ON A QUESTION OF GROSS CONCERNING UNIQUENESS OF ENTIRE FUNCTIONS

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In this paper, we prove that there exist two finite sets $S_1$ (with 1 element) and $S_2$ (with 3 elements) such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical. This answers a question posed by Gross. Examples are provided to show that this result is sharp.

1. INTRODUCTION

Let $h$ be a nonconstant entire function, and let $S$ be a subset of distinct elements in $\mathbb{C}$. Define (see [10])

$$E_h(S) = \bigcup_{a \in S} \{ z \mid h(z) = a, \text{ counting multiplicities} \},$$

$$\overline{E}_h(S) = \bigcup_{a \in S} \{ z \mid h(z) = a, \text{ ignoring multiplicities} \}.$$

Let $f$ and $g$ be two nonconstant entire functions, and let $S$ be a subset of distinct elements in $\mathbb{C}$. If $E_f(S) = E_g(S)$, we say $f$ and $g$ share the set $S$ CM (counting multiplicity). If $\overline{E}_f(S) = \overline{E}_g(S)$, we say $f$ and $g$ share the set $S$ IM (ignoring multiplicity). As a special case, let $S = \{a\}$, where $a \in \mathbb{C}$. If $E_f(\{a\}) = E_g(\{a\})$, we say $f$ and $g$ share the value $a$ CM. If $\overline{E}_f(\{a\}) = \overline{E}_g(\{a\})$, we say $f$ and $g$ share the value $a$ IM (see [2]).

In 1976, Gross asked the following question:

**QUESTION 1.** (See [1, Question 6].) Can one find two finite sets $S_j$ ($j = 1, 2$) such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In 1994, the present author [6] proved the following theorem, which answered the above Question 1 in the affirmative.
THEOREM A. Let $S_1 = \{\omega \mid \omega^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n \geq 5$, $a \neq 0$ and $a^{2n} \neq 1$. If $f$ and $g$ are entire functions such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

In [1] Gross wrote: "If the answer to Question 6 is affirmative, it would be interesting to know how large both sets would have to be."

Now it is natural to ask the following question:

QUESTION 2. What are the smallest cardinalities of $S_1$ and $S_2$ respectively, where $S_1$ and $S_2$ are two finite sets such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In this paper, we prove the following theorems, which answer Question 2.

THEOREM 1. Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where $a$ and $b$ are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If $f$ and $g$ are two entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

REMARK. Let $S_1 = \{0\}$ and $S_2 = \{2, -3, -6\}$. It is easy to see that $S_2 = \{\omega \mid \omega^2(\omega + 7) - 36 = 0\}$. From Theorem 1 we immediately obtain that if $f$ and $g$ are entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.

THEOREM 2. If $S_1$ and $S_2$ are two finite sets such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical, then $\max\{\#(S_1), \#(S_2)\} \geq 3$, where $\#(S)$ denotes the cardinality of the set $S$.

REMARK. From Theorem 2 we immediately obtain that the smallest cardinalities of $S_1$ and $S_2$ are 1 and 3 respectively, where $S_1$ and $S_2$ are two finite sets such that any two entire functions $f$ and $g$ satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical. This shows that Theorem 1 is sharp.

2. SOME LEMMAS

In this paper, we use the usual notations of Nevanlinna theory of meromorphic functions as explained in [3].

L EMMA 1. (See [9, Lemma 5].) Let $f$ and $g$ be two nonconstant meromorphic functions, and let $c_1$, $c_2$ and $c_3$ be three nonzero constants. If

\[ c_1f + c_2g = c_3, \]

then

\[ T(r,f) < N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{g}\right) + N(r,f) + S(r,f). \]
Let \( h \) be a nonconstant meromorphic function. We denote by \( N_2(r, h) \) the counting function of poles of \( h \), where a simple pole is counted once and a multiple pole is counted two times (see [7]).

**Lemma 2.** (See [7, Theorem 1].) Let \( F \) and \( G \) be two nonconstant meromorphic functions such that \( F \) and \( G \) share the value 1 CM. If

\[
\limsup_{r \to \infty} \frac{N_2(r, 1/F) + N_2(r, F) + N_2(r, 1/G) + N_2(r, G)}{T(r)} < 1,
\]

where \( T(r) = \max\{T(r, F), T(r, G)\} \), \( I \) denotes any set of infinite linear measure of \( 0 < r < \infty \), then \( F \equiv G \) or \( F \cdot G \equiv 1 \).

**Lemma 3.** Let

\[
F = \frac{f^2(f + a)}{b}, \quad G = \frac{g^2(g + a)}{b},
\]

where \( f \) and \( g \) are two nonconstant entire functions, \( a \) and \( b \) are two nonzero constants. Then \( F \cdot G \neq 1 \).

**Proof:** If \( F \cdot G \equiv 1 \), from (2.2) we have

\[
f^2(f + a)g^2(g + a) \equiv b^2.
\]

From this we know that 0 and \(-a\) are Picard exceptional values of \( f \), which is impossible. Thus \( F \cdot G \neq 1 \).

**Lemma 4.** Let \( f \) and \( g \) be two nonconstant entire functions which share the value 0 IM. If \( F \equiv G \), where \( F \) and \( G \) are given by (2.2), then \( f \equiv g \).

**Proof:** Since \( F \equiv G \), we have from (2.2)

\[
f^2(f + a) = g^2(g + a).
\]

Noting \( f \) and \( g \) share the value 0 IM, from (2.3) we know that \( f \) and \( g \) share 0 CM. From (2.3) we have

\[
f^3 - g^3 = -a(f^2 - g^2).
\]

If \( f^3 \neq g^3 \), from (2.4) we obtain

\[
g = \frac{a(h + 1)}{(h - u)(h - u^2)},
\]

where \( h = f/g \) and \( u = \exp((2\pi i)/3) \). From (2.5) we know that \( h \) is a nonconstant meromorphic function. Noting \( f \) and \( g \) share the value 0 CM, from \( h = f/g \) we know that 0 and \( \infty \) are Picard exceptional values of \( h \). Since \( g \) is a nonconstant entire function, from (2.5) we know that \( u \) and \( u^2 \) are Picard exceptional values of \( h \), which is impossible. Thus \( f^3 \equiv g^3 \) and \( f^2 \equiv g^2 \). From this, we get \( f \equiv g \).
3. Proof of Theorem 1

Let $F$ and $G$ be given by (2.2). Thus,

\begin{equation}
T(r, F) = 3T(r, f) + S(r, f), \quad T(r, G) = 3T(r, g) + S(r, g).
\end{equation}

Set

\begin{equation}
H = \frac{F'}{F - 1} - \frac{G'}{G - 1}.
\end{equation}

We discuss the following two cases.

**Case 1.** Suppose that $H = 0$. By integration we have from (3.2)

\begin{equation}
F - 1 = A(G - 1),
\end{equation}

where $A$ is a nonzero constant. We discuss the following two subcases.

**Case 1.1.** Assume that $A = 1$. From (3.3) we have $F \equiv G$. By Lemma 4 we get $f \equiv g$.

**Case 1.2.** Assume that $A \neq 1$. Suppose $0$ is not a Picard exceptional value of $f$ and $g$. Since $f$ and $g$ share the value $0$ CM, then there exists $z_0$ such that $f(z_0) = g(z_0) = 0$. From (2.2) we obtain $F(z_0) = G(z_0) = 0$. From this and (3.3) we get $A = 1$, which is a contradiction. Thus, $0$ is a Picard exceptional value of $f$ and $g$. From (3.3) we have

\begin{equation}
F - AG = 1 - A.
\end{equation}

From this we have

\begin{equation}
T(r, G) = T(r, F) + O(1).
\end{equation}

By Lemma 1, we obtain from (2.2), (3.1), (3.4) and (3.5)

\begin{align*}
3T(r, f) &\leq \overline{N}\left(r, \frac{1}{f + a}\right) + \overline{N}\left(r, \frac{1}{g + a}\right) + S(r, f) \\
&\leq 2T(r, f) + S(r, f),
\end{align*}

which is impossible.

**Case 2.** Suppose that $H \neq 0$. Then $F \neq G$. By $E_f(S_2) = E_g(S_2)$, we know that $F$ and $G$ share the value $1$ CM. From (3.2) we have

\begin{equation}
T(r, H) = m(r, H) + N(r, H) = S(r, F) + S(r, G).
\end{equation}
Since \( f \) and \( g \) share the value 0 CM, \( f \) and \( g \) have the same zeros. Let \( z_0 \) be a zero of \( f \) and \( g \). From (2.2) and (3.2) we know that \( z_0 \) is a zero of \( H \). From this and (3.6) we get

\[
N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{g}\right) \leq N\left(r, \frac{1}{H}\right) \leq S(r, F) + S(r, G).
\]

From this, (2.2) and (3.1) we obtain

\[
N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) \leq N\left(r, \frac{1}{f + a}\right) + N\left(r, \frac{1}{g + a}\right) + S(r, F) + S(r, G) \\
\leq \frac{1}{3} T(r, F) + \frac{1}{3} T(r, G) + S(r, F) + S(r, G).
\]

Thus,

\[
\limsup_{r \to \infty} \frac{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)}{T(r)} \leq \frac{2}{3} < 1,
\]

where \( T(r) = \max\{T(r, F), T(r, G)\} \), \( I \) denotes any set of infinite linear measure of \( 0 < r < \infty \). By Lemma 2, we obtain \( F \cdot G = 1 \). Again by Lemma 3, we get a contradiction.

This completes the proof of Theorem 1.

\[\square\]

4. PROOF OF THEOREM 2

4.1. SOME EXAMPLES.

Example 1. Let \( S_1 = \{a\} \) and \( S_2 = \{b\} \), where \( a \) and \( b \) are any two finite distinct complex numbers. Let

\[
f(z) = a + (b - a)e^{h(z)}, \quad g(z) = a + (b - a)e^{-h(z)},
\]

where \( h(z) \) is a nonconstant entire function. It is easy to show that \( E_f(S_j) = E_g(S_j) \) \((j = 1, 2)\), but \( f \not= g \).

Example 2. (See [8].) Let \( S_1 = \{a\} \) and \( S_2 = \{b_1, b_2\} \), where \( a, b_1 \) and \( b_2 \) are any three finite distinct complex numbers. Let

\[
f(z) = a + (b_1 - a)e^{h(z)}, \quad g(z) = a + (b_2 - a)e^{-h(z)},
\]

where \( h(z) \) is a nonconstant entire function. It is easy to show that \( E_f(S_j) = E_g(S_j) \) \((j = 1, 2)\), but \( f \not= g \).
EXAMPLE 3. (See [4].) Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, b_1$ and $b_2$ are any four finite distinct complex numbers satisfying $a_1 + a_2 \neq b_1 + b_2$. Let

$$f(z) = d + (d - a_1)e^{h(z)}, \quad g(z) = d + (d - a_2)e^{-h(z)},$$

where $h(z)$ is a nonconstant entire function, $d = (a_1a_2 - b_1b_2)/(a_1 + a_2 - b_1 - b_2)$. It is easy to show that $E_f(S_j) = E_g(S_j) \ (j = 1, 2)$, but $f \neq g$.

EXAMPLE 4. (See [5].) Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, b_1$ and $b_2$ are any four finite distinct complex numbers satisfying $a_1 + a_2 = b_1 + b_2$. Let $f(z)$ be a nonconstant entire function, $g(z) = a_1 + a_2 - f(z)$. It is easy to show that $E_f(S_j) = E_g(S_j) \ (j = 1, 2)$, but $f \neq g$.

4.2. PROOF OF THEOREM 2.

Suppose that $\max\{\#(S_1), \#(S_2)\} < 3$. We proceed to get a contradiction. If $\#(S_1) = \#(S_2) = 1$, from Example 1 we have a contradiction. If $\#(S_1) = 1$ and $\#(S_2) = 2$ or $\#(S_1) = 2$ and $\#(S_2) = 1$, from Example 2 we have again a contradiction. If $\#(S_1) = \#(S_2) = 2$, from Example 3 and Example 4 we can get a contradiction. This completes the proof of Theorem 2.

5. CONCLUDING REMARK

In fact, in Section 3 of this paper we proved the following theorem, which is an improvement of Theorem 1.

**Theorem 3.** Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where $a$ and $b$ are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If $f$ and $g$ are two entire functions satisfying $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then $f \equiv g$.

Proceeding as in the proof of Theorem 1, we can prove the following result, which is an extension of Theorem 3.

**Theorem 4.** Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^n(\omega + a) - b = 0\}$, where $n(\geq 2)$ is an integer, $a$ and $b$ are two nonzero constants such that the algebraic equation $\omega^n(\omega + a) - b = 0$ has no multiple roots. If $f$ and $g$ are two entire functions satisfying $E_f(S_1) = E_g(S_1)$ and $E_f(S_2) = E_g(S_2)$, then $f \equiv g$.

Let $n(\geq 2)$ be an integer, and let $a$ and $b$ be two nonzero constants. It is easy to show that if $b \neq (n^na^{n+1})/(n + 1)^{n+1}$, the algebraic equation $\omega^n(\omega + a) - b = 0$ has no multiple roots. Specially, if $b \neq 4a^3/27$, the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots.
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