Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 829–840

MODEL-COMPLETENESS AND ELEMENTARY PROPERTIES OF TORSION FREE ABELIAN GROUPS

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Introduction. The decidability of the elementary theory of abelian groups, and their complete classification by elementary properties (i.e. those formalizable in the lower predicate calculus (LPC) of formal logic), were established by W. Szmielew [**13**]. More general results were proved by Eklof and Fischer [**2**], and G. Sabbagh [**12**]. The rather formidable "high-power" techniques used in obtaining these remarkable results, and the length of the proofs (W. Szmielew's proof takes about 70 pages) triggered off several attempts at simplification. M. I. Kargapolov's proof [**3**] unfortunately turned out to be erroneous (cf. J. Mennicke's review in the Journal of Symbolic Logic, vol. 32, p. 535). At the meeting of the Canadian Mathematical Congress at Kingston, June, 1966, the present author outlined a very simple proof for the special case of *torsion free* groups. A detailed abstract was published in [**15**]. In this proof, we only use A. Robinson's model-completeness test [**8**] and a few rather elementary lemmas of algebraic nature, as well as some facts already proved in two previous papers [**10**; **14**] dealing with *ordered* groups. The technique used to ensure model-completeness is that of adjoining certain one-place atomic predicates $D_n(x)$, $n = 1, 2, \ldots,$ each distinguishing a subgroup (namely that of all elements which are divisible by n).

In [**4**], G. T. Kozlov and A. I. Kokorin generalized this method by taking up the "elementary theory of torsion-free groups, with a predicate that distinguishes a subgroup". In this manner they obtain W. Szmielew's result and seem to avoid Kargapolov's error. However, their proof is far more complicated than that of [**15**]. (It combines Robinson's model-completeness test with the Feferman-Vaught theorem and several rather difficult algebraic lemmas.)

In view of this, we consider it useful to publish our original proof [**15**] in full, thus providing a thorough, yet brief and elementary, analysis of torsion-free groups from the viewpoint of the lower predicate calculus (LPC) of formal logic. Though simple, it is strong enough to furnish *all* elementary theories for torsion free groups, which are both complete and model-complete (in this we supplement W. Szmielew's work which does not deal with model-completeness). Thus we obtain a complete elementary classification of torsion-free abelian groups.

Received January 31, 1973 and in revised form, April 9, 1973.

As a by-product, we obtain some algebraic results supplementing Prüfer's [**5, 6, 7**] and generalizing a theorem proved in [**14**]. We also extend to *all* torsion-free abelian groups A. Robinson's theorem [**8**, 3.1.5] in which model-completeness was proved for *divisible* torsion-free groups only.

The general case of *all* abelian groups will be left for a separate paper.

1. Terminology and notation. We recall some definitions from [**10**] and [**11**], with only minor adjustments.

1.1. Given an abelian group A and a positive integer p, we define the *p-th congruence invariant* of A, denoted $[p]A$ or briefly $[p]$, to be the maximum (possibly infinite) number of elements that can be selected from A in such a manner that they are mutually incongruent modulo p. (As usual, we write $a \equiv b \pmod p$ in A ("a is *p-congruent* with b") if and only if there is an element $x \in A$ such that $a = px + b$. If $b = 0$, i.e. $a = px$, we say that a is *divisible by* p (or *p-divisible*) in A.) Equivalently, $[p]A$ is the order of the quotient group A/pA where pA is the subgroup of all elements divisible by p in A. In the infinite case, we set $[p] = \infty$, without distinguishing between infinities of different cardinalities, and with the usual conventions as to inequalities and operations. If p is a prime, $[p]$ is called a *prime invariant* of A.

1.2. A *linear system* is any *finite* system of equations, inequalities ($\neq$), congruences and (or) incongruences of the form

$$\sum_{j=1}^{n} q_{ij}x_j = a_i, \quad \sum_{j=1}^{n} q_{kj}'x_j \equiv a_k' \pmod{r_k'}, \quad i, k = 1, 2, \ldots$$

(with $=$ and $\equiv$ possibly replaced by $\neq$ and $\not\equiv$, respectively) where q_{ij}, q_{kj}' are given integers; the x_j are unknowns; r_k' are positive integers; and a_i, a_k' are given elements of an abelian group A. The a_i, a_k' are called the *constants* of the system. Given two linear systems L and L', with constants in A, we say that L' is *stronger* than L (and L is *weaker* than L') if every solution of L' is also a solution of L. If L' is both weaker and stronger than L, the two systems are said to be *equivalent.*

1.3. As usual, a subgroup A_0 of A is said to be *pure* or *serving* in A if, for any positive integer r and any $a \in A_0$, the congruence $a \equiv 0 \pmod r$ holds in A_0 whenever it holds in A.

2. Some algebraic lemmas. We shall need a few purely algebraic lemmas. In all of them A is a torsion-free abelian group $\neq \{0\}$.

2.1. LEMMA. *For any prime* p, $[p^n]A = ([p]A)^n$, $n = 1, 2, \ldots$.

Proof. Let $A_n = p^nA$ (= subgroup of all elements divisible by p^n), $n = 0, 1, 2, \ldots$. We shall first of all show that

(2.1.1) $\quad [p^{n+1}]A_n = [p]A, \qquad n = 0, 1, 2, \ldots$.

https://doi.org/10.4153/CJM-1974-078-4 Published online by Cambridge University Press

(Here (in this proof only) $[p^{n+1}]A_n$ is the maximum number of elements that can be selected *from* A_n in such a manner that they are mutually p^{n+1}-incongruent *in* A. The term "p^{n+1}-incongruent elements" means "p^{n+1}-incongruent *in* A".)

In fact, the map $x \to p^n x$ carries A onto A_n, and is bijective since A is torsion-free. Also $x \in A$ is divisible by p if and only if $p^n x$ is divisible by p^{n+1}. Thus $x \not\equiv y \pmod{p}$ if and only if $p^n x \not\equiv p^n y \pmod{p^{n+1}}$. Hence one can select exactly as many mutually p-incongruent elements from A as there are p^{n+1}- incongruent elements in A_n. This proves (2.1.1).

We now prove 2.1 by induction. The lemma clearly holds for $n = 1$. Suppose it holds for some n. By definition, $[p^n]A$ equals the order of A/A_n, i.e. the number of distinct cosets of the form $a + A_n$ ($a \in A$). Clearly, the map $x \to a + x$ is a bijection of A_n onto $a + A_n$; moreover, it carries any two p^{n+1}-incongruent elements of A_n into such elements of $a + A_n$, and vice versa. Thus the maximum number of mutually p^{n+1}-incongruent elements in $a + A_n$ is the same as in A_n; i.e. it equals $[p^{n+1}]A_n = [p]A$, by (2.1.1). But, as we noted, A splits into exactly $[p^n]A$ cosets $a + A_n$ and, clearly, no two elements selected from *distinct* cosets can be p^n-congruent, let alone p^{n+1}-congruent. Thus the *total* maximum number of p^{n+1}-incongruent elements in A equals the number of cosets, $[p^n]A$, times $[p]A$. By our inductive hypothesis, then, $[p^{n+1}]A = [p^n]A \cdot [p]A = ([p]A)^n \cdot [p]A = ([p]A)^{n+1}$, and the induction is complete.

Our remaining lemmas deal with linear systems, as defined above. While the theory of linear equations and congruences in abelian groups is well established (cf. [**5**; **6**; **7**]), little has been done about systems in which also *incongruences* occur along with congruences (to be satisfied simultaneously). It is, however, this kind of system which is important for our purposes. The somewhat arduous lemmas proved below fill this gap, for torsion-free groups. The basic idea in these lemmas is to replace incongruences by *stronger* congruences in such a manner that the arising *stronger* linear system (containing *no* incongruences) is still solvable. It then follows that the original weaker system is solvable *a fortiori*.

2.2. LEMMA. *Let $a, a_1, a_2, \ldots, a_n \in A$. Let $k, k_1, k_2, \ldots, k_n$ be integers, with $k_i > k \geqq 0$, $i = 1, \ldots, n$. Let p be a prime, with $[p]A > n$. Then the following linear system in one unknown x is solvable in A:*

(2.2.1) $x \equiv a \pmod{p^k}, \qquad x \not\equiv a_i \pmod{p^{k_i}}, \qquad i = 1, 2, \ldots, n.$

Proof. As $k < k_i$, the congruence $x \equiv a_i \pmod{p^{k_i}}$ (if true) would imply $x \equiv a_i \pmod{p^k}$. Hence the *incongruence* $x \not\equiv a_i \pmod{p^k}$ implies $x \not\equiv a_i \pmod{p^{k_i}}$.

Now suppose that $a_i \not\equiv a \pmod{p^k}$ for some of the given a_i. For such a_i, the congruence $x \equiv a \pmod{p^k}$ in (2.2.1) *implies* $x \not\equiv a_i \pmod{p^k}$ and hence, as noted above, $x \not\equiv a_i \pmod{p^{k_i}}$. Thus the latter incongruence is *redundant* in (2.2.1) whenever $a \not\equiv a_i \pmod{p^k}$, and may be dropped without affecting the solutions of the system (if any). We assume that all such redundant incongruences have already been dropped, and so we have a system (2.2.1) in which $a_i \equiv a \pmod{p^k}$, $i = 1, 2, \ldots, n$. This means that there are elements $z_i \in A$ such that $a_i = a + p^k z_i$, $i = 1, 2, \ldots, n$. To fix ideas, let k_1 be the least of all k_i. Since $k < k_1$ and $[p]A > n$, Lemma 2.1 yields $[p^{k_1-k}]A = ([p]A)^{k_1-k} > n$. Thus, by the definition of $[p^{k_1-k}]$, one can find in A *more* than n elements that are mutually incongruent modulo p^{k_1-k}. Hence there is $z_0 \in A$ such that $z_0 \not\equiv z_i \pmod{p^{k_1-k}}$, $i = 1, 2, \ldots, n$, with the z_i as above. As A is torsion-free, we obtain $a_i = a + p^k z_i \not\equiv a + p^k z_0 \pmod{p^{k_i}}$. Thus $x = a + p^k z_0$ is a solution of (2.2.1) in A.

2.3. Lemma. *Let L be a linear system in one unknown x, of the form:*

$$
\begin{aligned}
&(2.3.1)\quad x \equiv a_i \pmod{p_i^{k_i}}, \qquad i = 1, 2, \ldots, m, \\
&(2.3.2)\quad x \not\equiv a_i \pmod{p_i^{k_i}}, \qquad i = m+1, m+2, \ldots, m'
\end{aligned}
\qquad (a_i \in A)
$$

where the k_i are integers > 0 and the p_i are primes. Then the following conditions (combined) suffice for L to have a solution in A:

(a) *The primes p_i in the congruences* (2.3.1) *are distinct;*

(b) *If some p_i occurs in both* (2.3.1) *and* (2.3.2), *then its exponent k_i in* (2.3.1) *is less than all its exponents in* (2.3.2)*; and*

(c) $[p_i]A = \infty$, $i = m+1, m+2, \ldots, m'$ (it suffices that $[p_i] > m' - m$).

Proof. Suppose that some $p_i = p$ occurs in one or several *incongruences* (2.3.2); say, in the first n of them:

$$(2.3.3)\quad x \not\equiv a_i \pmod{p^{k_i}},\ i = m+1, m+2, \ldots, n\ (n \leqq m').$$

Let $k = \min k_i$ $(i > m)$. Then, by assumption (c), we have $[p^k] = ([p])^k \geqq [p] = \infty > m' - m$; so, by the definition of $[p^k]$, there is $a_0 \in A$ such that $a_0 \not\equiv a_i \pmod{p^k}$, hence $a_0 \not\equiv a_i \pmod{p^{k_i}}$, $i = m+1, \ldots, n$. For *that* a_0, (2.3.3) is weaker than the single congruence $x \equiv a_0 \pmod{p^r}$, $r = \max k_i$; for it implies $x \equiv a_0 \pmod{p^{k_i}}$, hence $x \not\equiv a_i \pmod{p^{k_i}}$, $i = m+1, m+2, \ldots, n$, as required in (2.3.3).

Now, if p *does not* occur in (2.3.1), we replace (2.3.3) by $x \equiv a_0 \pmod{p^r}$ and include the latter in (2.3.1). This only strengthens L, *preserving condition* (a). If however, p does occur in some congruences (2.3.1), say in $x \equiv a_1 \pmod{p^{k_1}}$, then, by (b), k_1 is less than all k_i in (2.3.3). Thus by Lemma 2.2, there is an $a' \in A$ such that $a' \equiv a_1 \pmod{p^{k_1}}$ and $a' \not\equiv a_i \pmod{p^{k_i}}$, $i = m+1, \ldots, m'$. Clearly, both (2.3.3) and $x \equiv a_1 \pmod{p^{k_1}}$ are weaker than the single congruence $x \equiv a' \pmod{p^r}$ where $r = \max(k_{m+1}, \ldots, k_{m'})$. Thus we replace *both* by that single congruence. This again strengthens L and preserves (a). This process, when applied to *all* incongruences (2.3.2) transforms L into a stronger system L' of the form (2.3.1), *with all p_i distinct.* By a well-known elementary argument [**1**, p. 24] (which applies to all torsion-free abelian groups), L' has a solution in A. This completes the proof.

We shall say that a subgroup A_0 of A is *closed in A, with respect to linear systems L of a certain kind*, if any such system can be solved in A_0 whenever it has a solution in A and its constants are in A_0.

2.4. LEMMA. *If A_0 is a pure subgroup of A and if $[p]A_0 = [p]A$ for all primes, then A_0 is closed in A with respect to all systems of the form:*

$$\begin{aligned} &(2.4.1)\quad q_i x \equiv a_i \pmod{r_i}, \qquad i = 1, 2, \ldots, m; \\ &(2.4.2)\quad q_i x \not\equiv a_i \pmod{r_i}, \qquad i = m+1, m+2, \ldots, m' \end{aligned} \qquad (a_i \in A_0)$$

Proof. Let L be a system of that form, with a solution $x = c_0$ *in A*. We have to show that L is solvable *in A_0*, also. As A is torsion-free, we may assume that each q_i is prime to the corresponding r_i (otherwise, reduce by the common divisor d of r_i and q_i, noting that a_i too must be divisible by *d in both A and A_0* (by purity), since L *does* have a solution). Then (cf. [**1**, p. 23]) each congruence (hence also each incongruence) in L transforms into one in which $q_i = 1$, and a_i is replaced by some $na_i \in A_0$. Thus we may assume that all q_i equal 1, from the outset.

Moreover, every congruence, $x \equiv$ a (mod r), is equivalent to a system of the form $x \equiv a \pmod{p_j^{k_j}}$, $j = 1, \ldots, h$, where $r = p_1^{k_1} \ldots p_h^{k_h}$ is the prime-power decomposition of r. With the same notation, an *incongruence*, $x \not\equiv a$ (mod r), is equivalent to a *disjunction* composed of the incongruences $x \not\equiv a$ (mod $p_j^{k_j}$). (In other words, the incongruence $x \not\equiv a$ (mod r) holds for some x if and only if x satisfies *at least one* of the incongruences $x \not\equiv a$ (mod $p_j^{k_j}$). Thus, as $x = c_0$ is a solution of L, *one* such incongruence (at least) holds for $x = c_0$, and it *implies* $x \not\equiv a$ (mod r). Hence, substituting that incongruence for $x \not\equiv a$ (mod r), we only strengthen L, *retaining the solution* $x = c_0$. By applying this replacement process to all of (2.4.1) and (2.4.2), we thus replace L by a *stronger* system L' of the form (2.3.1)–(2.3.2), with all a_i in A_0 and with the same solution $x = c_0$ in A. In this manner all reduces to showing that L' can be solved *in A_0* as well.

For brevity, let $t_i = p_i^{k_i}$ in (2.3.1)–(2.3.2). Then, by our assumption and by Lemma 2.1, $[t_i]A_0 = [t_i]A$ for all i. Suppose that, for some $i = i_0$, $t_i = q < \infty$. Then one can find in A, as well as in A_0, exactly q (*but not more*) elements mutually incongruent mod t_{i_0}; let them be $e_0, e_1, \ldots, e_{q-1} \in A_0$. (Observe that, by the purity of A_0, the elements $e_0, e_1, \ldots, e_{q-1}$ are t_{i_0}-incongruent in *both* A_0 and A.) Then the element $c_0 \in A$ (constituting the solution of L') must be (t_{i_0})-congruent with one of the e_i; say, with e_0. This means that $x = c_0$ is also a solution of

$$(2.4.3)\quad x \equiv e_0 \pmod{t_{i_0}}, \qquad e_0 \in A_0.$$

This congruence is stronger than the incongruence $x \not\equiv a_{i_0}$ (mod t_{i_0}) occurring in L'; for, since c_0 satisfies *both*, $x \equiv e_0$ *implies* $x \equiv c_0$, hence $x \not\equiv a_{i_0}$ (modulo t_{i_0}). Thus, replacing that incongruence by (2.4.3), we only strengthen L', retaining the solution $x = c_0$. In this manner we remove from L' all

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incongruences (2.3.2) in which $[p_i] < \infty$, and then are left with a stronger system L'' *satisfying* 2.3(c). Moreover, the existence of a solution $x = c_0$ easily implies that every congruence in L'' is stronger than any other congruence or incongruence *in which the same prime* p_i *occurs with a smaller or equal exponent* k_i. These weaker congruences and incongruences may then be dropped from L'', without affecting any of its solutions in A or A_0 (the latter, by the purity of A_0). After this removal, L'' satisfies (in A as well as in A_0) the conditions (a), (b), (c) of 2.3, and its constants are still *in* A_0. Thus L'' (hence also the weaker original system L) has a solution *in* A_0. This proves the Lemma.

Note. The assumption $[p]A = [p]A_0$ was only used in the last part of the proof, to remove *incongruences* from L. Thus it is redundant if L contains no incongruences. The same remark applies to Propositions 2.6 and 3.7 below, based on 2.4.

2.5. Lemma. *For a subgroup A_0 to be closed in A, with respect to all linear systems L, it suffices that A_0 be closed with respect to those L which contain no equations* (*only inequalities, congruences and* (*or*) *incongruences*).

This was proved in [**14**, 3.9], for *ordered* groups. The same proof also applies to unordered torsion-free groups, so we omit it.

Given $c_0 \in A$, we define the c_0-*extension* of a subgroup $A_0 \subseteq A$ to be the subgroup of all elements $x \in A$ that satisfy equations of the form $tx = sc_0 + b$, with $b \in A_0$ and t, s integers ($t > 0$). As is known, it is the smallest *pure* subgroup of A containing both A_0 and c_0.

2.6. Lemma. *Let A_1 be the c_0-extension of a subgroup $A_0 \neq \{0\}$ of A. If A_0 is pure in A_1 and if $[p]A_0 = [p]A_1$ for all primes p, then A_0 is closed in A_1, with respect to all linear systems.*

The proof is quite similar to that of an analogous proposition [**14**, proposition 4.2], with minor adjustments. Let L be a linear system in n unknowns x_j, with its constants a_i in A_0, and let $(c_1, c_2, \ldots, c_n)$ be its solution *in* A_1. We have to show that L can already be solved *in* A_0. By 2.5, we may assume that L contains no *equations*.

Now, as A_1 is the c_0-extension of A_0 in A, the elements $c_j \in A_1$ satisfy some n equations of the form

$$(2.6.1) \quad t_j c_j = s_j c_0 + b_j \ (b_j \in A_0), \qquad j = 1, 2, \ldots, n,$$

where t_j and s_j are integers ($t_j > 0$). Since $(c_1, \ldots, c_n)$ is a solution, we may substitute the c_j for the x_j in L, and thus obtain a finite set of *correct* formulae. Using (2.6.1), we then eliminate $c_1, \ldots, c_n$ from these formulae. (This elimination can be carried out in any torsion-free group, as explained in [**14**, Footnote 8].

As L has no equations, this process yields a set of (true) formulae, of the form:

(2.6.2) $k_i c_0 \neq a_i'$, $\qquad i = 1, 2, \ldots, m_1,$

(2.6.3) $k_i c_0 \equiv a_i' \pmod{r_i'}$, $\qquad i = m_1 + 1, m_1 + 2, \ldots, m_2,$

(2.6.4) $k_i c_0 \not\equiv a_i' \pmod{r_i'}$ $\qquad i = m_2 + 1, m_2 + 2, \ldots, m_3,$

where k_i, r_i' are integers ($r_i' > 0$), and all a_i' are in A_0.

Next, replace $c_0, c_1, \ldots, c_n$ by *unknowns* $x_0, x_1, \ldots, x_n$ in formulas (2.6.1) through (2.6.4), thus obtaining a new linear system L' in $n + 1$ unknowns. In particular, equations (2.6.1) turn into

(2.6.1⁰) $t_j x_j = s_j x_0 + b_j$ ($b_j \in A_0$), $\qquad j = 1, 2, \ldots, n.$

Clearly $(c_0, c_1, \ldots, c_n)$ is a solution of L *in* A_1. Moreover, the entire process described above is *reversible* (A being torsion-free). Thus, if $(d_0, d_1, \ldots, d_n)$ is a solution of L', then $(d_1, \ldots, d_n)$ is a solution of L. Hence our problem reduces to showing that L' is solvable *in* A_0.

To achieve this, we use ($2.6.1^0$) to eliminate $x_1, \ldots, x_n$ from L', leaving only *one* unknown x_0. Then we replace equations $2.6.1^0$ by a set of exactly n congruences, $s_j x_0 + b_j \equiv 0 \pmod{t_j}$, $j = 1, \ldots, n$, with s_j, t_j, b_j as before. This yields a linear system L'' in *one* unknown x_0 only; L'' consists of the now added congruences, and of (2.6.2)–(2.6.4) (with c_0 replaced by x_0). Again, the process is reversible; so any solution of L'' yields one for L'. Moreover L'' has a solution $x_0 = c_0$ *in* A_1. As A_0 is pure in A_1, and as $[p]A_0 = [p]A_1$ for all primes p (by assumption), Lemma 2.4 yields a solution, *in* A_0, of the *partial* linear system L''' arising from L'' by dropping from it the *inequalities* $k_i x_0 \neq a_i$, $i = 1, \ldots, m_1$, and thus consisting of congruences and incongruences only (indeed, A_0 is closed in A_1 with respect to such systems). Finally, as A_0 is torsion-free and not $\{0\}$, L''' must even have *infinitely many* solutions in A_0; for if $a \in A_0$ is a solution, so also is any element of the form $a + rz$ where $z \in A_0$ and r is a common multiple of all r_i' and all (non-zero) k_i occurring in (2.6.3) and (2.6.4). Thus some of these solutions must also satisfy the (*finitely many*) inequalities (2.6.2). This yields a solution of *all* of L'' in A_0, as required. Thus the lemma is proved.

Lemma 2.6 concludes the preliminary algebraic part of this paper. We now pass to the metamathematical part, with the aim of proving the model-completeness theorem (Theorem 3.6).

3. The model-completeness theorem. As in [**10**], we formalize the concept of an abelian group $\neq \{0\}$ by a system of axioms in the lower predicate calculus (LPC), based on two atomic relations: the binary relation $E(x,y)$ (read: "x is equal to y"), and the ternary relation $S(x,y,z)$ (read: "z is the sum of x and y"). For model-completeness, our language will also include certain unary atomic relations $D_n(x)$, $n = 1, 2, \ldots$ (see below). We write "$\sim$", "$\wedge$", "$\vee$", "$\cdot\supset\cdot$" and "$\cdot\equiv\cdot$" for "not", "and", "or", "implies" and "is logically equivalent", respectively. "$(\exists x)$" and "(x)" are the existential and the universal quantifiers.

3.1. AXIOMS OF EQUALITY (or EQUIVALENCE):

(a) $(x)E(x,x)$.

(b) $(x)(y)[E(x,y) \cdot\supset\cdot E(y,x)]$.

(c) $(x)(y)(z)[E(x,y) \wedge E(y,z) \cdot\supset\cdot E(x,z)]$.

(d) $(\exists x)(\exists y)[\sim E(x,y)]$.

3.2. GROUP AXIOMS:

(a′) $(x)(y)(\exists z)\, S(x,y,z)$.

(b′) $(x)(y)(z)(w)[S(x,y,z) \wedge S(x,y,w) \cdot\supset\cdot E(z,w)]$.

(c′) $(x)(y)(z)[S(x,y,z) \cdot\supset\cdot S(y,x,z)]$.

(d′) $(u)(v)(w)(x)(y)(z)[S(u,v,w) \wedge S(w,x,y) \wedge S(v,x,z) \cdot\supset\cdot S(u,z,y)]$.

(e′) $(u)(v)(w)(x)(y)(z)[S(u,v,w) \wedge E(u,x) \wedge E(v,y) \wedge E(w,z) \cdot\supset\cdot S(x,y,z)]$.

(f′) $(x)(y)(\exists z)\, S(x,z,y)$.

Here the axioms (a′), (b′) express the fact that the group is closed under addition and that the sum is unique; axioms (c′), (d′) give the commutative and associative laws; (e′) expresses the substitutivity of the equality relation with respect to addition, and (f′) ensures the existence of the inverse. To make the group torsion-free, we now add the following sequence of axioms (which, in ordinary language, state that $nx = 0$ implies $x = 0$):

3.3. AXIOMS EXCLUDING TORSION:

$$(x_1)(x_2)\ldots(x_n)\{[S(x_1, x_1, x_2) \wedge S(x_1, x_2, x_3) \wedge S(x_1, x_3, x_4) \wedge \ldots \wedge S(x_1, x_{n-1}, x_n) \wedge S(x_n, x_n, x_n)] \cdot\supset\cdot S(x_1, x_1, x_1)\},$$
$$n = 2, 3, \ldots .$$

The system of axioms introduced above (3.1 through 3.3) is neither complete nor model-complete, mainly because it does not specify the *prime invariants* (cf. 1.1) of the group under consideration. To achieve both completeness and model completeness, we first of all introduce a sequence of new atomic predicates $D_n(x)$ (read: "x is divisible by n"), and the following additional sequence of axioms:

3.4. AXIOMS DEFINING THE PREDICATES $D_n(x)$:

$$(x)\{D_n(x) \cdot\equiv\cdot (\exists y_1)(\exists y_2)\ldots(\exists y_{n-1})[S(y_1, y_1, y_2) \wedge S(y_1, y_2, y_3) \wedge \ldots \wedge S(y_1, y_{n-2}, y_{n-1}) \wedge S(y_1, y_{n-1}, x)]\}, \qquad n = 2, 3, 4, \ldots .$$

(It should be stressed that we treat Formulae 3.4 not as definitions, but as *axioms*, and the predicates D_n as *atomic* ones in our language. It is the adjunction of these atomic predicates that ensures the model-completeness of the system of axioms here constructed. In this respect, cf. also Note 1 below.)

Next, we fix an arbitrary infinite sequence $\{m_n\}$ where each m_n is a non-negative integer or ∞, and add yet another sequence of axioms:

3.5. Axioms Specifying the Prime Invariants:

$$[p_n] = (p_n)^{m_n}, \qquad n = 1, 2, 3, \ldots$$

where $\{p_n\}$ is the ascending sequence of all primes, and the $[p_n]$ are the corresponding prime invariants of the group, which thus are specified by the formulae (3.5). It is understood that these formulae are only abbreviations of their formal representation in the LPC, as explained in [**10**, p. 233].

Any system consisting of all these axioms (3.1 through 3.5), for some particular choice of the m_n in 3.5., shall be called a *system of axioms for a torsion-free abelian group with specified prime invariants*. Clearly, there are exactly $2^{\aleph_0}$ such systems, each corresponding to a particular choice of the sequence $\{m_n\}$.

Note 1. All such systems are *consistent* since they have *models*, as is shown in [**14**, Theorem 2.5]. Moreover, they exhaust all possible cases since the invariants $[p_n]$ *always* have the form indicated in (3.5) (cf. [**14**, Theorem 2.6.]); in particular, the special case of a *divisible* group is obtained by choosing all m_n equal to 0.

We shall need some more definitions and facts from [**10**].

A consistent system K of axioms in the LPC is said to be *complete* if, for every elementary statement X (i.e. one formulated in the LPC), either X or its negation, $\sim X$, is deducible from K.

A statement Y is said to be *primitive* if it has the form

$$Y = (\exists y_1)(\exists y_2) \ldots (\exists y_n)\ (Z)$$

where Z is a conjunction of atomic formulas (in our case, formulas of the form $E(x, y)$, $S(x, y, z)$ and $D_n(x)$) and (or) their negations.

The following proposition [**10**, 1.6], due to A. Robinson, may be accepted as a *definition* of model-completeness:

A consistent system K of axioms in the LPC is model-complete if and only if, for every pair of models, A and A_0, of K (where A is an extension of A_0), any primitive statement which holds in A and is defined in A_0, holds in A_0 as well.

Note 2. By definition, A is an *extension of* A_0 if $A_0 \subseteq A$ and if every *atomic* statement which holds in A and is defined in A_0, holds in A_0 as well. In our case, A is a torsion-free group, and A_0 its subgroup. Moreover, A_0 must be *pure* in A because the predicates D_n are *atomic*, and so by the above definition, $D_n(a)$ holds in A_0 whenever it holds in A and is defined in A_0 (i.e. $a \in A_0$).

Finally, from [**10**, 1.8], we recall that a *countable model-complete system of axioms K is also complete in the ordinary sense, if:*

(a) *Any two countable models of K, which have no constants in common other than those of K (if any), can be embedded in a joint extension M;* and

(b) *K has infinite models only.*

We can now establish our main result.

3.6. Theorem. *Let K be a system of axioms for a torsion-free abelian group with specified prime invariants. Then K is model-complete and complete.*

Proof. Let A_0 be a model of K, and A its extension; so A_0 is a *pure* subgroup of A, by Note 2. Let Y be a primitive statement true in A and defined in A_0, in terms of the atomic predicates E, S and D_n, $n = 2, 3, \ldots$. In ordinary language, Y means that a certain finite system of equations, inequalities, congruences and (or) incongruences of the form

$$(3.6.1)\quad \begin{array}{lll} \alpha = \beta, & \alpha + \beta = \gamma, & \alpha \equiv 0 \pmod r, \\ \alpha \neq \beta, & \alpha + \beta \neq \gamma, & \alpha \not\equiv 0 \pmod r, \end{array}$$

has a solution. (Note that formulae (3.6.1) only *typify* the equations, inequalities etc., which may occur in the system any finite number of times. The letters α, β, γ stand for constants from A_0 or the "unknowns" (the n bound variables y_k in Y)).

Now, as noted above, the model-completeness of K will be established if we show that (3.6.1) has a solution in A_0 assuming that it has one in A. (Let a given solution be $(c_1, c_2, \ldots, c_n)$, $c_i \in A$.) To achieve this, we introduce a sequence of n subgroups, $A_1, A_2, \ldots, A_n$, where A_i is the (c_i)-extension of A_{i-1} in A $(i = 1, 2, \ldots, n)$, so that each A_i is pure in A, by our definition of the (c_0)-extension (see § 2). As was noted above, A_0 is pure in A, as well. It follows that *A_0 is pure in A_1, and A_{i-1} is pure in A_i.* (Indeed, if a $\in A_{i-1}$ and if $a \equiv 0 \pmod r$ in A_i then, certainly, $a \equiv 0 \pmod r$ *in* A, hence also in A_{i-1}, by the purity of A_{i-1} in A.)

This implies that any two p-incongruent elements of A_{i-1} are also p-incongruent in A_i. Thus, for any p, the maximum number of p-incongruent elements in A_i cannot be *less* than in A_{i-1}. In other words, $[p]A_0 \leqq [p]A_1 \leqq \ldots \leqq [p]A_n \leqq [p]A$. Moreover, by assumption, A_0 and A are models of *the same* system K, and so satisfy *the same* sequence of axioms (3.5). It follows that $[p]A_0 = [p]A = [p]A_i$, $i = 1, \ldots, n$, for each prime p. Hence recalling that (3.6.1) has a solution $(c_1, \ldots, c_n)$ in A_n, and applying Lemma 2.6 successively n times, we infer that (3.6.1) has a solution in A_{n-1}, hence in A_{n-2}, in $A_{n-3}, \ldots$, and in A_0, as required. Thus K is model-complete. Its completeness now follows exactly as it was done for ordered groups in Theorem 4.6 of [**10**]. In fact, the system K is countable and contains no constants. We now use [**14**, Theorem 6.1]. (This theorem reads as our Corollary 3.9 below, but is limited to *countable* groups.) By that theorem, any two disjoint countable models of K can be embedded in a common extension; and any model-complete system K with these properties is also complete in the ordinary sense, by Theorem 1.8 of [**10**], quoted above. Thus all is proved.

The recursive part of this proof (dealing with $A_1, \ldots, A_n$) also yields an algebraic result:

3.7 COROLLARY. *A subgroup $A_0 \neq \{0\}$ of a torsion-free abelian group A is closed in A, with respect to all linear systems, if and only if A_0 is pure in A and $[p]A_0$ for each prime p.*

Indeed, the proof given above works also with (3.6.1) replaced by *any* linear system L. Thus the conditions are sufficient. We omit the easy proof of their necessity.

Notes. (3) Theorem 3.6 contains Robinson's Theorems 3.1.5 and 4.3.2 in [**6**a] as special cases. To obtain them, one only has to choose the particular system K in which all m_n in (3.5) are 0 (this yields the divisible case).

(4) Prüfer [**5**] showed that any pure subgroup A_0 of an abelian group A is closed in A, with respect to all systems of linear *equations*. For torsion-free groups, our Corollary 3.7 extends Prüfer's result to *all* linear systems, as defined in §1. (Cf. the Note following Lemma 2.4.)

The completeness of K can also be expressed as follows:

3.8 COROLLARY. *Two torsion-free abelian groups A and B, other than $\{0\}$, are elementarily equivalent if and only if $[p]A = [p]B$ for each prime p.*

This is W. Szmielew's result when restricted to torsion-free groups, and simplified accordingly. Though this certainly falls short of the general theorem of W. Szmielew, the simplicity of the new proof seems to justify the singling out of this special case, treated here more simply and thoroughly than in [**4**].

From Corollary 3.8 we obtain a classification of torsion-free abelian groups by their elementary properties, along the same lines as that of certain ordered groups, given at the end of [**10**]. As we have noted, there are exactly $2^{\aleph_0}$ different systems K, each corresponding to a particular choice of the exponents m_n in (3.5), i.e. of the prime invariants $[p_n]$. By 3.8, each such choice yields a class of elementarily equivalent groups. Thus there are exactly $2^{\aleph_0}$ such classes. In other words, apart from elementarily equivalent "copies", there are exactly $2^{\aleph_0}$ torsion-free abelian groups. The divisible case is only one of them (all divisible torsion-free abelian groups are elementarily equivalent).

Our next corollary generalizes Theorem 6.1 of [**14**], proved there for *countable* groups only. It may serve as an example of a useful application of meta-mathematical methods to algebra (where ordinary algebraic methods require much more effort). In fact, we have:

3.9. COROLLARY. *Let A and B be two disjoint torsion-free abelian groups other* *than* $\{0\}$, *with* $[p]A = [p]B$ *for each prime* p. *Then there is a torsion-free abelian group* M *which contains* A *and* B *as pure subgroups and has the same prime invariants:* $[p]M = [p]A = [p]B$, *for each prime* p.

Indeed, A and B have no constants in common and are models of one and the same complete system K described in 3.6. Thus, by Robinson's Theorem 4.2.2 proved in [**8**], there is a model M of K which is an extension of both A and B. This implies that $[p]M = [p]A = [p]B$ and that both A and B are pure in M (as was explained in Note 2). Thus all is proved.

Note 5. By the same argument, Theorem 6.2 of [**14**] (dealing with *ordered* groups) extends to arbitrary (not necessarily countable) "regularly dense" groups. Of course, when embedding A and B in M, we must identify their zero-elements.

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