A NOTE ON HENSELIAN VALUATION RINGS

Otto Endler

(received October 25, 1967)

Let $K$ be a field and $\overline{K}$ its algebraic closure. A valuation ring $A$ of $K$ is called henselian, if there is only one valuation ring $C$ of $\overline{K}$ which lies over $A$ (i.e. such that $C \cap K = A$) or, equivalently, if Hensel's Lemma is valid for $K, A$ (see [5], F). In the following, we shall consider only rank one valuation rings.

Let $L|K$ be an algebraic field extension, $B$ a valuation ring of $L$, and $A = B \cap K$ the valuation ring of $K$ lying under $B$. If $A$ is henselian, then obviously so is $B$. It is natural to ask for conditions such that the converse is true, i.e. that $B$ henselian implies $A$ henselian. This is true, for instance, whenever $L|K$ is purely inseparable (see [2], (10.7)). We intend to show that also each of the following conditions is sufficient:

1) $L|K$ is normal and $L_s \neq L$

2) $[L:K] < \infty$ and $L_s \neq L$

where $L_s$ is the separable closure of $L$ and $[\cdot : \cdot]_s$ the separability degree. Condition 2) is weaker than the condition

2') $[L:K] < \infty$ and $[L_s : L] = \infty$

recently presented by Ribenboim([4], C), which in turn is weaker than the condition

2'') $[L:K] < \infty$, and for every $n \geq 1$ there exists exactly one separable extension $K_n|K$ of degree $n$,

used by Kaplansky and Schilling [3], Theorem 4.

To prove the sufficiency of condition 1) (theorem 1) we shall need only the conjugation theorem for valuation rings and the following well known fact (see [3], Theorem 2, or [2], (27.7)): Any field having


185
more than one henselian valuation ring is separably closed. To prove
the sufficiency of condition 2) (theorem 2) we shall need only theorem 1
and Artin-Schreier's theorem in a slightly generalized form. Hence,
our proof will be much easier than the proof of the analogous theorem
in [4].

Before proving these theorems, we want to mention two examples
which show that none of the conditions

i) \( L_s \neq L \); ii) \( [L_s : L] = \infty \); iii) \( L|K \) is normal; iv) \( [L:K]_s < \infty \)

alone is sufficient.

Example 1. Let \( A \) be the p-adic valuation ring of \( K = \mathbb{Q} \), and
let \( L \) be the decomposition field of some valuation ring \( C \) of \( K_a \)
which lies over \( A \). Then \( L_s = K_a, [L_s : L] = \infty \) \( B = C \cap L \) is henselian,
but \( A = B \cap K \) is non-henselian.

Example 2. Let \( L \) (resp. \( K \)) be the field of all algebraic (resp.
real algebraic) numbers. Then \( L|K \) is normal, \( [L:K]_s = [L:K] = 2 < \infty \),
every valuation ring of \( L \) is henselian, but no valuation ring of \( K \) is
henselian (see [4], A).

Now we prove:

**THEOREM 1.** Let \( L|K \) be normal and \( L_s \neq L \). If \( B \) is a
henselian valuation ring of \( L \), then \( A = B \cap K \) is a henselian
valuation ring of \( K \).

**Proof.** Let \( C \) be the unique valuation ring of \( L_a \) lying over \( B \).
Suppose that \( A \) is not henselian. Then there exists some valuation
ring \( C' \neq C \) of \( L \) which lies over \( A \), and we have \( B' \neq B \), where
\( B' = C' \cap L \). There exists some \( K \)-automorphism \( \sigma \) of \( L_a \) such
that \( C = \sigma C' \), and we have \( \sigma L = L \) since \( L|K \) is normal. Since \( B \) is the
only henselian valuation ring of \( L \), there exists some valuation ring
\( C'' \neq C' \) of \( L_a \) which lies over \( B' \), and we have \( \sigma C'' \neq C \),
\( \sigma C'' \cap L = \sigma (C'' \cap L) = \sigma B' = \sigma (C' \cap L) = C \cap L = B \), which is a
contradiction.

**COROLLARY.** Let \( A \) be a non-henselian valuation ring of \( K \), \( L|K \)
a separable extension, and \( B \) a henselian valuation ring of \( L \) lying
over \( A \). Then \( L_s = K_s \) is the least field that contains \( L \) and is
normal over \( K \).

**Proof.** \( L_s = K_s \) is obviously normal over \( K \). On the other hand,
let \( N|K \) be normal and \( L \subseteq N \subseteq L_a \). Then the unique valuation ring

186
In particular, applying this corollary to a henselisation $(L, B)$ of a non-henselian valued field $(K, A)$, we see that no field between $L$ and $L_s$ and distinct from $L_s$ is normal over $K$.

We shall use Artin-Schreier's theorem in the following form:

**LEMMA.** Let $S$ be a separably closed field and let $K$ be a subfield of $S$ such that $1 < [S:K] < \infty$. Then $K$ is really closed, $S$ is algebraically closed, and $S = K(\sqrt{-1})$.

**Proof.** The algebraic closure $S_a$ of $S$ is purely inseparable over the field $L = \{a \in S | a$ separable over $K\}$, and the fixed field $K'$ of the Galois group of $S_a/K$ is purely inseparable over $K$. We have $S_a = L' \cdot K'$, hence $[S_a : K'] \leq [L : K]_s = [S : K]_s < \infty$. On the other hand $[S_a : K]_s > [S : K]_s > 1$, hence $S_a \neq K'$. By Artin-Schreier's theorem (see [1], theorem 4), $K'$ is really closed, $S_a$ is algebraically closed, and $S_a = K'(\sqrt{-1})$. Since $K'$ has characteristic zero, we have $S_a = S = L$ and $K' = K$.

Now we prove:

**THEOREM 2.** Let $L/K$ be algebraic, $[L : K]_s < \infty$, and $L_s \neq L$. If $B$ is a henselian valuation ring of $L$, then $A = B \triangleleft K$ is a henselian valuation ring of $K$.

**Footnotes**

1) A henselisation of a valued field $(K, A)$ is a valued field $(L, B)$, consisting of the decomposition field $L$ over $K$ of some valuation ring $C$ of $K$ that lies over $A$ and the valuation ring $B = C \triangleleft L$. The henselisation of $(K, A)$ is unique up to an $K$-isomorphism. In particular, $(L, B) = (K, A)$ if and only if $A$ is henselian. (See [5], F).

2) I was told by Mr. Ribenboim that another proof of theorem 2 was communicated to him by Mr. Neukirch, some months ago. For the case of a perfect field $K$ see J. Neukirch, Bonner Math. Schriften Nr. 25, (4.12).
Proof. Without loss of generality we may assume \([L:K]_s > 1\).

Let \(N\) be the least field that contains \(L\) and is normal over \(K\); then
\([L:K] < [N:K] < \infty\). Suppose that \(N_s = N\); then from the lemma it follows that
\([N:K] = 2\), hence \([N:L] = [N:K][L:K]^{-1} < 1\), hence
\(L = N = N_s \supseteq L_s\), in contradiction to \(L_s \neq L\); therefore \(N_s \neq N\).

Since the valuation ring \(C\) of \(N\) that lies over \(B\) is henselian, we conclude from theorem 1 that \(A\) is henselian.

**COROLLARY.** Let \(A\) be a non-henselian valuation ring of \(K\), \(L|K\) an algebraic extension, and \(B\) a henselian valuation ring lying over \(A\). If \([L:K] < \infty\), then \(K\) is really closed, \(L\) is algebraically closed, and \(L = K(\sqrt{-1})\).

Proof. \([L:K] > 1\), since \(L|K\) is not purely inseparable. If \([L:K] < \infty\), then \(L_s = L_s\) by theorem 2. Now the corollary results from the lemma.

Applying this corollary to a henselisation \((L, B)\) of a non-henselian valued field \((K, A)\), we see that \(L|K\) is never finite unless \(K\) is really closed and \(L = L_s = L_s A = K(\sqrt{-1})\). One should note that the field \(L\) of a henselisation \((L, B)\) of a non-henselian valued field \((K, A)\) may be separably closed also in other cases. Indeed, this happens whenever \(K\) has a henselian valuation ring \(A_h\) (since then \(L\) has more than one henselian valuation ring). One knows that in this case any valuation ring \(A \neq A_h\) of \(K\) is saturated, i.e. its value group \(\Gamma_A\) is divisible and its residue field \(\Lambda_A\) is algebraically closed (see [2], (27.6)). Moreover, we prove:

**THEOREM 3.** Let \(A\) be a valuation ring of \(K\) such that \(\Lambda_A\) has characteristic zero, and let \((L, B)\) be a henselisation of \((K, A)\). Then:

\[L\text{ is algebraically closed} \iff A\text{ is saturated}.

Proof. \(L\) is the decomposition field over \(K\) of some valuation ring \(C\) of \(K\) that lies over \(A\). Let \(M\) be the inertia field of \(C\) over \(K\). Then \(K_A \supseteq M\) and \(M|L\) are Galois extensions, with Galois groups \(G_1\) and \(G_2\), say. \(G_1\) is isomorphic to the character group of the value group extension \(\Gamma_C/\Gamma_A\) and \(G_2\) is isomorphic to the Galois group of the Galois extension \(\Lambda_C/\Lambda_A\) (see [2], (20.1) and (20.20)). \(A\) is saturated if and only if \(\Gamma_C = \Gamma_A\) and \(\Lambda_C = \Lambda_A\) (see [2], (22.7)). These equations hold if and only if \(G_1\) and \(G_2\) are trivial, if and only if \(L = K_A\).
REFERENCES


Mathematisches Institut der Universität Bonn (Germany)