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AN IDENTITY PROPERTY FOR 2-COMPLEX PAIRS

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Abstract. An identity property defined for a pair of 2-complexes (Y, X) first arose in 1993 within a strategy for constructing a counterexample of infinite type to Whitehead's Asphericity Conjecture. In this note we make use of the theory of pictures to characterize a more general right N-identity property, where $N < \pi_1 Y$. We also define combinatorial asphericity (CA) for the pair (Y, X) and determine a test for (CA) in the case that Y is obtained from X by the addition of a single 2-cell. This test can be used to determine an explicit generating set for $\pi_2 Y$.

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1. Introduction. In this note we study an identity property for a pair of 2-complexes, (Y, X), where Y is obtained from X by the addition of 2-cells. This property is a natural generalization of the (absolute) identity property for a 2-complex and first arose in the context of a question of J.H.C. Whitehead [**15**]: is every subcomplex of a connected, aspherical 2-complex itself aspherical? Much research has been conducted regarding this still unanswered question (see [**4**] for a good survey of this research). One strategy for building a counterexample to the Whitehead conjecture (which asserts the answer to his question is "yes") is to construct an infinite chain of 2-complexes $X_1 \subset X_2 \subset \cdots X_i \subset X_{i+1} \subset \cdots$ in which $\pi_2 X_1$ is not trivial, but $\pi_2(X_i) \to \pi_2(X_{i+1})$ is trivial for each $i \geq 1$ (see [**12**]). In [**8**], Dyer introduces the identity property within a strategy for constructing such a chain of spaces. In this strategy, the identity property is used to replace the homotopy requirement $(\pi_2 X_i \xrightarrow{0} \pi_2 X_{i+1})$ with a homological one that is perhaps more accessible. For the reader's convenience we describe this strategy in some detail below, in Section 3.

We may recover the (absolute) identity property for a 2-complex Y by considering the pair $(Y, Y^{(1)})$, where $Y^{(1)}$ is the 1-skeleton of Y. This absolute identity property is the traditional way to detect asphericity of a 2-complex. More generally, if N is a subgroup of the fundamental group of Y, the right N-identity property approximates asphericity. That is, Y has the right N-identity property if and only if $\pi_2 Y \to H_2 Y_N$ is trivial, where Y_N is the cover of Y corresponding to N. The right N-identity property has been the focus of much attention (see [**7**] for a good survey article), and it arises naturally in the context of the Whitehead conjecture. A deep result of J.F. Adams [**1**] says that a subcomplex X of an aspherical 2-complex has the right P-identity property, for some perfect subgroup P of the fundamental group of X.

In this paper we use pictures to define a right N-identity property for the pair (Y, X), where N is a subgroup of $\pi_1 Y$. We will always view our 2-complexes as arising from group presentations in the standard way, and it is often convenient to discuss the identity property in terms of a pair of group presentations $(\mathcal{Q}, \mathcal{P})$ where $\mathcal{Q}$ is obtained from $\mathcal{P}$ by the addition of relators. In Section 3 we present various characterizations of the N-identity property that extend known characterizations of the absolute identity property. In Section 4 we make use of a Cockcroft property on certain disk pictures to determine a combinatorial group-theoretic characterization of the identity property.

In the last two sections we generalize to 2-complex pairs (Y, X) the notion of weakening the asphericity of a 2-complex to combinatorial asphericity. Simple mixed pictures called *relative dipoles* play an important role here. We define (Y, X) to be combinatorially aspherical (CA) if $\pi_2 Y$ is generated (over $\pi_2 X$) by a set of relative dipoles. In the case Y is obtained from X by adding a single 2-cell, we prove a simple test for determining an explicit set of generators for $\pi_2 Y$ (over $\pi_2 X$).

2. Pictures. If Y is the model of the presentation $\mathcal{Q} = \langle\, \mathbf{x} : \mathbf{r}, \mathbf{s}\,\rangle$, then elements of $\pi_2 Y$ are represented by spherical pictures over $\mathcal{Q}$. We refer the reader to [**5**] and [**13**] for two thorough treatments of pictures, but we outline some key features here.

Any oriented transverse path γ in a picture $\mathbf{B}$ defined over $\mathcal{Q}$ determines a word, $\omega(\gamma) \in F(\mathbf{x})$, from the labels of the arcs it traverses. If γ_i and γ_i' are distinct transverse paths from the global basepoint of $\mathbf{B}$ to the basepoint of a disk Δ_i in $\mathbf{B}$, then they may determine distinct words in F. However, an important feature of pictures is that these words determine the same element of the group presented. This fact is essentially due to a pictorial version of van Kampen's Lemma.

LEMMA 2.1. (van Kampen) *Suppose $\mathcal{P} = \langle\, \mathbf{x} : \mathbf{r}\,\rangle$ presents the group G. For any word $w \in F(\mathbf{x})$, there exists a picture $\mathbf{B}$ over $\mathcal{P}$ with $\partial\mathbf{B} = w$ if and only if w has trivial image in G.*

If $\mathbf{B}$ is a picture over $\mathcal{Q}$, $\partial\mathbf{B}$ denotes the word in F spelled by the arcs traversed along the topological boundary of $\mathbf{B}$. If no arc meets the topological boundary, then $\mathbf{B}$ is called a *spherical picture*, and its boundary label is $1 \in F$. Two pictures are *equivalent* if one can be transformed to the other by a sequence of allowable moves. These moves are of three types: insertion/deletion of a floating arc, bridge move, and insertion/deletion of a folding pair. (See [**5**], [**13**], for details.) We multiply two pictures by forming their disjoint union, and we invert a picture by taking its mirror image while changing the sign of each disk label. With these operations the equivalence classes of pictures form a group. The normal subgroup generated by spherical pictures is abelian. We write $\mathbf{P} + \mathbf{Q}$ for the product of two spherical pictures, and $\mathbf{B} \cdot \mathbf{D}$ for the product of two arbitrary pictures. Similarly, $-\mathbf{P}$ denotes the inverse of a spherical picture $\mathbf{P}$, while $\mathbf{B}^{-1}$ denotes the inverse of a disk picture $\mathbf{B}$.

If $\mathbf{P}$ is a spherical picture over $\mathcal{Q}$ we let $[\mathbf{P}]$ denote the element of $\pi_2 Y$ it represents. The left $\mathbf{Z}H$-module structure of $\pi_2 Y$ is induced by the following F-action on spherical pictures. For $w \in F$, $w \cdot \mathbf{P}$ is the spherical picture obtained by encircling $\mathbf{P}$ with arcs whose labels spell w. Then we have the well-defined H-action $\bar{w} \cdot [\mathbf{P}] = [w \cdot \mathbf{P}]$, where $\bar{w}$ is the image of w in H.

Suppose $C_*(\tilde{Y})$ is the chain complex of the universal cover $\tilde{Y}$ of Y. The standard $\mathbf{Z}H$-module injection $\mu : \pi_2 Y \to C_2(\tilde{Y})$ can be described in terms of spherical pictures as follows. Suppose $\mathbf{P}$ over $\mathcal{Q}$ represents $[\mathbf{P}] \in \pi_2 Y$, and that $\mathbf{P}$ has k disks, $\Delta_1, \Delta_2, \cdots, \Delta_k$, with disk Δ_i getting the label $\omega(\Delta_i)^{\epsilon_i}$ ($\epsilon_i = \pm 1$, and $\omega(\Delta_i)$ is a relator of $\mathcal{Q}$) for $i = 1, 2, \cdots, k$. A transverse path γ_i from the global basepoint of $\mathbf{P}$ to the basepoint of Δ_i determines a word $\omega(\gamma_i)$ from the arcs it traverses. Let h_i be the image of this word in the fundamental group H. Then

$$\mu([\mathbf{P}]) = \sum_{i=1}^{k} \epsilon_i h_i c^2_{\omega(\Delta_i)}.$$

3. The N-identity property for (Y,X). A spherical picture over $\mathcal{Q} = \langle\, \mathbf{x} : \mathbf{r}, \mathbf{s}\,\rangle$ can have some disks labeled by relators in $\mathbf{r}$, and some with labels from $\mathbf{s}$. We will call the former disks $\mathbf{r}$-disks, and the latter disks $\mathbf{s}$-disks. Let $N < H$ be any subgroup of H, and suppose N_F is the pre-image of N in the free group $F(\mathbf{x})$.

Definition 3.1. The pair (Y, X) (or $(\mathcal{Q}, \mathcal{P})$) has the *right (resp. left) N-identity property* if every spherical picture over $\mathcal{Q}$ has a pairing of its $\mathbf{s}$-disks $(i \leftrightarrow j)$ such that

$$\omega(\Delta_i) = \omega(\Delta_j);$$
$$\epsilon_i \neq \epsilon_j;$$
$$N_F\omega(\gamma_i) = N_F\omega(\gamma_j) \text{ (resp. } \omega(\gamma_i)N_F = \omega(\gamma_j)N_F).$$

Implicit in the definition is a set of transverse paths $\{\gamma_i\}$ to the disk basepoints of disks with labels in $\mathbf{s}$. If the definition is satisfied for a particular set of transverse paths then it is satisfied for any such set. This is a consequence of van Kampen's lemma.

If $\mathbf{r}$ is empty (i.e., if X is the 1-skeleton of Y), then this definition matches the defintion given in [**7**] of the right N-identity property for the two-complex Y.

If $N \triangleleft H$ is normal, then the left and right identity properties coincide. In this case, we will refer to the *N-identity property*. If (Y, X) has the $\{1\}$-identity property, then we say that (Y, X) has the *identity property*. It is this identity property that appreared in [**8**]. If $X = Y^{(1)}$, it is well known that the pair $(Y, Y^{(1)})$ has the identity property if and only if the 2-complex Y is aspherical.

Note that if the inclusion induced map $i_\# : \pi_2 X \to \pi_2 Y$ is surjective, then (Y, X) has the identity property, for if this map is surjective, then any spherical picture over $\mathcal{Q}$ is equivalent to one without $\mathbf{s}$-disks. By virtue of the allowable moves on pictures, it follows that any picture over $\mathcal{Q}$ has the requisite pairing of its $\mathbf{s}$-disks. Thus, for instance, if Y is aspherical and X is any subcomplex containing $Y^{(1)}$, then (Y, X) has the identity property.

For a specific example, consider $\mathcal{P} = \langle a, b : [a, b]\rangle$ and $\mathbf{s} = \{a^3\}$, so that $\mathcal{Q} = \langle a, b : [a, b], a^3\rangle$. Then (Y, X) has the H-identity property. Indeed, in the case $N = H$, the third condition in the definition is superfluous. Thus, $(\mathcal{Q}, \mathcal{P})$ has the H-identity property if and only if the $\mathbf{s}$-disks of any spherical picture $\mathbf{P}$ over $\mathcal{Q}$ can be paired so that disks in each pair have the same label but opposite orientation. We will say such a picture has *parity in* $\mathbf{s}$. In view of the well-known $\mathbf{Z}H$-module generators of $\pi_2 Y$ depicted in Figure 1, it follows that every spherical picture over $\mathcal{Q}$ has parity in $\mathbf{s}$.

We observe from the definition that if $N_1 \subset N_2$ and $(\mathcal{Q}, \mathcal{P})$ has the right N_1-identity property, then $(\mathcal{Q}, \mathcal{P})$ has the right N_2-identity property as well.

Let $p : \tilde{Y} \to Y$ denote the universal covering of Y, and consider $p^{-1}(X) = X_L$, the covering of X associated to the normal subgroup $L = \langle\langle \mathbf{s} \rangle\rangle_G$. The homology sequence of the pair $(\tilde{Y}, X_L)$ yields the following exact sequence of left $\mathbf{Z}H$-modules:

$$\begin{array}{ccccccccc} & & & H_2(\tilde{Y}) & & H_2(\tilde{Y}, X_L) & & H_1(X_L) & \\ & & & \| & & \| & & \| & \\ 1 & \to & H_2 X_L & \xrightarrow{i} & \pi_2 Y & \xrightarrow{j} & \bigoplus_{s\in\mathbf{s}} \mathbf{Z}Hc_s^2 & \xrightarrow{\psi} & H_1(L) & \to & 1. \end{array} \qquad (1)$$

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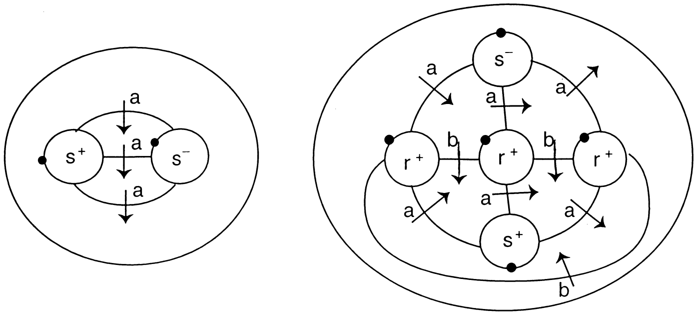

Figure 1

From now on, we will let $\mathbf{Z}H^{\mathbf{s}}$ denote $\bigoplus_{s\in\mathbf{s}}\mathbf{Z}Hc_s^2$. We have a left H-action on $H_1(L)$, induced by the conjugation action of G on L. That is, for $h\in H$, $h\cdot\bar{s}[L,L]=g\bar{s}g^{-1}[L,L]$, where $g\in G$ has image $h\in H$, and $\bar{s}$ is the image of s in G. This action makes $H_1(L)$ into a left $\mathbf{Z}H$-module, called the *relative relation module* associated to $(\mathcal{Q},\mathcal{P})$.

The maps in (1) are as follows. For $s\in\mathbf{s}$, $\psi(c_s^2)=\bar{s}[L,L]$. The map $j:\pi_2Y\to\mathbf{Z}H^{\mathbf{s}}$ is the composition

$$j=\rho\circ\mu:\pi_2Y\to C_2(\tilde{Y})=\mathbf{Z}H^{\mathbf{r}}\oplus\mathbf{Z}H^{\mathbf{s}}\to\mathbf{Z}H^{\mathbf{s}}$$

where ρ is projection onto the $\mathbf{s}$-coordinates, and μ is the map defined in the previous section.

For the third map in (1) note that if $\alpha\in\mathbf{Z}H^{\mathbf{r}}$ has image in H_2X_L, then $(\alpha,0)\in\mathbf{Z}H^{\mathbf{r}}\oplus\mathbf{Z}H^{\mathbf{s}}$ has image in $H_2(\tilde{Y})$. Let $i(\alpha)=[\mathbf{P}_\alpha]$, where the spherical picture $\mathbf{P}_\alpha$ is a representative of the unique class $[\mathbf{P}_\alpha]$ in π_2Y with $\mu([\mathbf{P}_\alpha])=[(\alpha,0)]$.

Given the subgroup $N<H=\pi_1Y$, let Y_N denote the cover of Y associated to N. Let $N_G=\iota_\#^{-1}(N)$ be the pre-image of N in G, and build X_{N_G}, the cover of X with respect to N_G. Then X_{N_G} is a subcomplex of Y_N, and the pair (Y_N,X_{N_G}) covers (Y,X).

Definition 3.2. Let $N<\pi_1Y$. The pair (Y,X) is N-*Cockcroft* if the composite map

$$\pi_2(Y)\xrightarrow{j}\mathbf{Z}H^{\mathbf{s}}=H_2(\tilde{Y},X_L)\xrightarrow{\rho_N}H_2(Y_N,X_{N_G})$$

is trivial, where ρ_N is the induced map from the projection $(\tilde{Y},X_L)\to(Y_N,X_{N_G})$ of covers.

As with the identity property, we recover the N-Cockcroft property for a 2-complex Y from the N-Cockcroft property for the pair $(Y,Y^{(1)})$.

For any group N, *the augmentation ideal* $IN=\ker(\epsilon:\mathbf{Z}N\to\mathbf{Z})$, where $\epsilon(\sum n_ih_i)=\sum n_i$, $n_i\in\mathbf{Z}, h_i\in N$. Then $IN\cdot\mathbf{Z}H$ is the right ideal of $\mathbf{Z}H$ consisting of all finite sums $\sum a_ib_i$, $(a_i\in IN, b_i\in\mathbf{Z}H)$.

We may identify the quotient module $\mathbf{Z}H/IN\cdot\mathbf{Z}H$ with $\mathbf{Z}\otimes_N\mathbf{Z}H$ by the natural isomorphism $\overline{b}\mapsto 1\otimes b$ where $\overline{b}$ denotes the image in $\mathbf{Z}H/IN\cdot\mathbf{Z}H$ of an element b in $\mathbf{Z}H$ (see [**6**, p. 34]). Also, one can check that $\mathbf{Z}H/IN\cdot\mathbf{Z}H\cong\mathbf{Z}[N\backslash H]$ where $N\backslash H$ denotes the set of right cosets of N, and that $\mathbf{Z}[N\backslash H]\cong\mathbf{Z}[N_F\backslash F]$ are naturally identified.

PROPOSITION 3.3. *The following statements are equivalent for the pair* (Y, X).

1. (Y, X) *has the right* N*-identity property*.

2. *The* $\mathbf{s}$*-coefficients of any spherical picture* $\mathbf{P}$ *over* $\mathcal{Q}$ *lie in the right ideal* $IN \cdot \mathbf{Z}H$.

3. (Y, X) *is* N*-Cockcroft*.

We remark that this proposition is an immediate generalization of parts of Theorem 4.2 in [**7**], and the proof of Theorem 4.2 may be adapted to this more general setting.

Proof. (1) $\Rightarrow$ (2) Let $\mathbf{P}$ over $\mathcal{Q}$ be an arbitrary spherical picture. For $s \in \mathbf{s}$, let b_s denote the coefficient of c_s^2 in $j([\mathbf{P}]) \in \mathbf{Z}H^s$. We must show that $b_s \in IN \cdot \mathbf{Z}H$. Suppose γ is a set of transverse paths to the disk basepoints of $\mathbf{P}$. As usual, let $\omega(\gamma_i)$ be the word in F determined by the path γ_i to the basepoint of Δ_i, and h_i this word's image in H. Then $b_s = \sum \epsilon_k h_k$, where the sum runs over all disks Δ_k labelled by $s^{\epsilon_k}(\epsilon_k = \pm 1)$. If $(\mathcal{Q},\mathcal{P})$ has the right N-identity property then there exists a pairing of the s-disks $(i \leftrightarrow j)$ such that $\epsilon_i = -\epsilon_j$ and $h_i = n_j h_j$ for some $n_j \in N$. We may then rewrite b_s as $\sum \epsilon_j(1 - n_j)h_j$ where the sum includes one term for each pair of disks labelled by s. Thus, $b_s \in IN \cdot \mathbf{Z}H$.

(2) $\Rightarrow$ (1) We can tensor (1) by $\mathbf{Z}\otimes_{\mathbf{Z}N}$ – to obtain the next exact sequence

$$\mathbf{Z} \otimes_N H_2 X_L \stackrel{1\otimes i}{\to} \mathbf{Z} \otimes_N \pi_2 Y \stackrel{1\otimes j}{\to} \mathbf{Z} \otimes_N \mathbf{Z}H^{\mathbf{s}} \stackrel{1\otimes \psi}{\to} \mathbf{Z} \otimes_N H_1(L) \to 1.$$

Since $\mathbf{Z} \otimes_N \mathbf{Z}H^{\mathbf{s}} \cong (\mathbf{Z}H/IN \cdot \mathbf{Z}H)^{\mathbf{s}}$, condition (2) implies the map $1 \otimes j$ is trivial, and hence $1 \otimes \psi$ is an isomorphism. But $\mathbf{Z} \otimes_N \mathbf{Z}H^{\mathbf{s}} \cong \oplus_{s\in\mathbf{s}}(\mathbf{Z}[N_F \backslash F])c_s^2$, and $\mathbf{Z} \otimes_N H_1(L) \cong \mathbf{Z} \otimes_N L/[L, L] \cong L/[N_G, L]$, so that

$$\oplus_{s\in\mathbf{s}}(\mathbf{Z}[N_F \backslash F])c_s^2 \cong L/[N_G, L].$$

This map is given on basis elements by $c_s^2 \mapsto \bar{s}[N_G, L]$.

Now consider $\mathbf{P}$ over $\mathcal{Q}$, and suppose $j([\mathbf{P}]) = \sum \epsilon_i \bar{b}_i c_{s_i}^2$ where $\epsilon_i = \pm 1$, $b_i \in G$, and $\bar{b}_i$ is its image in H. From sequence (1), $\psi(j[\mathbf{P}]) = 0$ implies that $\sum \epsilon_i b_i \bar{s}_i b_i^{-1}[L, L] = 0$ in $H_1(L)$, so that $\sum 1 \otimes \epsilon_i b_i \bar{s}_i b_i^{-1}[L, L] = 0$ in $\mathbf{Z} \otimes_N H_1(L)$. Condition (2) implies that this element's pre-image in $\oplus_{s\in\mathbf{s}}(\mathbf{Z}[N_F \backslash F])c_s^2$ is trivial. In particular,

$$\sum_i \epsilon_i(N_F f_i)c_{s_i}^2 = 0,$$

where f_i is the pre-image of b_i in F. So, for each s, the partial sum $\sum_{s_i = s} \epsilon_i(N_F f_i)c_{s_i}^2$ is trivial and the pairing sought necessarily exists.

(2)$\Longleftrightarrow$(3) This equivalence follows from an analysis of the chain complex of the pair (Y_N, X_{N_G}). We may identify $C_2(Y_N)$ with $\mathbf{Z} \otimes_N C_2(\bar{Y}) \cong \mathbf{Z} \otimes_N \mathbf{Z}H^{\mathbf{r}\cup\mathbf{s}}$. Similarly, $C_2(X_{N_G}) \cong \mathbf{Z} \otimes_{N_G} \mathbf{Z}G^{\mathbf{r}}$, from which we may identify $H_2(Y_N, X_{N_G}) \cong C_2(Y_N, X_{N_G})$ with $\mathbf{Z} \otimes_N \mathbf{Z}H^{\mathbf{s}}$. Thus, $(\mathcal{Q}, \mathcal{P})$ is N-Cockcroft if and only if $\pi_2(Y) \stackrel{j}{\to} \mathbf{Z}H^{\mathbf{s}} \to (\mathbf{Z}H/IN \cdot \mathbf{Z}H)^{\mathbf{s}}$ is trivial; that is, if and only if $j([\mathbf{P}])$ has coefficients living in $IN \cdot \mathbf{Z}H$ for each $\mathbf{P}$ over $\mathcal{Q}$

One checks that the pair $(\mathcal{Q}, \mathcal{P})$ has the left N-identity property if and only if the $\mathbf{s}$-coefficients of any spherical picture over $\mathcal{Q}$ live in the left ideal $\mathbf{Z}H \cdot IN$ of $\mathbf{Z}H$. In fact, $(\mathcal{Q}, \mathcal{P})$ has the left N-identity property if and only if the **s**-coefficients of each spherical picture **P** in a *module generating set* of $\pi_2 Y$ live in $\mathbf{Z}H \cdot IN$. We may restrict our attention to a π_2 generating set in this case precisely because $\pi_2 Y$ is a left $\mathbf{Z}H$-module. See [**14**] for a discussion of this left/right distinction in the absolute case.

For example, suppose $\mathcal{P} = \langle \mathbf{x}, t : [t, u] \rangle$ where $u \in F(\mathbf{x})$ is non-trivial, and t is a letter not in $\mathbf{x}$. Let $\mathbf{s} = \{tu\}$, so that $\mathcal{Q} = \langle \mathbf{x}, t : [t, u], tu \rangle$. We remark that $\mathcal{Q}$ is Tietze equivalent to $\langle \mathbf{x} : 1 \rangle$. Thus, $\mathcal{Q}$ is not Cockcroft, and H is isomorphic to the free group on $\mathbf{x}$.

If Y is the model on $\mathcal{Q}$, then one can check that the spherical picture over $\mathcal{Q}$ in Figure 2 generates $\pi_2 Y$ as left $\mathbf{Z}H$-module.

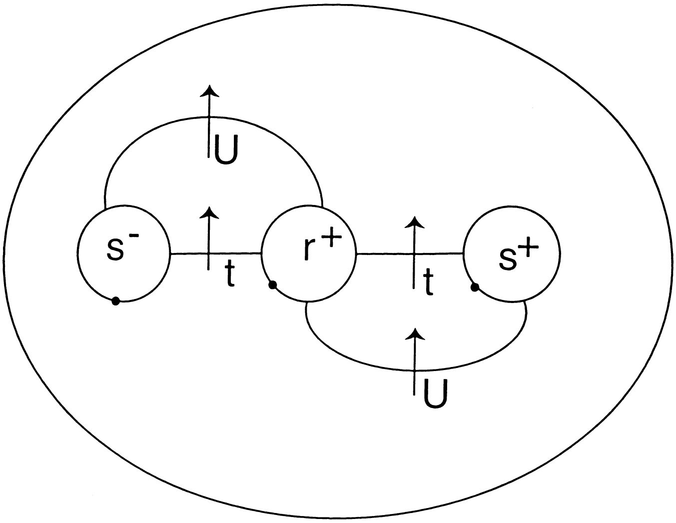

Figure 2

Note that $j([\mathbf{P}]) = (\bar{u} - 1)c_s^2$, where $\bar{u}$ is the image of u in H, from which we see that (Y, X) has the left N-identity property for a subgroup $N < H$ if and only if $\bar{u} \in N$. On the other hand, (Y, X) has the right N-identity property if and only if $h(\bar{u} - 1) \in IN \cdot \mathbf{Z}H$ for each $h \in H$. Furthermore, (Y, X) is $\langle\langle \bar{u} \rangle\rangle_H$-Cockcroft here, though Y itself is not Cockcroft.

In general, if $N = \{1\}$ then $IN \cdot \mathbf{Z}H$ is trivial. Thus, Proposition 3.3 ensures that $(\mathcal{Q}, \mathcal{P})$ has the identity property if and only if the map j in (1) is trivial. Consider the following diagram with commutative square, obtained from sequence (1) by adding the vertical Hopf sequence:

$$
\begin{array}{ccccccccccc}
 & & \pi_2 X & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
1 & \to & H_2 X_L & \xrightarrow{i} & \pi_2 Y & \xrightarrow{j} & \mathbf{Z}H^{\mathbf{s}} & \xrightarrow{\psi} & H_1(L) & \to & 1. \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & H_2(L) & \xrightarrow{\phi} & \pi_2 Y / im(\pi_2 X) & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 1 & & & & & & & &
\end{array} \qquad (2)
$$

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One checks that the map ϕ is injective, and that ϕ is surjective if and only if j is trivial. Then, the following proposition holds.

PROPOSITION 3.4. *The following statements are equivalent.*
1. (Y, X) *has the identity property.*
2. *The map* $\phi : H_2(L) \to \pi_2 Y/im(\pi_2 X)$ *is a* $\mathbf{Z}H$*-module isomorphism.*
3. *The map* $\psi : \mathbf{Z}H^{\mathbf{s}} \to H_1(L)$ *is a* $\mathbf{Z}H$*-module isomorphism.*

REMARK 3.5. For the reader's convenience, we now consider the strategy for constructing a counterexample to the Whitehead conjecture, as given in [**8**]. Suppose X is a non-aspherical 2-complex modelled on $\mathcal{P} = \langle\, \mathbf{x} : \mathbf{r}\,\rangle$ that is L_1-Cockcroft, for some non-trivial normal subgroup $L_1 \lhd G = \pi_1 X$.

Consider a strictly increasing sequence of normal subgroups of G

$$\{1\} = L_0 < L_1 < \cdots < L_n < L_{n+1} < \cdots < G$$

and nested sets of elements of $F(\mathbf{x})$

$$\emptyset = \mathbf{s}_0 \subset \mathbf{s}_1 \subset \cdots \mathbf{s}_n \subset \mathbf{s}_{n+1} \cdots$$

such that $L_n = \langle\langle \mathbf{s}_n \rangle\rangle_G$. Consider the family of group presentations $\mathcal{P}_n = \langle\, \mathbf{x} : \mathbf{r}, \mathbf{s}_n \,\rangle$ for each integer $n \geq 0$. ($\mathcal{P}_0 = \mathcal{P}$.) Suppose further that for each $n \geq 1$

(i) the pair $(\mathcal{P}_n, \mathcal{P})$ has the identity property; and

(ii) the map $H_2 X_{L_n} \to H_2 X_{L_{n+1}}$ induced by the injection $L_n \to L_{n+1}$ is trivial.

Having such a sequence of presentations, we can construct an infinite counter-example to the Whitehead conjecture as follows.

Let $X_n = X \cup \{c_s^2 : s \in \mathbf{s}_n\}$. If $p : \tilde{X}_n \to X_n$ is the universal cover, then $p^{-1}(X) = X_{L_n}$.

Since $(\mathcal{P}_n, \mathcal{P})$ has the identity property, $H_2 X_{L_n} \cong \pi_2 X_n$ for each n. This fact, condition (ii), and the commutative diagram

$$\begin{array}{ccc} H_2 X_{L_n} & \to & \pi_2 X_n \\ \downarrow & \circ & \downarrow \\ H_2 X_{L_{n+1}} & \to & \pi_2 X_{n+1} \end{array}$$

ensure that $\pi_2 X_n \to \pi_2 X_{n+1}$ is trivial at each stage. In this way, we build $X_\infty = \bigcup X_n$, an aspherical 2-complex having the non-aspherical subcomplex, X.

4. A group-theoretic gharacterization. In this section we require our subgroup $N \lhd H$ to be normal. Suppose $\mathbf{P}$ is a spherical picture over $\mathcal{Q}$. We may focus directly upon the $\mathbf{s}$-disks of $\mathbf{P}$ as follows: consider a set of paths $\{\gamma_i\}$ to the $\mathbf{r}$-disks of $\mathbf{P}$ (see Figure 3). We assume that no two paths intersect except at the global basepoint. In [**13**] such a set of paths is called a *spray*, except that here we're restricting a spray to the $\mathbf{r}$-disks. Next, cut along the boundary of this spray to obtain a disk picture having all the $\mathbf{s}$-disks of $\mathbf{P}$ (if any), and boundary label reading

$$\prod \omega(\gamma_i)\omega(\Delta_i)^{\epsilon_i}\omega(\gamma_i)^{-1},$$

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where the product runs over all **r**-disks Δ_i, and $\omega(\Delta_i)^{\epsilon_i} \in \mathbf{r} \cup \mathbf{r}^{-1}$ is the label of Δ_i. In other words, we obtain a picture over the presentation $\mathcal{Z} = \langle\, \mathbf{x} : \mathbf{s} \,\rangle$ whose boundary is in R. This leads to the following notion.

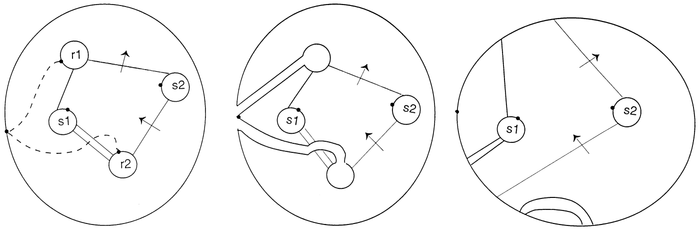

Figure 3

DEFINITION 4.1. Suppose $\mathcal{P} = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ presents the group G, $N \lhd G$ is a normal subgroup, and **w** is a set of words in $F = F(\mathbf{x})$. Let N_F be the pre-image of N in F, and $R = \langle\langle r \rangle\rangle_F$. Then $\mathcal{P}$ is *N-Cockcroft (rel* **w***)* if and only if any disk picture over $\mathcal{P}$ with boundary label in $W = \langle\langle \mathbf{w} \rangle\rangle_F$ has a pairing $(i \leftrightarrow j)$ of its disks such that

$$\omega(\Delta_i) = \omega(\Delta_j), \quad \epsilon_i \neq \epsilon_j, \text{ and } \omega(\gamma_i)\omega(\gamma_j)^{-1} \in N_F.$$

We say that $\mathcal{P}$ is *Cockcroft (rel* **w***)* if $\mathcal{P}$ is G-Cockcroft (rel **w**). We remark that a notion very similar to this (in the case $N = G$) was introduced in [**2**] to study Cockcroft properties of pictures arising from group constructions. The following key lemma is a generalization of a fact stated in [**2**] for the case $N = G$.

LEMMA 4.2. *With the notation as in 4.1, $\mathcal{P}$ is N-Cockcroft (rel* **w***) if and only if $\mathcal{P}$ is N-Cockcroft and $W \cap R \subset [R, N_F]$.*

Proof. First, suppose $\mathcal{P}$ is N-Cockcroft (rel **w**). Since $1 \in W$ it follows that $\mathcal{P}$ has the N-identity property; that is, all *spherical* pictures over $\mathcal{P}$ have the appropriate pairing of its disks. Thus, $\mathcal{P}$ is N-Cockcroft. We must show that $W \cap R \subset [R, N_F]$.

Consider $v \in W \cap R$. Since $v \in R$, van Kampen's Lemma guarantees a picture **B** over $\mathcal{P}$ having v as its boundary label. Since $v \in W$, the disks of **B** have the prescribed pairing of Definition 4.1. We must show this forces $\partial \mathbf{B} = v \in [R, N_F]$.

Consider any spray of paths $\gamma = \{\gamma_1, \cdots, \gamma_k\}$ to the disk basepoints in **B**, where γ_i attaches to disk Δ_i. It is well known (see [**13**]) that the word determined by this spray, $\prod \omega(\gamma_i)\omega(\Delta_i)^{\epsilon_i}\omega(\gamma_i)^{-1}$, is freely equal to the boundary label of the picture **B**. That is,

$$\partial \mathbf{B} = \prod \omega(\gamma_i)\omega(\Delta_i)^{\epsilon_i}\omega(\gamma_i)^{-1}.$$

But the words in this product are paired according to Definition 4.1. Armed with this pairing, one checks that the product is trivial in $F/[R, N_F]$. (An easy way to see this is to choose a spray of paths in **B** so that paired disks are adjacent in the (clockwise) sequence of paths comprising the spray.)

Conversely, consider a picture **B** over $\mathcal{P}$ having boundary label $\partial\mathbf{B} \in W$. We show that **B** has the prescribed pairing. By van Kampen's Lemma, $\partial\mathbf{B} \in R$, and so $\partial\mathbf{B} \in W \cap R \subset [R, N_F]$. It follows that $\partial\mathbf{B}$ is freely equal to a word v of the form

$$v = \prod_{i=1}^{k} f_i[w_i, u_i]f_i^{-1}$$

where each $f_i \in F$, each $u_i \in N_F$ and each $w_i \in R$.

Now, consider the picture **D** in Figure 4 having boundary label equal to v.

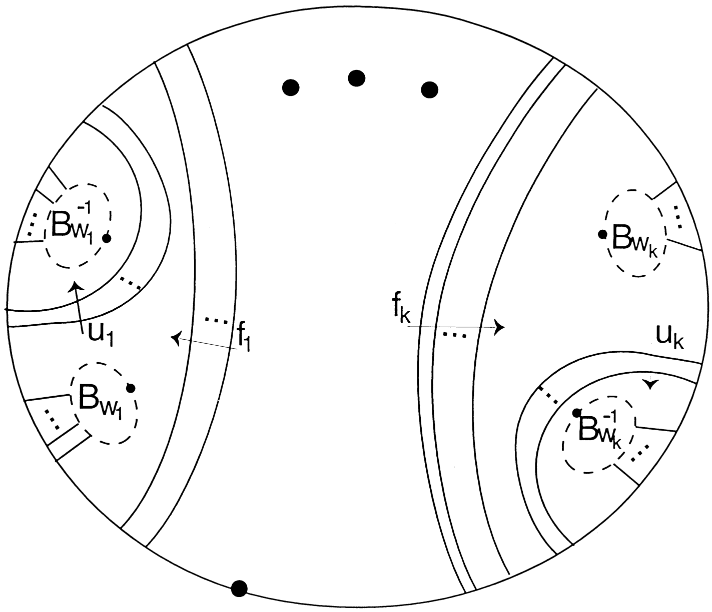

Figure 4

Each subpicture $\mathbf{B}_{w_i}$ of **D** is a picture over $\mathcal{P}$ associated to $w_i \in R$ (this picture has boundary label identically equal to w_i). Notice that by pairing each disk of $\mathbf{B}_{w_i}$ with its mirror image in $\mathbf{B}_{w_i}^{-1}$, the disks of **D** can be paired to satisfy the conditions of Definition 4.1. Indeed, we may always choose a path from a disk of $\mathbf{B}_{w_i}$ to its mirror image in $\mathbf{B}_{w_i}^{-1}$ whose associated word is a conjugate of $u_i \in N_F$, and hence in N_F, since N is normal.

Let **Q** be the spherical picture associated to $\mathbf{B} \cdot \mathbf{D}^{-1}$. Since $\mathcal{P}$ is N-Cockcroft, **Q** has an appropriate pairing of its disks. Now the disks of **D** may be paired appropriately as indicated above, so the subpicture **B** has a pairing of its disks as well. •

Recall, Z is the 2-complex modelled on $\mathcal{Z}$, $A = \pi_1 Z$, and set $M = \langle\langle r \rangle\rangle_A$. Let $\pi_A : A \to H$ and $\pi_F : F \to H$ denote the inclusion induced maps on the fundamental groups $\pi_1 Z \to \pi_1 Y$ and $\pi_1 Y^{(1)} \to \pi_1 Y$, respectively.

THEOREM 4.3. *Suppose N is normal subgroup of H, and let $N_A = \pi_A^{-1}(N)$ and $N_F = \pi_F^{-1}(N)$. The following statements are equivalent.*

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1. $(\mathcal{Q}, \mathcal{P})$ *has the N-identity property;*
2. $\mathcal{Z}$ *is* N_A*-Cockcroft (rel* $\mathbf{r}$*); and*
3. $\mathcal{Z}$ *is* N_A*-Cockcroft and* $R \cap S \subset [S, N_F]$.

Proof. (1)⇒(2) Suppose **B** is a picture over $\mathcal{Z}$ with boundary label in R. Then there is a disk picture **D** over $\mathcal{P}$ with boundary label identically equal to $\partial\mathbf{B}$ by van Kampen's lemma. Then $\mathbf{B}{\cdot}\mathbf{D}^{-1}$ is equivalent to a spherical picture over $\mathcal{Q}$, and since $(\mathcal{Q}, \mathcal{P})$ has the N-identity property, this spherical picture has the appropriate pairing of its **s**-disks. Thus, the subpicture **B**, which contains all the **s**-disks of $\mathbf{B}{\cdot}\mathbf{D}^{-1}$ must have a pairing of its **s**-disks satisfying the conditions of Definition 4.1. Indeed the pairing that works for the spherical picture $\mathbf{B}{\cdot}\mathbf{D}^{-1}$ restricts to a suitable pairing for **B**.

(2)⇒(1): Suppose **P** is a spherical picture over $\mathcal{Q}$. We may cut the **r**-disks from **P** as in Figure 3 to form a disk picture over $\mathcal{Z}$ with boundary in R. (If **P** has no **r**-disks, then the original picture is unchanged.) This new picture has a prescribed pairing by assumption. Moreover, this pairing can be used in **P** to see that the original picture **P** has an appropriate pairing.

(2) $\Longleftrightarrow$ (3): This follows as a corollary to the above lemma, since the pre-image of N_A in the free group F is N_F.

Corollary 4.4. *The following statements are equivalent.*
1. $(\mathcal{Q}, \mathcal{P})$ *has the identity property;*
2. $\mathcal{Z}$ *is M-Cockcroft (rel* $\mathbf{r}$*); and*
3. $\mathcal{Z}$ *is M-Cockcroft and* $R \cap S \subset [S, RS]$.

Proof. The pre-image of the trivial normal subgroup $\{1\}$ in A is M, and the pre-image of M in F is RS. The corollary is a restatement of Theorem 4.3 in this special case.

Example 4.5. Consider any 2-relator presentation $\mathcal{Z} = \langle \mathbf{x} : u, v\rangle$ where $u, v \in F(\mathbf{x}) = F$. Let $\mathcal{P} = \langle \mathbf{x} : [u, v]^n\rangle$ and $\mathcal{Q} = \langle \mathbf{x} : [u, v]^n, u, v\rangle$, for $n \geq 2$. Let $r = [u, v]^n$, $R = \langle\langle r\rangle\rangle_F$, $s_1 = u, s_2 = v$ and $S = \langle\langle\{s_1, s_2\}\rangle\rangle_F$. Since $r \in S \cap [S, S]$, it follows that $R \subset S \cap [S, RS]$, and the group-theoretic condition of the corollary holds. So long as $\mathcal{Z}$ is M-Cockcroft (e.g., let $\mathcal{Z}$ be any two relator, aspherical presentation), then $(\mathcal{Q}, \mathcal{P})$ has the identity property.

We observe further that the inclusion induced map $\pi_2 X \to \pi_2 Y$ is trivial, while both homotopy groups are non-zero. As the model of a one relator presentation, $\pi_2 X$ is generated as a $\mathbf{Z}G$-module by a complete dipole. However, this dipole dissolves in the presence of the new **s**-disks. To see this, note that the root $[u, v]$ of the relator r is contained in S. It follows that X is L-Cockcroft. Thus $\pi_2 X \to \pi_2 Y$ is trivial. Finally, $\pi_2 Y$ is not trivial. In the case $n = 2$, a non trivial spherical picture over $\mathcal{Q}$ is given in Figure 5. We remark that this picture also demonstrates that Y itself is not Cockcroft.

In [**8**], Dyer's strategy for constructing a counterexample of infinite type to Whitehead's Conjecture is to begin with a 2-complex pair (Y, X) for which (Y, X) has the identity property and $0 \neq \pi_2 X \to \pi_2 Y$ is trivial. To be able to extend this pair to a longer chain of suitable 2-complexes, it is necessary for Y to be Cockcroft. That is, if Y is not Cockcroft then there exists no 2-complex Y' containing Y for which the inclusion induced map $\pi_2(i) : \pi_2 Y \to \pi_2 Y'$ is trivial. We have seen that the

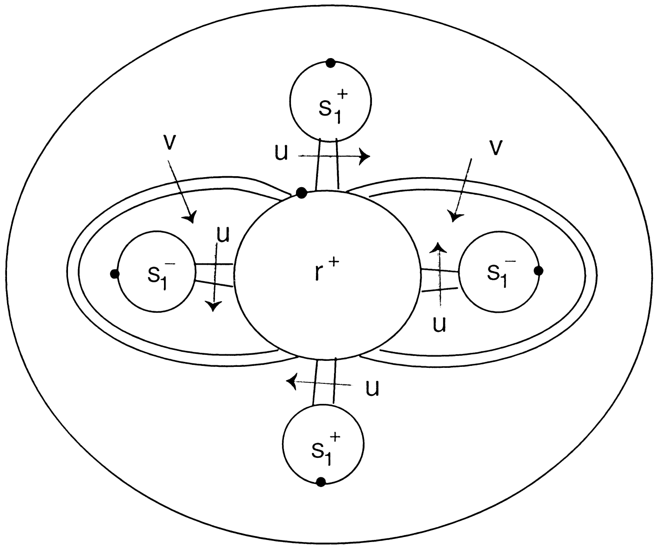

Figure 5

above example satisfies the first two conditions, but is not delicate enough to satisfy the third. It would be of considerable interest to find non trivial examples (where $\pi_2 X \neq 0$) of presentations that satisfy the three conditions of the following corollary.

Corollary 4.6. *The following sets of conditions are equivalent.*

$$\left\{ \begin{array}{c} \pi_2 X \xrightarrow{0} \pi_2 Y \\ (Y, X) \text{ has the identity property} \\ Y \text{ is Cockcroft} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} X \text{ is } L-\text{Cockcroft} \\ Z \text{ is } M-\text{Cockcroft} \\ R \cap S \subset [S, RS] \cap [R, F] \end{array} \right\}.$$

Proof. For $(\mathcal{Q}, \mathcal{P})$ to have the identity property *and* Q to be Cockcroft, we must have $j([\mathbf{P}]) = 0$ for all spherical pictures over $\mathcal{Q}$ *and* all these pictures must have parity in r, for all $r \in \mathbf{r}$. That is, $\mathcal{Z}$ must be M-Cockcroft (rel $\mathbf{r}$) and $\mathcal{P}$ must be G-Cockcroft (rel $\mathbf{s}$). The result now follows from Corollary 4.4 and Lemma 4.2.

5. Combinatorial asphericity. Certain relations always hold in the relative relation module $H_1(L)$. Let $C_G(s)$ denote the centralizer of the image of the word s in G. (For each $s \in \mathbf{s}$ we let s also denote this word's image in the group G; context will make clear whether we're viewing $s \in F$ or $s \in G$.)

For any $g \in C_G(s)$, let $\bar{g}$ be its image in H. Then

$$(\bar{g} - 1) \cdot s[L, L] = gsg^{-1}[L, L] - s[L, L] = s[L, L] - s[L, L] = 0.$$

We call the set $\{(\bar{g} - 1) \cdot s[L, L] : s \in \mathbf{s}, g \in C_G(s)\}$ the set of *trivial identities* in the relative relation module $H_1(L)$, and we say $(\mathcal{Q}, \mathcal{P})$ (or (Y, X)) has the *generalized identity property* if $H_1(L)$ with generators $\{s[L, L] : s \in \mathbf{s}\}$ is defined by the trivial identities.

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The generalized identity property reduces to the identity property in the case that the image of $C_G(s)$ in H is trivial for each s. We have observed that if (Y, X) has the identity property then $\pi_2 Y \cong H_2 X_L$. We turn to relative dipoles to study $\pi_2 Y$ in the presence of the generalized identity property.

Suppose **P** is a spherical picture over $\mathcal{Q}$ containing exactly 2 **s**-disks and possibly some **r**-disks. Then **P** is called a *relative* **s**-*dipole*, or simply an **s**-*dipole*, if the two **s**-disks are labelled by the same element of **s** but with opposite signs. Notice the π_2 generators in Figure 1 are both **s**-dipoles.

If $\mathcal{B}_{\mathcal{P}}$ is a set of spherical pictures over $\mathcal{P}$ that generates $\pi_2 X$ as left $\mathbf{Z}G$-module, then a set $\mathcal{B}$ of spherical pictures over $\mathcal{Q}$ *generates* $\pi_2 Y$ *over* $\mathcal{P}$ if $\mathcal{B} \cup \mathcal{B}_{\mathcal{P}}$ generates $\pi_2 Y$ as left $\mathbf{Z}H$-module. Two spherical pictures **P** and **Q** over $\mathcal{Q}$ are called *equivalent (rel* $\mathcal{B}_{\mathcal{P}}$*)* if $[\mathbf{P}] - [\mathbf{Q}] \in im(\iota_\# : \pi_2 X \to \pi_2 Y)$.

We will call the pair $(\mathcal{Q}, \mathcal{P})$ (or (Y, X)) *combinatorially aspherical*, denoted (CA), if $\pi_2 Y$ is generated over $\mathcal{P}$ by a set of **s**-dipoles.

Example 5.1. Suppose $\langle \mathbf{x} : \mathbf{s} \rangle$ is a (CA) presenation of an infinite group. We construct a relative (CA) pair as follows. Let t be a letter not in **x**, and let $u \in F(\mathbf{x})$ be a word having infinite order in $\langle \mathbf{x} : \mathbf{s} \rangle$. Let $w = tut$ and $r_x = [x, w]$ for each $x \in \mathbf{x}$. Finally, set $\mathbf{r} = \{r_x : x \in \mathbf{x}\}$, and consider the pair $\mathcal{P} = \langle \mathbf{x}, t : \mathbf{r} \rangle$ and $\mathcal{Q} = \langle \mathbf{x}, t : \mathbf{r}, \mathbf{s} \rangle$. It follows from work on generalized graphs of groups in [**2**] (see also [**5**]) that $\pi_2 Y$ is generated over $\mathcal{P}$ by π_2 generators of $\langle \mathbf{x} : \mathbf{s} \rangle$ (which are dipoles since this presentation is (CA)) and one additional spherical pictures for each $s \in \mathbf{s}$. In particular, if $s = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$, with each $x_i \in \mathbf{x}$, each $\epsilon_i = \pm 1$, then the relator s contributes the π_2 generator pictured in Figure 6. Thus, $(\mathcal{Q}, \mathcal{P})$ is (CA).

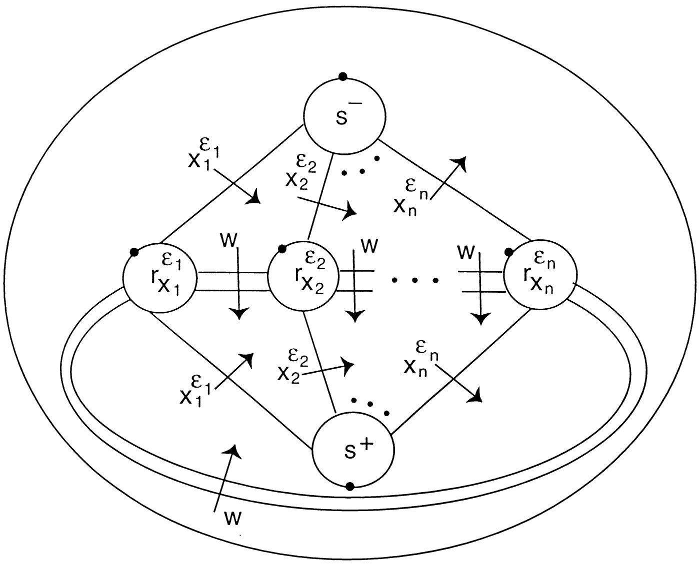

Figure 6

In [**2**] it is proved that w has infinite order in H, and it follows that these examples $(\mathcal{Q}, \mathcal{P})$ *do not* have the identity property. Thus, in the relative setting (as in the absolute setting), combinatorial asphericity is a weaker notion than the identity property.

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We summarize the above discussion with the following proposition.

PROPOSITION 5.2. *Suppose* $\langle \mathbf{x} : \mathbf{s} \rangle$ *is (CA), and* $u \in F(\mathbf{x})$ *has infinite order in the group presented. If* $\mathcal{P} = \langle \mathbf{x}, t \ : \ [x, tut] \ \ (x \in \mathbf{x}) \ \rangle$, *and* $\mathcal{Q} = \langle \mathbf{x}, t \ : \ [x, tut] \ (x \in \mathbf{x}) \ , \mathbf{s} \ \rangle$, *then* $(\mathcal{Q}, \mathcal{P})$ *is (CA).*

For each $1 \neq w \in C_G(s)$, for each $s \in \mathbf{s}$, we may construct a *basic* **s**-*dipole* $\mathbf{P}_w$ as depicted in Figure 7.

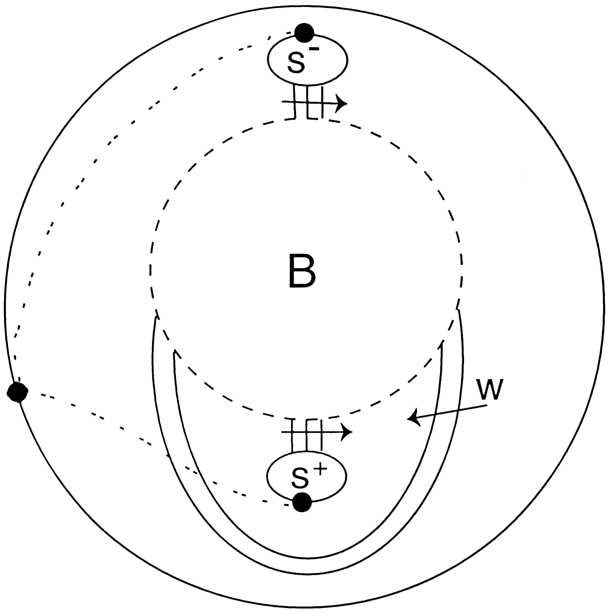

Figure 7

In this construction one may choose the word $W \in F$ representing w, and the subpicture $\mathbf{B}$ of $\mathbf{P}_w$ having boundary label $WsW^{-1}s^{-1} \in R$ (such a picture $\mathbf{B}$ must exist by van Kampen's Lemma). In general, such choices will lead to inequivalent basic **s**-dipoles. However $\mathbf{P}_w$ is unique modulo $\pi_2 X$ in the sense described in the following lemma.

LEMMA 5.3. *Let* $\mathcal{B}_{\mathcal{P}}$ *be a set of spherical pictures over* $\mathcal{P}$ *that generates* $\pi_2 X$. *Suppose* $w \in C_G(s)$, *for* $s \in \mathbf{s}$, *and* $\mathbf{P}_w$ *is a basic* **s**-*dipole formed from the word* $W \in F$ *representing* w *and the picture* $\mathbf{B}$ *over* $\mathcal{P}$ *with* $\partial \mathbf{B} = [W, s]$. *If* $\mathbf{P}'_w$ *is a second basic* **s**-*dipoles formed from* $W' \in F$ *representing* w *and* $\mathbf{B}'$ *with* $\partial \mathbf{B}' = [W', s]$, *then* $\mathbf{P}_w$ *and* $\mathbf{P}'_w$ *are equivalent (rel* $\mathcal{B}_{\mathcal{P}}$*).*

Proof. Since W and W' both represent the element w of G, van Kampen's Lemma ensures the existence of a picture $\mathbf{D}$ over $\mathcal{P}$ with $\partial \mathbf{D} = W^{-1}W'$. Now consider $\mathbf{P}_w - \mathbf{P}'_w$ (Figure 8 (a)). Into this picture we may insert the trivial picture $\mathbf{D}{\cdot}\mathbf{D}^{-1}$ as depicted in Figure 8 (b). Now we make a series of bridge moves to split off the **s**-disks (Figure 8 (c)) and then fold them from the picture (Figure 8 (d)). That is, $\mathbf{P}_w - \mathbf{P}'_w$ is equivalent to a spherical picture over $\mathcal{P}$. Thus, any two basic **s**-dipoles associated to $w \in C_G(s)$ are equivalent (rel $\mathcal{B}_{\mathcal{P}}$).

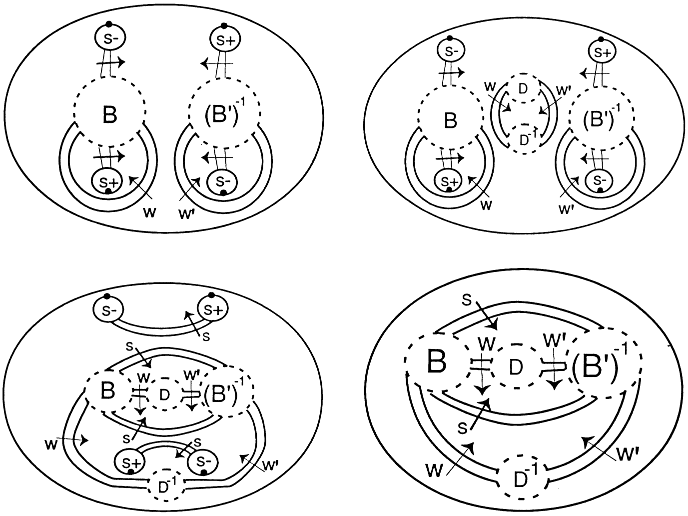

Figure 8

For each $s \in \mathbf{s}$, construct one basic $\mathbf{s}$-dipole $\mathbf{P}_w$ for each $w \in C_G(s)$. Let $\mathcal{D}_s^*$ denote this set of basic $\mathbf{s}$-dipoles, and let $\mathcal{D}^* = \bigcup_{s\in\mathbf{s}} \mathcal{D}_s^*$. Finally, let $J(\mathcal{D}^*)$ denote the submodule of $\pi_2 Y$ generated by $\mathcal{D}^*$.

Lemma 5.4. *If* $\mathbf{P}$ *is any* $\mathbf{s}$*-dipole over* $\mathcal{Q}$*, then* $[\mathbf{P}] \in J(\mathcal{D}^*)$.

Proof. An arbitrary $\mathbf{s}$-dipole $\mathbf{P}$ contains the local configuration of Figure 9 (a), where we assume a path to the negatively oriented s-disk determines the word $V \in F$. Consider the picture $V^{-1} \cdot \mathbf{P}$ depicted in Figure 9 (b).

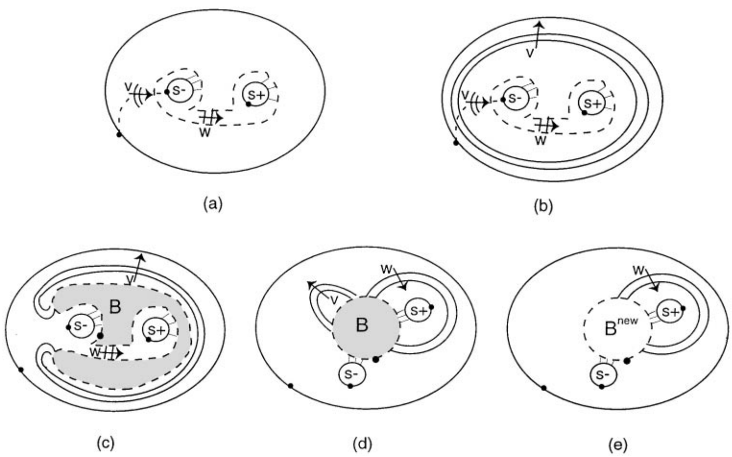

Figure 9

By bridge moves we may open a path connecting the global basepoint to the basepoint of the negatively oriented disk so that the two basepoints are in the same region. See Figure 9 (c). We may contain all the $\mathbf{r}$-disks of this new picture within the

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shaded subpicture having boundary label from the designated "basepoint" identically equal to $sV^{-1}VWs^{-1}W^{-1}$.

After planar isotopy, we may view our picture as in Figure 9 (d), and upon including the arcs labelled by V into the shaded subpicture we obtain a basic **s**-dipole. By Lemma 5.3, this dipole is equivalent (rel $\mathcal{B}_{\mathcal{P}}$) to the **s**-dipole $\mathbf{P}_w$ in $\mathcal{D}^*$ where w is the image of W in G. It follows that the original **s**-dipole **P** is equivalent (rel $\mathcal{B}_{\mathcal{P}}$) to $V \cdot \mathbf{P}_w$, so $[\mathbf{P}] \in J(\mathcal{D}^*)$.

Now suppose that $\mathcal{A}_s \subset G$ is a generating set for $C_G(s)$, and $\mathcal{D}_s = \{\mathbf{P}_w : w \in \mathcal{A}_s\}$ for each $s \in \mathbf{s}$. Set $\mathcal{D} = \bigcup_{s\in\mathbf{s}} \mathcal{D}_s$ and $J(\mathcal{D})$ to be the submodule of $\pi_2 Y$ generated by $\mathcal{D}$.

Lemma 5.5. $J(\mathcal{D}) = J(\mathcal{D}^*)$.

Proof. Clearly $J(\mathcal{D}) \subset J(\mathcal{D}^*)$. The reverse inclusion follows from two facts. First, for $w \in \mathcal{A}_s$, $\mathbf{P}_{w^{-1}}$ is equivalent to $(w^{-1} \cdot \mathbf{P}_w)^{-1}$. Second, if $w_1, w_2 \in C_G(s)$ and $w = w_1^{\epsilon_1} w_2^{\epsilon_2}$, then $\mathbf{P}_w$ is equivalent to $(\mathbf{P}_{w_1^{\epsilon_1}}) + w_1^{\epsilon_1} \cdot (\mathbf{P}_{w_2^{\epsilon_2}})$ (rel $\mathcal{B}_{\mathcal{P}}$). Both equivalences can be checked directly by performing moves on pictures, and the second is demonstrated schematically in Figure 10.

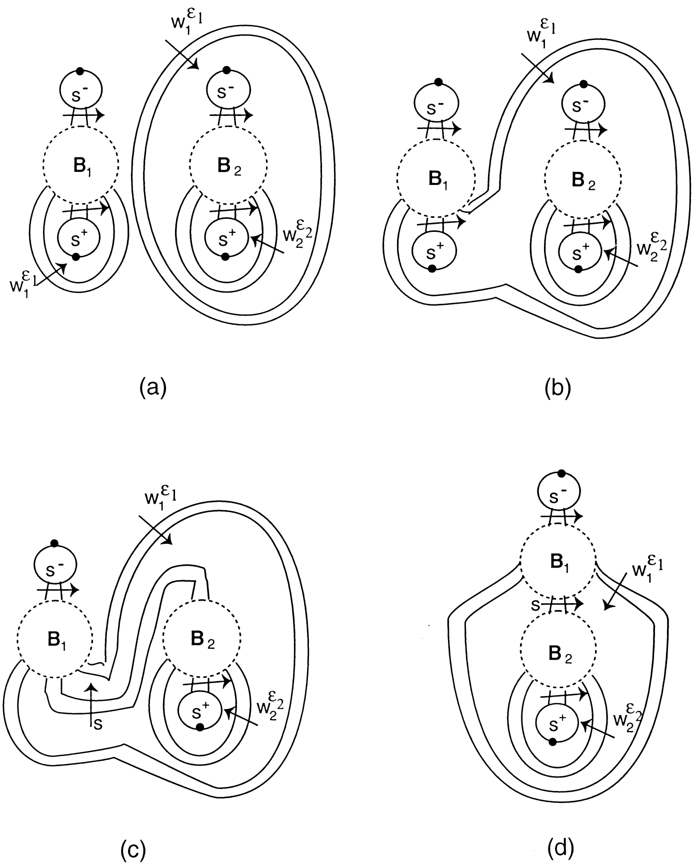

Figure 10

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Proposition 5.6. *$(\mathcal{Q}, \mathcal{P})$ is (CA) if and only if $\pi_2 Y$ is generated over $\mathcal{P}$ by $\mathcal{D}$.*

Proof. If $\pi_2 Y$ is generated by the set $\mathcal{B}_{\mathcal{P}} \cup \mathcal{D}$, then $(\mathcal{Q}, \mathcal{P})$ is (CA) by definition. Conversely, if $(\mathcal{Q}, \mathcal{P})$ is (CA) then $\pi_2 Y$ is generated by the set $\mathcal{B}_{\mathcal{P}}$ together with some set of **s**-dipoles. But the two previous lemmas ensure that any **s**-dipole in this set is equivalent (rel $\mathcal{B}_{\mathcal{P}}$) to an element of the submodule of $\pi_2 Y$ generated by $\mathcal{D}$. The result follows.

We remark that different pictures in $\mathcal{D}$ may be equivalent. For instance, if $\mathcal{P} = \langle a, b : [a, b] \rangle$ and $s = ab^{-1}$, then $C_G(s) = G$ is generated by a and b. One may check that the basic s-dipoles $\mathbf{P}_a$ and $\mathbf{P}_b$ determined by these words are equivalent. Nonetheless, we can find minimal generating sets in some interesting cases. For the remainder of this paper, we assume the set $\mathcal{D}$ has been chosen, and consists of one basic **s**-dipole $\mathbf{P}_w$ for each $w \in \mathcal{A}_s$ in a generating set for $C_G(s)$, for each $s \in \mathbf{s}$.

Proposition 5.7. *If $(\mathcal{Q}, \mathcal{P})$ is (CA) and $N < H$, then $(\mathcal{Q}, \mathcal{P})$ has the right N-identity property if and only if N contains the normal subgroup $K = \langle\langle \{\mathcal{A}_s : s \in \mathbf{s}\} \rangle\rangle_H$ of H.*

Proof. If $(\mathcal{Q}, \mathcal{P})$ is (CA) then $\pi_2 Y$ is generated by the set $\mathcal{B}_{\mathcal{P}} \cup \mathcal{D}$. The image in j of any picture in this set is either trivial or of the form $(\bar{w}_s - 1)c_s^2$ for some $s \in \mathbf{s}$, $w_s \in \mathcal{A}_s$. Thus, any picture $\mathbf{P}$ over $\mathcal{Q}$ has image in j of the form

$$j(\mathbf{P}) = \sum h_i(\bar{w}_{s_i} - 1)c_{s_i}^2,$$

where $h_i \in H$, $w_{s_i} \in \mathcal{A}_{s_i}$, $s_i \in \mathbf{s}$. With respect to a subgroup $N < H$, $\mathbf{P}$ has the prescribed pairing of Definition 3.1 if and only if

$$N_F u_i w_{s_i} = N_F u_i$$

where $u_i \in F$ represents $h_i \in H$. That is, $\mathbf{P}$ has the prescribed pairing if and only if $u_i w_{s_i} u_i^{-1} \in N_F$. It follows that all pictures over $\mathcal{Q}$ have the prescribed pairing if and only if $K \subset N$.

For instance, let $\mathcal{P} = \langle a, b : [a, b] \rangle$ and $s = a^3$. Then $(\mathcal{Q}, \mathcal{P})$ is (CA) since $\pi_2 Y$ is generated by **s**-dipoles (as shown in Figure 1). Since G is abelian in this case, the centralizer $C_G(s) = G$ is generated by a and b. The normal subgroup of H generated by a and b is H itself, so the above result ensures that $(\mathcal{Q}, \mathcal{P})$ has the right N-identity property if and only if $N = H$.

Proposition 5.8. *If $(\mathcal{Q}, \mathcal{P})$ is (CA) then $(\mathcal{Q}, \mathcal{P})$ has the generalized identity property.*

Proof. According to the exactness of sequence (1), $H_1(L)$ is isomorphic to the $\mathbf{Z}H$-module generated by $\{c_s^2 : s \in \mathbf{s}\}$ and defined by the relations $j([\mathbf{P}]) = 0$ for all pictures in a generating set of $\pi_2 Y$. In the present situation, we may assume $\pi_2 Y$ is generated by the pictures in $\mathcal{B}_{\mathcal{P}} \cup \mathcal{D}$. Since any picture $\mathbf{P}$ over $\mathcal{P}$ has no **s**-disks, $j([\mathbf{P}])$ *is* 0 for elements of $\mathcal{B}_{\mathcal{P}}$. Moreover, if $\mathbf{P} \in \mathcal{D}$ then $j([\mathbf{P}]) = (\bar{w} - 1) \cdot c_s^2$, where w is in a generating set of $C_G(s)$, $s \in \mathbf{s}$. That is, each $\mathbf{P} \in \mathcal{D}$ determines a trivial identity $j([\mathbf{P}]) = 0$. Thus, $H_1(L)$ is defined by trivial identities, and $(\mathcal{Q}, \mathcal{P})$ has the generalized identity property.

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EXAMPLE 5.9. Consider the pair $\mathcal{P} = \langle \mathbf{x} : [u, v]^2 \rangle$, $\mathcal{Q} = \langle \mathbf{x} : [u, v]^2, u, v \rangle$. We know from Example 4.5 that $(\mathcal{Q}, \mathcal{P})$ has the identity property, so it necessarily has the generalized identity property. However, the spherical picture of Figure 5 is a generator of $\pi_2 Y$ which is not equivalent to a sum of **s**-dipoles and pictures over $\mathcal{P}$. It follows that $(\mathcal{Q}, \mathcal{P})$ is not (CA), and the converse to the above theorem is false.

We have the following partial converse to Proposition 5.8.

PROPOSITION 5.10. *If* $H_2(L) = 0$ *and if* $(\mathcal{Q}, \mathcal{P})$ *has the generalized identity property then* $(\mathcal{Q}, \mathcal{P})$ *is (CA).*

Proof. We show that any spherical picture **P** over $\mathcal{Q}$ is in $J(\mathcal{B}_{\mathcal{P}} \cup \mathcal{D}^*)$. Since $(\mathcal{Q}, \mathcal{P})$ has the generalized identity property, $j([\mathbf{P}]) = \sum h_i(\bar{w}_i - 1)c_{s_i}^2$ where $h_i \in H$, $w_i \in C_G(s_i)$, and $\bar{w}_i$ is the image of w_i in H. trivial identities in $H_1(L)$. We may associate to this sum a natural picture $\mathbf{P}'$ over $\mathcal{Q}$ that is a sum of **s**-dipoles. In particular, the term $h_i(\bar{w}_i - 1)c_{s_i}^2$ gives rise to the **s**-dipole $v_i \cdot \mathbf{P}_{w_i}$ where v_i represents h_i in the free group, and $\mathbf{P}_{w_i}$ is the **s**-dipole in $\mathcal{D}^*$ associated to w_i. Let $\mathbf{P}' = \sum v_i \cdot \mathbf{P}_i$. Then $j([\mathbf{P}]) = j([\mathbf{P}'])$ and $[\mathbf{P} - (\mathbf{P}')] \in ker\ j = im\ i$, where $i : H_2(X_L) \to \pi_2 Y$ is from the fundamental sequence (1). Now, the Hopf sequence $\pi_2 X \to H_2 X_L \to H_2 L \to 0$ ensures that $H_2 L$ is isomorphic to $H_2 X_L / im(\pi_2 X)$. If $H_2 L = 0$ then every spherical picture over $\mathcal{Q}$ in the image of $H_2 X_L$ actually comes from $\pi_2 X$. Thus, $[\mathbf{P} - \mathbf{P}'] \in \mathrm{im}(\pi_2 X \to \pi_2 Y)$, and **P** is equivalent (rel $\mathcal{P}$) to a spherical picture whose image is in $J(\mathcal{D}^*)$. It follows that $[\mathbf{P}] \in J(\mathcal{B}_{\mathcal{P}} \cup \mathcal{D})$.

6. A test for (CA). Consider $\mathcal{Q} = \langle\, \mathbf{x} : \mathbf{r}, \mathbf{s} \,\rangle$. For each s in **s**, let $\exp_s(\mathbf{P})$ = the number of s^+ disks in **P** - the number of s^- disks in **P**. Then $\mathcal{Q}$ has parity in **s** if and only if $\exp_s(\mathbf{P}) = 0$ for each $s \in \mathbf{s}$ and each **P** over $\mathcal{Q}$. Recall, $\mathcal{Q}$ has parity in **s** if and only if $(\mathcal{Q}, \mathcal{P})$ has the H-identity property. This is true if and only if $(\mathcal{Q}, \mathcal{P})$ is Cockcroft.

A subset **c** of a (multiplicative) abelian group C is called *linearly independent* if

$$\prod_{i=1}^{k} c_i^{n_i} = 1 \text{ implies } n_i = 0 \text{ for each } i$$

where $n_i \in \mathbf{Z}$, and $c_1, c_2, ..., c_k$ are distinct elements of **c**.

LEMMA 6.1. *If* **s** *determines a linearly independent set of elements in* $H_1(G)$, *then* $(\mathcal{Q}, \mathcal{P})$ *is Cockcroft.*

Proof. Take any spherical picture **P** over $\mathcal{Q}$. Assume **P** has **s**-disks. Then any spray to the **s**-disks determines an equation

$$\prod_{\mathbf{s}-\text{disks in } \mathbf{P}} w_i s_i^{\epsilon_i} w_i^{-1} \overset{F}{=} w$$

where $w \in R$, and $w_i \in F$ for each i. Viewing this equation in G, we obtain

$$\prod_{\mathbf{s}-\text{disks in } \mathbf{P}} w_i s_i^{\epsilon_i} w_i^{-1} \overset{G}{=} 1$$

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and modulo $[G, G]$ we have

$$\prod_{\mathbf{s}-\text{disks in } \mathbf{P}} s_i^{\epsilon_i} \overset{H_1(G)}{=} 1.$$

Grouping common bases,

$$\prod_{s \in \mathbf{s}} s^{\exp_s(\mathbf{P})} \overset{H_1(G)}{=} 1.$$

The linear independence condition implies that $\exp_s(\mathbf{P}) = 0$ for each $s \in \mathbf{s}$.

In the case $\mathbf{s}$ consists of a single word, we have the following simple sufficient condition for $(\mathcal{Q}, \mathcal{P})$ to be (CA).

THEOREM 6.2. *Suppose $\mathcal{P} = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ presents the group G and the word $s \in F(\mathbf{x})$ lives in the center of G and has infinite order in $H_1(G)$. If we let $\mathcal{Q} = \langle \mathbf{x} : \mathbf{r}, s\rangle$ then $(\mathcal{Q}, \mathcal{P})$ is (CA), and $\pi_2 Y$ is generated (over $\mathcal{P}$) by the set $\mathcal{D}$ of basic* **s**-*dipoles.*

Proof. Since s has infinite order in $H_1(G)$, $\mathcal{Q} = \langle \mathbf{x} : \mathbf{r}, s\rangle$ has parity in $\mathbf{s} = \{s\}$ by Lemma 6.1. Thus, the image $j([\mathbf{P}])$ of any spherical picture $\mathbf{P}$ over $\mathcal{Q}$ has the form

$$\Sigma h_i(\bar{g}_i - 1)c_s^2$$

where $h_i \in H$, $g_i \in G$, and $\bar{g}_i$ is its image in H. Since the image of s is in the center of G, $C_G(s) = G$, and $j([\mathbf{P}])$ is mapped by ψ to a consequence of trivial identities. Since this holds for all $\mathbf{P}$ over $\mathcal{Q}$, $(\mathcal{Q}, \mathcal{P})$ has the generalized identity property. Furthermore, the two conditions on s ensure that L, the normal closure of s in G, is infinite cyclic. Thus, $H_2(L)$ is trivial and $(\mathcal{Q}, \mathcal{P})$ is (CA) by Proposition 5.10.

EXAMPLE 6.3. Suppose $\mathcal{P} = \langle a, b : [a, b]\rangle$ and $\mathcal{Q} = \langle a, b : [a, b], s\rangle$ where s is any non-trivial word that does not set $a \overset{H}{=} b$. Since G is abelian, $C_G(s) = G$ is generated by a and b, and by our choice of s these words determine distinct elements of H. Also, any such s has infinite order in $G = H_1(G)$ so $(\mathcal{Q}, \mathcal{P})$ is (CA) by Theorem 6.2. Finally, $\mathcal{P}$ is aspherical, so $\pi_2 Y$ is generated by two distinct basic **s**-dipoles $\mathbf{P}_a$ and $\mathbf{P}_b$.

For instance, consider the presentation $\mathcal{Z} = \langle a, b : s = a^2b^{-3}\rangle$, of the (2, 3) torus knot group. Let $\mathcal{P} = \langle a, b : r = [a, b]\rangle$, and $\mathcal{Q} = \langle a, b : r, s\rangle$. Then $(\mathcal{Q}, \mathcal{P})$ is (CA) and $\pi_2 Y$ is generated by two s-dipoles $\mathbf{P}_a$ and $\mathbf{P}_b$, as seen in Figure 11. The pictures are formed from simples choice for the disk pictures $\mathbf{B}_a$ and $\mathbf{B}_b$.

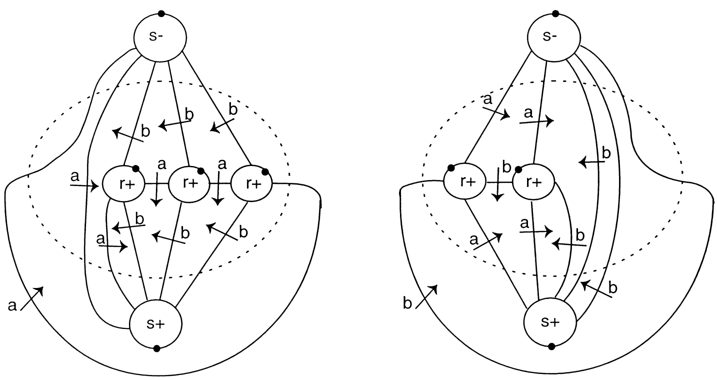

Figure 11

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Example 6.4. Consider the presentation $\mathcal{P} = \langle a, b : a^6 = b^3, b^3 = (ab)^2 \rangle$ of the group G, and let $\mathbf{s} = \{a^6\}$. Then $\mathcal{Q} = \langle a, b : a^6 = b^3, b^3 = (ab)^2, a^6 \rangle$ presents H.

In G, s can be expressed as a product of a's and as a product of b's, so s is in the center of G. Furthermore, the abelianization of G is the infinite cyclic group generated by a, so a^6 has infinite order in $H_1(G)$. Then by Theorem 6.2, $(\mathcal{Q}, \mathcal{P})$ is (CA).

To find generators for $\pi_2 Y$ (over $\mathcal{P}$) first note that $C_G(s) = G$ since s is in the center of G. This centralizer is generated by a and b, so $\pi_2 Y$ is generated (over $\mathcal{P}$) by two s-dipoles $\mathbf{P}_a$ and $\mathbf{P}_b$, as shown in Figure 12.

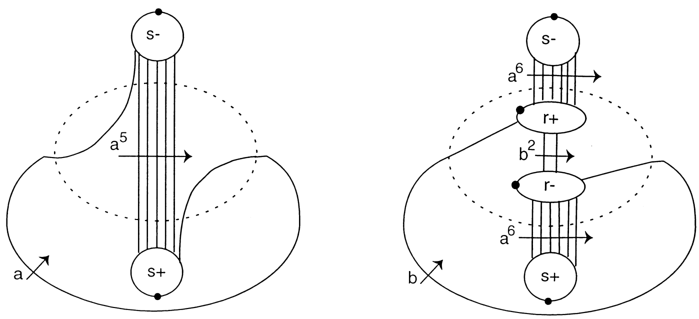

Figure 12

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REFERENCES

1. J. F. Adams, A new proof of a theorem of W.H. Cockcroft, *J. London Math. Soc.* **30** (1955), 482–488.

2. Y. G. Baik and S. J. Pride, Generators of the second homotopy module of presentations arising from group constructions, preprint (University of Glasgow, 1992).

3. W. A. Bogley, Unions of Cockcroft two-complexes, *Proc. Edinburgh Math. Soc. (2)* **37** (1994), 317–324.

4. W. A. Bogley, Whitehead's asphericity question, in *Two-dimensional homotopy and combinatorial group theory* (London Math. Society Lecture Note Series, **197** (1993), edited by C. Hog-Angeloni, W. Metzler and A. J. Sieradski), 309–334.

5. W. A. Bogley and S. J. Pride, Calculating generators of π_2, in *Two-dimensional homotopy and combinatorial group theory* (London Math. Society Lecture Note Series, **197** (1993), edited by C. Hog-Angeloni, W. Metzler and A. J. Sieradski), 157–188.

6. K. A. Brown, *Cohomology of groups* (Springer-Verlag, 1982).

7. M. N. Dyer, Crossed modules and π_2 homotopy modules, in *Two-dimensional homotopy and combinatorial group theory* (London Math. Society Lecture Note Series, **197** (1993), edited by C. Hog-Angeloni, W. Metzler and A. J. Sieradski), 125–156.

8. M. N. Dyer, Adding 2-cells in a delicate way, preprint (University of Oregon, 1993).

9. N. D. Gilbert and J. Howie, Threshold subgroups for Cockcroft 2-complexes, *Comm. Algebra* **23** (1995), 255–275.

10. M. A. Gutièrrez and J. G. Ratcliffe, On the second homotopy group, *Quart. J. Math. Oxford Ser. (2)* **32** (1981), 45–55.

11. J. Harlander, Minimal Cockcroft subgroups, *Glasgow Math. J.* **36** (1994), 87–90.

12. J. Howie, Some remarks on a problem of J.H.C. Whitehead, *Topology* **22** (1983), 475–485.

13. S. J. Pride, Identities among relations of group presentations, in *Group theory from a geometrical viewpoint, Trieste 1990* (edited by E. Ghys, A. Haefliger, and A. Verjovsky), (World Scientific Publishing, 1991).

14. S. J. Pride, Examples of a presentation which are minimally Cockcroft in several different ways, *J. Pure Appl. Algebra* **88** (1993), 199–204.

15. J. H. C. Whitehead, On adding relations to homotopy groups, *Ann. of Math.* **42** (1941), 1197–1239.