CENTRAL IDEMPOTENTS IN \( p \)-ADIC GROUP RINGS

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Abstract

We provide character-free proofs of some results on idempotents in \( p \)-adic group rings, centering around Brauer’s Second Main Theorem on Blocks.


The main purpose of this paper is to provide character-free proofs of some (known) results on central idempotents in \( p \)-adic group rings of finite groups. The results we have in mind are all directly or indirectly related to Brauer’s Second Main Theorem on Blocks. Thus we also prove a character-free version of the Second Main Theorem, using ideas of Puig [18, 19].

In the following, \( \mathcal{O} \) will denote a complete discrete valuation ring with algebraically closed residue field \( \mathbb{F} \) of prime characteristic \( p \), and \( \alpha \mapsto \bar{\alpha} \) will denote the standard epimorphism \( \mathcal{O} \to \mathbb{F} \).

Unless stated otherwise, the algebras we will consider are all associative with identity element, and free of finite rank over their ring of coefficients (\( \mathcal{O} \) or \( \mathbb{F} \)). For such an algebra \( A \), we denote by \( JA \) its Jacobson radical, by \( ZA \) its center, by \( UA \) its group of units and by \( [A, A] \) its \( ZA \)-submodule consisting of all finite sums of elements of the form \( [a, b] = ab - ba \) \((a, b \in A)\).

For a finite group \( G \), \( \mathcal{O}G \) and \( \mathbb{F}G \) denote the group algebras of \( G \) over \( \mathcal{O} \) and \( \mathbb{F} \), respectively. There is a useful map \( \lambda : \mathcal{O}G \to \mathcal{O} \) defined in the following way: If \( a = \sum_{g \in G} \alpha_g g \in \mathcal{O}G \) with \( \alpha_g \in \mathcal{O} \) for \( g \in G \) then \( \lambda(a) = \alpha_1 \). It is
well-known that $\lambda$ vanishes on $[\mathcal{O} G, \mathcal{O} G]$.

We will have to use properties of $G$-algebras. We therefore recall that a $G$-algebra over $\mathcal{O}$ is a pair consisting of an $\mathcal{O}$-algebra $A$ and a homomorphism $\phi$ from $G$ into the automorphism group $\text{Aut}(A)$ of $A$. We write $^g a$ instead of $(\phi(g))(a)$ for $g \in G$ and $a \in A$ and define

$$A^H := \{a \in A : ^h a = a \text{ for } h \in H\},$$

for every subgroup $H$ of $G$. If $K$ is a subgroup of $H$ and $b \in A^K$ then $^{hk} b = ^h b$ for $h \in H$ and $k \in K$. We therefore write $^{hk} b$ instead of $^h b$. The transfer map $\text{Tr}^H_K : A^K \rightarrow A^H$ is then defined by $\text{Tr}^H_K(b) := \sum_{hK \in H/K} ^{hk} b$ for $b \in A^K$; here $H/K$ denotes the set of cosets $hK$ ($h \in H$). We set $A^H_K := \text{Tr}^H_K(A^K)$. Then $A^H_K$ is an ideal in $A^H$.

The group algebra $\mathcal{O} G$ will be considered as a $G$-algebra in such a way that $^g a = gag^{-1}$ for $g \in G$ and $a \in \mathcal{O} G$. In this case the map

$$\mathcal{O} G \rightarrow \mathbb{F} C_G(Q), \quad \sum_{g \in G} \alpha_k g \mapsto \sum_{g \in C_G(Q)} \bar{\alpha}_k g,$$

restricts to a homomorphism $\text{Br}_Q : (\mathcal{O} G)^Q \rightarrow \mathbb{F} C_G(Q)$ which is called the Brauer homomorphism with respect to $Q$, for any $p$-subgroup $Q$ of $G$.

We will prove Brauer's Second Main Theorem on Blocks in the following form.

**Theorem 1.** Let $G$ be a finite group, $u$ a $p$-element in $G$, $s$ a $p$-regular element in $C_G(u)$ and $e$ an idempotent in $Z\mathcal{O} G$. We denote by $e_u$ the unique idempotent in $Z\mathcal{O} C_G(u)$ such that $\text{Br}_u(e) = \text{Br}_u(e_u)$. Then $eus \equiv e_u us \pmod{[\mathcal{O} G, \mathcal{O} G]}$.

We note that the existence and uniqueness of $e_u$ follow from lifting theorems for idempotents.

In order to get from Theorem 1 the Second Main Theorem in its usual form (see 5.4.1 in [16], for example) we just apply an irreducible character $\chi$ to the congruence above (using the fact that $\chi$ vanishes on $[\mathcal{O} G, \mathcal{O} G]$).

The reader may wish to consult the references for other proofs of the Second Main Theorem.

**Proof.** We may assume that $u \neq 1$ and wish to show that $(e - e_u)su \in [\mathcal{O} G, \mathcal{O} G]$. Since $s$ is a linear combination of idempotents in $\mathcal{O} \langle s \rangle$ it suffices to show that $(e - e_u)fu \in [\mathcal{O} G, \mathcal{O} G]$ for any idempotent $f$ in $\mathcal{O} C_G(u)$. By the
definition of $e_u$, the idempotent $(e - e_u)f \in (\mathcal{O}G)^{(u)}$ is contained in the kernel of $\text{Br}(u)$. But it is well-known (and easy to see) that $\text{Ker}(\text{Br}(u))$ is the sum of the two ideals $(J\mathcal{O})(\mathcal{O}G)^{(u)}$ and $(\mathcal{O}G)^{(u)}(v)$ (where $v := u^p$) of $(\mathcal{O}G)^{(u)}$. Since $0$ is the only idempotent contained in $(J\mathcal{O})(\mathcal{O}G)^{(u)}$ it therefore follows from Rosenberg’s lemma (see 5.1 in [13], for example) that $(e - e_u)f \in (\mathcal{O}G)^{(u)}$. Hence, by Puig’s version of Green’s indecomposability theorem (see [18] or Theorem 9 below), there is an idempotent $j$ in $(\mathcal{O}G)^{(v)}$ orthogonal to $gj$ for $g \in (u) \setminus (v)$ such that $(e - e_u)f = \text{Tr}_{(v)}^{(u)}(j)$. Thus

$$ju = ju - j(uv)u = j(ju) - (ju)j \in [\mathcal{O}G, \mathcal{O}G]$$

and

$$(e - e_u)fu = \text{Tr}_{(v)}^{(u)}(j) \in \text{Tr}_{(v)}^{(u)}([\mathcal{O}G, \mathcal{O}G] \cap (\mathcal{O}G)^{(v)}) \subseteq [\mathcal{O}G, \mathcal{O}G],$$

as we wanted to show.

We would like to add some related results on idempotents. We start with a simple lemma due to Cliff [3]. Similar results can also be found in Oliver [17] and Taylor [20]. The analogous fact in prime characteristic goes back to Brauer. For a recent account, see Külshammer [12].

**Lemma 2.** Let $A$ be the free $\mathcal{O}$-algebra in generators $x_1, \ldots, x_k$, and let $m$ and $n$ be non-negative integers such that $m \leq n$. Then $(x_1 + \cdots + x_k)^p = a + b + c$ where $a \in p^mA$, $b$ is a sum of $p^{a-m+1}$-th powers of monomials in $x_1, \ldots, x_k$, and $c \in [A, A]$.

**Proof.** We write $(x_1 + \cdots + x_k)^p$ as the sum of the $k^p$ different terms $y_1, \ldots, y_{p^k}$ with $y_1, \ldots, y_{p^k} \in \{x_1, \ldots, x_k\}$. The cyclic group $Z = \langle z \rangle$ of order $p^k$ acts on the set of these terms in such a way that $z(y_1y_2\cdots y_{p^k}) = y_2 \cdots y_{p^k}y_1$. It is obvious that terms in the same $Z$-orbit lie in the same coset modulo $[A, A]$. Hence, if $B$ is a $Z$-orbit containing at least $p^m$ elements then $\sum_{b \in B} b$ is contained in $p^mA + [A, A]$. On the other hand, if $B$ is a $Z$-orbit containing less than $p^m$ elements then, for $b \in B$, the stabilizer of $b$ in $Z$ has order at least $p^{n-m+1}$. This means that $b$ is of the form $b = (y_1 \cdots y_{p^{n-1}})^{p^{n-m+1}}$, and the result follows.

We wish to apply Lemma 2 to group algebras. Thus let $G$ be a finite group, and let $K$ be a conjugacy class of $G$. We call $K$ $p$-regular if it consists of $p$-regular elements, and $p$-singular otherwise. For a subset $X$ of $G$, we set $X^+ := \sum_{g \in X} g \in \mathcal{O}G$.

The following result is due to Cliff [3].
PROPOSITION 3. Let $G$ be a finite group, and let $e$ be an idempotent in $\mathcal{O}G$. We write $e = \sum_{g \in G} \varepsilon_e g$ with $\varepsilon_e \in \mathcal{O}$ for $g \in G$. Then $\sum_{g \in L} \varepsilon_e = 0$ for every $p$-singular conjugacy class $L$ of $G$.

PROOF. Let $L$ be a $p$-singular conjugacy class of $G$. It suffices to show that $\sum_{g \in L} \varepsilon_e \in \mathcal{O}^{p^n}$ for every positive integer $m$. We therefore fix a positive integer $m$ and choose a positive integer $n \geq m$ such that $g^{p^{n-m+1}} = 1$ for every $p$-element $g \in G$.

Let $A$ be the free $\mathcal{O}$-algebra in $|G|$ generators $x_g \ (g \in G)$. There is a unique homomorphism of algebras $\phi : A \rightarrow \mathcal{O}G$ satisfying $\phi(x_g) = \varepsilon_e g$ for $g \in G$. Thus

$$\phi\left(\left(\sum_{g \in G} x_g\right)^{p^n}\right) = \left(\sum_{g \in G} \phi(x_g)\right)^{p^n} = e^{p^n} = e.$$

We write $(\sum_{g \in G} x_g)^{p^n} = a + b + c$ with $a, b, c$ as in Lemma 2. Then $e = \phi(a) + \phi(b) + \phi(c)$ where $\phi(a) \in \mathcal{O}^{p^n}G$, $\phi(b)$ is a linear combination of $p$-regular elements in $G$, and $\phi(c) \in [\mathcal{O}G, \mathcal{O}G]$. Hence

$$\sum_{g \in L} \varepsilon_e = \lambda(e(L^{-1})^+) = \lambda(\phi(a)(L^{-1})^+) + \lambda(\phi(b)(L^{-1})^+) + \lambda(\phi(c)(L^{-1})^+)$$

where $\lambda(\phi(a)(L^{-1})^+) \in \mathcal{O}^{p^n}$, $\lambda(\phi(b)(L^{-1})^+) = 0$ since $L$ is $p$-singular, and $\lambda(\phi(c)(L^{-1})^+) = 0$ since $\phi(c)(L^{-1})^+ \in [\mathcal{O}G, \mathcal{O}G]$. Thus $\sum_{g \in L} \varepsilon_e \in \mathcal{O}^{p^n}$ as we wished to show.

An immediate consequence of Proposition 3 is the following result.

COROLLARY 4. Let $G$ be a finite group, $s$ a $p$-regular element in $G$ and $e$ an idempotent in $\mathcal{O}G$. We write $es = \sum_{g \in G} \alpha_g g$ with $\alpha_g \in \mathcal{O}$ for $g \in G$. Then $\sum_{g \in L} \alpha_g = 0$ for every $p$-singular conjugacy class $L$ of $G$.

PROOF. The $p$-regular element $s \in G$ is a linear combination of idempotents in $\mathcal{O}\langle s \rangle$, so $es$ is a linear combination of idempotents in $\mathcal{O}G$, and the result follows from Proposition 3.

We recall that every element $g \in G$ can be written uniquely in the form $g = us$ where $u$ is a $p$-element in $G$ and $s$ is a $p$-regular element in $G$ such that $us = su$. Then $u$ is called the $p$-factor of $g$, and $s$ is called the $p$-regular factor.
of \( g \). Two elements in \( G \) are said to be contained in the same \( p \)-section of \( G \) if their \( p \)-factors are conjugate in \( G \).

The following result is known to be a consequence of the Second Main Theorem (see [11], for example). We present a proof using the ideas above.

**Proposition 5.** Let \( K \) be a conjugacy class of \( G \), and let \( e \) be an idempotent in \( ZG \). Then \( K^+e \) is a linear combination of elements contained in the same \( p \)-section as \( K \). In particular, \( e \) is a linear combination of \( p \)-regular elements in \( G \).

**Proof.** We write \( K^+e = \sum_{g \in G} \alpha_g g \) with \( \alpha_g \in \mathcal{O} \) for \( g \in G \). Then \( \alpha_g = \lambda(K^+eg^{-1}) \) for \( g \in G \), so it suffices to show that \( \lambda(K^+eg^{-1}) = 0 \) whenever \( g \) is not contained in the same \( p \)-section as \( K \). We fix such an element \( g \) and denote by \( u \) the \( p \)-factor and by \( s \) the \( p \)-regular factor of \( g^{-1} \), so that \( g^{-1} = us = su \). Moreover, we denote by \( e_u \) the unique idempotent in \( ZG(u) \) such that \( \text{Br}(u)(e) = \text{Br}(u)(e_u) \). Then, by Theorem 1, \( eus \equiv e_uus \pmod{[\mathcal{O}G, \mathcal{O}G]} \), so \( K^+eg^{-1} \equiv K^+e_u g^{-1} \pmod{[\mathcal{O}G, \mathcal{O}G]} \), and therefore

\[
\lambda(K^+eg^{-1}) = \lambda(K^+e_u g^{-1}) = \lambda((K \cap C_G(u))^+e_u g^{-1})
\]

since \( e_u g^{-1} \in \mathcal{O}C_G(u) \). We write \( e_us = \sum_{h \in C_G(u)} \beta_h h \) with \( \beta_h \in \mathcal{O} \) for \( h \in C_G(u) \). Then, by Corollary 4, \( \sum_{h \in L} \beta_h = 0 \) for every \( p \)-singular conjugacy class \( L \) of \( C_G(u) \). But, since \( g \) and \( K \) are contained in different \( p \)-sections, \((K \cap C_G(u))u \) is a union of \( p \)-singular conjugacy classes of \( C_G(u) \). Thus we have

\[
\lambda((K \cap C_G(u))^+e_u g^{-1}) = \sum_{h \in (K \cap C_G(u))u} \beta_h = 0
\]

as we wanted to show.

Let \( u \) be a \( p \)-element in \( G \), and let \( K \) be a conjugacy class of \( G \) contained in the same \( p \)-section of \( G \) as \( u \). Then the elements in \( K \) with \( p \)-factor \( u \) form a conjugacy class \( K_u \) of \( C_G(u) \), and the map \( K \mapsto K_u \) is a bijection between the set of conjugacy classes of \( G \) contained in the same \( p \)-section of \( G \) as \( u \) and the set of conjugacy classes of \( C_G(u) \) contained in the same \( p \)-section of \( C_G(u) \) as \( u \).

Let \( e \) be an idempotent in \( ZG \), and let \( e_u \) be the unique idempotent in \( ZG(u) \) such that \( \text{Br}(u)(e) = \text{Br}(u)(e_u) \). We wish to compare \( K^+e \) and \( K^+_u e_u \).

The following result can be found in Iizuka [11].
THEOREM 6. Let $G$ be a finite group, $u$ a $p$-element in $G$, $K$ a conjugacy class of $G$ contained in the $p$-section of $u$ in $G$, and $K_u := \{ g \in K : g$ has $p$-factor $u \}$. Let $e$ be an idempotent in $\mathcal{O}G$, and let $e_u$ be the unique idempotent in $\mathcal{O}C_G(u)$ such that $Br_u(e) = Br_u(e_u)$. We write $K^+e = \sum_{g \in G} \alpha_g g$ and $K^+_u e_u = \sum_{h \in C_G(u)} \beta_h h$ with $\alpha_g, \beta_h \in \mathcal{O}$ for $g \in G$ and $h \in C_G(u)$. Then $\alpha_g = 0$ if $g$ is not contained in the $p$-section of $u$, and $\alpha_{us} = \beta_{us}$ for any $p$-regular element $s$ in $C_G(u)$.

PROOF. The first assertion follows from Proposition 5. Thus let $s$ be a $p$-regular element in $C_G(u)$. Then $\alpha_{us} = \lambda(K^+e us^{-1})$ and $\beta_{us} = \lambda(K^+_u e_u us^{-1})$. As in the proof of Proposition 5, we have

$$\lambda(K^+e us^{-1}) = \lambda(K^+_u e_u us^{-1}) = \lambda((K \cap C_G(u))^+ u^{-1} e_u s^{-1}).$$

If $L$ is a conjugacy class of $C_G(u)$ contained in $(K \cap C_G(u)) \setminus K_u$ then $Lu^{-1}$ is a $p$-singular conjugacy class of $C_G(u)$, so $\lambda(L^+ u^{-1} e_u s^{-1}) = 0$ by Corollary 4. Thus

$$\lambda((K \cap C_G(u))^+ u^{-1} e_u s^{-1}) = \lambda(K^+_u u^{-1} e_u s^{-1}) = \beta_{us}$$

as we wanted to show.

The theorem implies that one can compute $K^+e$ from $K^+_u e_u$ and vice versa. As an application, we mention the following result taken from Broué [2].

PROPOSITION 7. Let $G$ be a finite group, $u$ a $p$-element in $G$ and $U$ the $p$-section of $G$ containing $u$. Let $B$ be a block of $\mathcal{O}G$ with block idempotent $e$ and defect group $D$. If $u$ is not conjugate in $G$ to an element in $D$ then $K^+e = 0$ for every conjugacy class $K$ of $G$ contained in $U$.

PROOF. If $u$ is not conjugate to an element in $D$ then $Br_u(e) = 0$. But then $e_u = 0$, in the notation of Theorem 6. Thus $K^+_u e_u = 0$ for every conjugacy class $K$ of $G$ contained in $U$, and therefore $K^+e = 0$ by Theorem 6.

The following result also appears in Broué [2].

PROPOSITION 8. Let $G$ be a finite group, $u$ a $p$-element in $G$ and $U$ the $p$-section of $G$ containing $u$. Moreover, let $B$ be a block of $\mathcal{O}G$ with block idempotent $e$ and defect group $D$. Then the following statements are equivalent:

(1) $K^+e \in \mathcal{O}UG$ for every conjugacy class $K$ of $G$ contained in $U$;
(2) $u$ is not conjugate in $G$ to an element in $Z(D)$.
PROOF. Suppose first that \( u \in Z(D) \). By Brauer’s First Main Theorem on Blocks, \( Br_D(e) \) is a block idempotent in \( FN_G(D) \) with defect group \( D \). Since \( Br_D(e)^2 = Br_D(e) \), Proposition 5 implies that there is a \( p \)-regular conjugacy class \( S \) of \( N_G(D) \) with defect group \( D \) such that \( S^+Br_D(e) \notin JZFN_G(D) \). We choose \( s \in S \) and note that \( S \subseteq C_G(D) \). Moreover, we denote by \( L \) the conjugacy class of \( N_G(D) \) containing \( us = su \), and by \( \nu : FN_G(D) \to \mathbb{F}[N_G(D)/D] \) the natural epimorphism. It is easy to see that

\[
\nu(L^+) = \left| N_G(D) \cap C_G(s) : N_G(D) \cap C_G(us) \right| \nu(S^+) \neq 0.
\]

Since the kernel of \( \nu \) is nilpotent this means that

\[
L^+ - \left| N_G(D) \cap C_G(s) : N_G(D) \cap C_G(us) \right| S^+ \in JZFN_G(D),
\]

so \( Br_D(e)L^+ \notin JZFN_G(D) \). If \( K \) denotes the conjugacy class of \( G \) containing \( us \) then \( K \) has defect group \( D \), and \( K \cap C_G(D) = L \). Thus

\[
Br_D(K^+e) = Br_D(K^+)Br_D(e) = L^+Br_D(e) \notin JZFN_G(D),
\]

so \( K^+e \notin JZ\theta G \).

Now suppose conversely that \( u \) is not conjugate to an element in \( Z(D) \), and let \( K \) be a conjugacy class of \( G \) contained in \( U \). If \( Q \) denotes a defect group of \( K \) then

\[
K^+e \in (\theta G)_Q^G \cap (\theta G)_D^G \subseteq \sum_{R < D} (\theta G)^G_R + (J\theta)Z\theta G,
\]

so \( K^+e \in \left( \sum_{R < D} (\theta G)^G_R + (J\theta)Z\theta G \right) \cap Z\theta Ge \subseteq JZ\theta G \).

Appendix: Green’s theorem à la Puig

The theorem we need is the following one.

**Theorem 9.** Let \( P \) be a finite \( p \)-group, let \( A \) be a \( P \)-algebra, and let \( i \) be an idempotent in \( A_1^p \). Then there is an idempotent \( j \) in \( A \) orthogonal to \( g j \) for \( g \in P \setminus \{1\} \) such that \( i = Tr_1^p(j) \).

The theorem proved by Puig in [18] is more general, but this version suffices for our purposes.

In the proof of Theorem 1, the theorem is applied with \( P = \langle u \rangle / \langle v \rangle \), \( A = (\theta G)^{(v)} \) and \( i = (e - e_u)f \).
PROOF. Since $i A_i^p = (i A_i)^p$ we may replace $A$ by $i A_i$ and therefore assume that $i = 1_A$. We denote by $M_P(A)$ the $\mathcal{O}$-algebra consisting of all matrices of degree $|P|$ with coefficients in $A$. It will be convenient to index rows and columns of elements in $M_P(A)$ by elements in $P$. Then $M_P(A)$ becomes a $P$-algebra over $\mathcal{O}$ in such a way that $(^z m)_{y,z} = x (m_{x^{-1}y, x^{-1}z})$ for $x, y, z \in P$ and $m \in M_P(A)$.

For $a \in A$, we define $\delta(a) \in M_P(A)$ by $\delta(a)_{x,y} := a$ if $x = y = 1$, and $\delta(a)_{x,y} = 0$ otherwise. Then $\delta : A \rightarrow M_P(A)$ is a (non-unitary) monomorphism of algebras.

For $a \in A$, we define $\theta(a) := \text{Tr}_1^p (\delta(a)) \in M_P(A)^p$. Since

$$
\theta(a)_{x,y} = \sum_{z \in P} (^z \delta(a))_{x,y} = \sum_{z \in P} (\delta(a)_{x^{-1}z, y^{-1}z})
$$

for $a \in A$ and $x, y \in P$ we have $\theta(a)_{x,y} = ^x a$ if $x = y$, and $\theta(a)_{x,y} = 0$ otherwise. Thus $\theta : A \rightarrow M_P(A)^p$ is a unitary homomorphism of algebras.

We write $1_A = \text{Tr}_1^p (c)$ with $c \in A$ and define $\alpha(a) \in M_P(A)$ for $a \in A$ by $\alpha(a)_{x,y} := (^x c) a$ for $x, y \in P$. Then $\alpha : A \rightarrow M_P(A)$ is a homomorphism of $P$-algebras since

$$(\alpha(a)\alpha(b))_{x,y} = \sum_{z \in P} \alpha(a)_{x,z} \alpha(b)_{z,y} = \sum_{z \in P} (^z c) a (^z c) b = (^x c) a b = \alpha(ab)_{x,y}$$

and

$$(^x \alpha(a))_{y,z} = x \alpha(a)_{x^{-1}y, x^{-1}z} = x (^z c) a = (^x c) (^z a) = \alpha(^z a)_{y,z}$$

for $x, y, z \in P$ and $a \in A$. Moreover, $\alpha$ is injective since

$$\sum_{x \in P} \alpha(a)_{x,1} = \sum_{x \in P} (^x c) a = a$$

for $a \in A$. Finally, $\alpha(A) = \alpha(1) M_P(A) \alpha(1)$ since

$$(\alpha(1)m\alpha(1))_{x,y} = \sum_{u,v \in P} \alpha(1)_{x,u} m_{u,v} \alpha(1)_{v,y}$$

$$= \sum_{u,v \in P} (^x c) m_{u,v} (^v c) = \alpha \left( \sum_{u,v \in P} m_{u,v} (^v c) \right)_{x,y}$$

for $m \in M_P(A)$ and $x, y \in P$. 

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For \( x \in P \), we define \( \gamma(x) \in M_P(A) \) by \( \gamma(x)_{y,z} = 1 \) if \( z = yx \), and \( \gamma(x)_{y,z} = 0 \) otherwise. Then

\[
(\gamma(u)\gamma(v))_{x,y} = \sum_{z \in P} \gamma(u)_{x,z} \gamma(v)_{z,y} = \gamma(v)_{xu,y} = \gamma(uv)_{x,y}
\]

and

\[
(\gamma(v))_{x,y} = \gamma(\gamma(v)_{ux,u^{-1}y}) = \gamma(v)_{u^{-1}x,u^{-1}y} = \gamma(v)_{x,y}
\]

for \( u, v, x, y \in P \). Since \( \gamma(1) = 1, \gamma : P \to UM_P(A)^P \) is a homomorphism of groups.

If \( m \in M_P(A)^P \) then \( m_{x,y} = (\gamma m)_{x,y} = \gamma(m_{z^{-1},x,z^{-1}}) \) for \( x, y, z \in P \). Thus

\[
\left( \sum_{z \in P} \theta(m_{1,z})\gamma(z) \right)_{x,y} = \sum_{u,z \in P} \theta(m_{1,z})_{x,u} \gamma(z)_{u,y} = \sum_{u \in P} \theta(m_{1,u^{-1}})_{x,u} = \gamma(m_{1,x^{-1}}) = m_{x,y}
\]

for \( x, y \in P \), so \( m = \sum_{z \in P} \theta(m_{1,z})\gamma(z) \) and \( M_P(A)^P = \sum_{z \in P} \theta(A)\gamma(z) \).

Moreover,

\[
(\gamma(x)\theta(a)\gamma(x^{-1}))_{y,z} = \sum_{u,v \in P} \gamma(x)_{y,u} \theta(a)_{u,v} \gamma(x^{-1})_{v,z} = \theta(a)_{yx,zx} = \theta(a)_{y,z}
\]

for \( x, y, z \in P \) and \( a \in A \), so \( \gamma(x)\theta(a) = \theta(a)\gamma(x) \) for \( a \in A \) and \( x \in P \).

Suppose that \( \sum_{x \in P} \theta(a_x)\gamma(x) = 0 \) for some orthogonal primitive idempotents \( a_1, \ldots, a_r \) in \( A \). Then we have \( 1_{M_P(A)} = \gamma(1) = \gamma(e_1) + \cdots + \gamma(e_r) \) with pairwise orthogonal primitive idempotents \( e_1, \ldots, e_r \) in \( A \). Since \( \alpha(1) \) is an idempotent in \( M_P(A)^P \) there are a subset \( J \) of \( \{1, \ldots, r\} \) and a unit \( w \) in \( M_P(A)^P \) such that \( \alpha(1) = w(\sum_{i \in J} \theta(e_i)) \) (see 2.10 in [13], for example). Then \( \gamma' := w(\sum_{i \in J} \delta(e_i)) \) is an idempotent in \( M_P(A) \) orthogonal to \( xj' \) for \( g \in P \setminus \{1\} \) such that \( \alpha(1) = Tr_{P}^p(j') \); in particular,
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\[ j' \in \alpha(1)M_p(A)\alpha(1) = \alpha(A), \]
so \( j' = \alpha(j) \) for an idempotent \( j \) in \( A \) satisfying the required properties.

The seemingly technical calculations of the proof are by now standard tools in ring theory (see Cohen and Montgomery [4], for example).

It remains to prove Green's indecomposability theorem in the following version.

**Proposition 10.** Let \( P \) be a finite \( p \)-group, \( A \) a \( P \)-algebra over \( \mathcal{O} \) and \( AP \) the corresponding skew group algebra of \( P \) over \( A \). Then every primitive idempotent in \( A \) remains primitive in \( AP \).

**Proof.** Let \( i \) be a primitive idempotent in \( A \). Then \( i + JA \) is a primitive idempotent in \( A/JA \). Since \( JA \) is a \( P \)-invariant ideal of \( A \), it generates a nilpotent ideal \( (JA)(AP) = (AP)(JA) \) of \( AP \) such that \( AP/(JA)(AP) \) is isomorphic to \( (A/JA)P \), the skew group algebra of \( P \) over the \( P \)-algebra \( A/JA \). Since it suffices to prove that \( i + JA \) is primitive in \( (A/JA)P \), we may assume that \( JA = 0 \).

In this case \( A \) is a direct product of complete matrix algebras over \( \mathbb{F} \) permuted by \( P \). If \( A \) is isomorphic to \( A_1 \times A_2 \) with \( P \)-algebras \( A_1, A_2 \) then \( AP \) is isomorphic to \( A_1P \times A_2P \). Thus we may assume that \( A = B_1 \times \cdots \times B_q \) with complete matrix algebras \( B_1, \ldots, B_q \) over \( \mathbb{F} \) transitively permuted by \( P \). We denote by \( Q \) the stabilizer of \( B_1 \) in \( P \) and by \( g_1, \ldots, g_q \) a set of representatives for the cosets \( gQ \) in \( P \). Then the map

\[ \text{Mat}(B_1Q) \longrightarrow AP, \quad [b_{ij}] \longmapsto \sum_{i,j=1}^q g_i b_{ij} g_j^{-1}, \]

is easily seen to be an isomorphism of algebras. In particular, any primitive idempotent in \( B_1Q \) remains primitive in \( AP \). Thus we may assume that \( A \) itself is a complete matrix algebra over \( \mathbb{F} \).

For \( g \in G \), there is an element \( u_g \in UA \) such that \( g a = u_g a u_g^{-1} \) for \( a \in A \), by the Skolem-Noether theorem. Moreover, \( u_g u_h u_{gh}^{-1} \in UZA = UF1_A \) for \( g, h \in P \), and the map \( (g, h) \mapsto u_g u_h u_{gh}^{-1} \) is a 2-cocycle of \( P \) with values in \( UZA = UF1_A \). Since \( P \) is a \( p \)-group we have \( H^2(P, UF) = 1 \), so we may assume that \( u_g u_h = u_{gh} \) for \( g, h \in P \). But then the map

\[ A \otimes_{\mathbb{F}} \mathbb{F}P \longrightarrow AP, \quad a \otimes g \longmapsto au_g^{-1}g, \]
is an isomorphism of algebras; in particular, \( AP/J(AP) \) is isomorphic to \( A \), and the result follows.

The result is known to hold, more generally, for crossed products instead of skew group algebras. Essentially the same proof works in this more general situation.

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