c_1}\int_s^{c_1}\left[\exp\int_a^u m(w)\,dw\right]n(u)\,du\,dt\\
&= f_1(y(c_1))\int_a^{c_1}\max_{t\le s\le c_1}\int_s^{c_1}\left[\exp\int_t^u m(w)\,dw\right]n(u)\,du\,dt \qquad (2.8)\\
&\le f_1(y(c_1))\int_a^{c_1}[\exp L(t, c_1; m)]\int_t^{c_1} n^+(u)\,du\,dt.
\end{aligned}$$

In the linear case, the inequality provided by the extremes of (2.8) is what improves inequality (7) of Fink and St.Mary [3].

By (2.5) we also obtain

$$1 < f_1(y(c_2))\int_{c_2}^{b}[\exp R(c_2, t; m)]\int_{c_2}^{t} n^+(u)\,du\,dt. \tag{2.9}$$

Thus with

$$Q = \max\{f_1(y(c_1)), f_1(y(c_2))\}, \tag{2.10}$$

by (2.8) and (2.9), using different variables of integration and multiplying we have

$$\begin{aligned}
1 &< Q^2\int_a^{c_1}\int_{c_2}^{b}\{\exp[L(u, c_1; m)+R(c_2, v; m)]\}\left(\int_u^{c_1}\int_{c_2}^{v} n^+(x)n^+(z)\,dz\,dx\,dv\,du\right.\\
&\le 4^{-1}Q^2\int_a^{c_1}\int_{c_2}^{b}\left\{\exp\left[\int_u^v m - I(u, v; m)\right)\right\}\left(\int_u^v n^+\right)^2 dv\,du \qquad (2.11)\\
&< 4^{-2}Q^2\{\exp[S(a, b; m)-I(a, b; m)]\}\left(\int_a^b n^+\right)^2(b-a)^2.
\end{aligned}$$

The inequalities follow from the definitions and properties of L, R, S and I, along with modifications of the argument used to establish (2.7).

In the linear case, where $Q=1$, by taking square roots of (2.11) we obtain (1.8).

An interesting question is whether (1.8) may be improved to

$$(b-a)S(a, b; n)-4\exp\{(\tfrac{1}{2})[I(a, b; m)-S(a, b; m)]\} > 0?$$

By our method of using (2.4) and (2.5), to answer this question in the affirmative, even when (1.9) holds, it appears that in (2.8) we need the inequality

$$\max_{t\le s\le c_1}\int_s^{c_1}\left[\exp\int_t^u m(w)\,dw\right]n(u)\,du \le [\exp L(t, c_1; m)]R(t, c_1; n),$$

which for general functions m and n is not true.

We now summarize the above results.

THEOREM 2.1. *Let y be a solution of* (1.1) *satisfying $y'(c)=0$ and $y(c)>0$ for some $c\in J$. Then for $x<c$, $(x>c)$, as long as y is positive and monotone increasing on $(x, c]$, (monotone decreasing on $[c, x)$), the inequalities in* (2.2), ((2.3)), *provide*

https://doi.org/10.4153/CMB-1974-088-2 Published online by Cambridge University Press

lower bounds on $y(x)$ *which are expressed in terms of* $y(c)$ *and integral functionals, as defined by* (1.6), *involving the coefficients r and p of* (1.1). *They are strict if* $y(x) < y(c)$.

As a result, inequalities (2.4), ((2.5)), *provide implicit lower bounds on the distance from c to the first possible zero a*, (*b*), *of y lying to the left*, (*right*), *of c. They improve previous results when* (2.6) *and* (1.9) *hold.*

Inequalities (2.4) *and* (2.5) *in turn yield Lyapunov inequalities concerning the distance between consecutive zeros* $a < b$ *of a solution y of* (1.1), *or of* (1.2), *which is positive on* (a, b). *The first inequalities, provided by* (2.7), *relate to* (1.1) *and assume condition* (1.9). *The second provided by* (2.11), *relate to* (1.2), *and does not assume condition* (1.9), *and they improve previous results when inequality* (1.11) *is strict.*

References

1. S. B. Eliason, *A Lyapunov inequality for a certain second order non-linear differential equation*, J. London Math. Soc., (2), **2**, (1970), 461–466.

2. ——, *Comparison theorems for second order nonlinear differential equations*, Quart. Appl. Math., **35**, (1971), 148–156.

3. A. M. Fink, and D. F. St.Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc., **21**, (1969), 640–642.

4. P. Hartman, *Ordinary Differential Equations*, New York, 1964, 345–346, 401.

5. A. Levin, *On linear second order differential equations*, Soviet Math. Dokl., **4**, (1963), 1814–1817.

6. A. Liapounoff, *Problème générale de la stabilité du mouvement*, Annals of Mathematics Study 17, Princeton University Press, (1949).

7. Z. Nehari, *On an inequality of Lyapunov*, Studies in Mathematical Analysis and Related Topics, Stanford University Press, (1962), 256–261.

8. D. F. St.Mary, *Some oscillation and comparison theorems for* $(r(t)y')' + p(t)y = 0$, J. Diff. Eq., **5**, (1969), 314–323.

University of Oklahoma
Norman, Oklahoma 73069

https://doi.org/10.4153/CMB-1974-088-2 Published online by Cambridge University Press