Canad. Math. Bull. Vol. **52** (4), 2009 pp. 627–636

On L^1-Convergence of Fourier Series under the MVBV Condition

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Abstract. Let $f \in L_{2\pi}$ be a real-valued even function with its Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, and let $S_n(f,x)$, $n \geq 1$, be the n-th partial sum of the Fourier series. It is well known that if the nonnegative sequence $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} \|f - S_n(f)\|_L = 0 \text{ if and only if } \lim_{n\to\infty} a_n \log n = 0.$$

We weaken the monotone condition in this classical result to the so-called mean value bounded variation (MVBV) condition. The generalization of the above classical result in real-valued function space is presented as a special case of the main result in this paper, which gives the L^1-convergence of a function $f \in L_{2\pi}$ in complex space. We also give results on L^1-approximation of a function $f \in L_{2\pi}$ under the MVBV condition.

1 Introduction

Let $L_{2\pi}$ be the space of all complex-valued integrable functions $f(x)$ of period 2π equipped with the norm $\|f\|_L = \int_{-\pi}^{\pi} |f(x)|dx$. Denote the Fourier series of $f \in L_{2\pi}$ by $\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$,and its partial sum $S_n(f,x)$ by $\sum_{k=-n}^{n} \hat{f}(k)e^{ikx}$. When $f(x) \in L_{2\pi}$ is a real-valued even function, then the Fourier series of f has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.1}$$

correspondingly, its partial sum $S_n(f,x)$ is $\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$.

The following two classical convergence results can be found in many monographs (see [1] and [9], for example).

Result One: If a nonnegative sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n=1}^{\infty} b_n \sin nx$ converges uniformly if and only if $\lim_{n\to\infty} nb_n = 0$.

Result Two: Let $f \in L_{2\pi}$ be an even function and (1.1) be its Fourier series. If the sequence $\{a_n\}_{n=0}^{\infty}$ is nonnegative, decreasing, and $\lim_{n\to\infty} a_n = 0$, then

$$\lim_{n\to\infty} \|f - S_n(f)\|_L = 0 \text{ if and only if } \lim_{n\to\infty} a_n \log n = 0.$$

Received by the editors October 5, 2006.

The first author's research is supported in part by an NSERC RCD grant from St. Francis Xavier University and in part by AARMS of Canada. The second author's research is supported by NSERC of Canada. The third author's research was done as a W. F. James Chair Professor at St. Francis Xavier University. His research is also supported in part by NSF of China under grant number 10471130.

AMS subject classification: Primary: 42A25; secondary: 41A50.

Keywords: complex trigonometric series, L^1 convergence, monotonicity, mean value bounded variation.

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https://doi.org/10.4153/CMB-2009-061-6 Published online by Cambridge University Press

These results have been generalized by weakening the monotone conditions of the coefficient sequences. They have also been generalized to the complex valued function spaces. The most recent generalizations of Result One can be found in [8] where the monotonic condition is finally weakened to the MVBV condition (Mean Value Bounded Variation condition, see Corollary 2.8 for the definition), and it is proved to be the weakest possible condition we can have to replace the monotone condition in Result One. The process of generalizing Result Two can be found in many papers, for example, see [2–7]. In this paper, we will weaken the monotone condition in Result Two (and all its later generalized conditions, see [8] for the relations between these conditions), to the MVBV condition in the complex-valued function spaces (see Definition 2.1) in Theorem 2.2, and give the generalization in real valued function spaces as a special case of Theorem 2.2 in Corollary 2.8. Like the important role that the MVBV condition plays in generalizing Result One, although we are not able to prove it here, we propose that Theorem 2.2 is the ultimate generalization of Result Two, *i.e.,* the MVBV condition is also the weakest possible condition we can have to replace the monotone condition in Result Two. We also discuss, under the MVBV condition, the L^1-approximation rate of a function $f \in L_{2\pi}$ in the last section.

Throughout this paper, we always use $C(x)$ to indicate a positive constant depending upon x only, and use C to indicate an absolute positive constant. They may have different values in different occurrences.

2 L^1 Convergence

In this section, we first give the definition of the MVBV condition, or the class *MVBVS*, and then prove our main result on L^1-convergence of the Fourier series of a complex-valued function $f(x) \in L_{2\pi}$ whose coefficients form a sequence in the class *MVBVS*.

Definition 2.1 Let $\mathbf{c} := \{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying $c_n \in K(\theta_1) := \{z : |\arg z| \le \theta_1\}$ for some $\theta_1 \in [0, \pi/2)$ and all $n = 0, 1, 2, \ldots$. If there is a number $\lambda \ge 2$ such that

$$\sum_{k=m}^{2m} |\Delta c_k| := \sum_{k=m}^{2m} |c_{k+1} - c_k| \le C(\mathbf{c}) \frac{1}{m} \sum_{k=[\lambda^{-1}m]}^{[\lambda m]} |c_k|$$

holds for all $m = 1, 2, \ldots$, then we say that the sequence $\mathbf{c}$ is a *Mean Value Bounded Variation Sequence*, *i.e.,* $\mathbf{c} \in$ *MVBVS*, in the complex sense, or the sequence $\mathbf{c}$ satisfies the MVBV condition.

Our main result is the following theorem.

Theorem 2.2 *Let $f(x) \in L_{2\pi}$ be a complex-valued function. If the Fourier coefficients $\hat{f}(n)$ of f satisfy that $\{\hat{f}(n)\}_{n=0}^{+\infty} \in$ MVBVS and*

$$\lim_{\mu \to 1^+} \limsup_{n \to \infty} \sum_{k=n}^{[\mu n]} |\Delta \hat{f}(k) - \Delta \hat{f}(-k)| \log k = 0, \tag{2.1}$$

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where

$$\Delta\hat{f}(k) = \hat{f}(k+1) - \hat{f}(k), \quad \Delta\hat{f}(-k) = \hat{f}(-k-1) - \hat{f}(-k), \quad k \geq 0.$$

Then $\lim_{n\to\infty} \|f - S_n(f)\|_L = 0$ *if and only if* $\lim_{n\to\infty} \hat{f}(n)\log|n| = 0$.

In order to prove Theorem 2.2, we present the following four lemmas.

Lemma 2.3 *Let* $\{c_n\} \in MVBVS$, *then for any given* $1 < \mu < 2$, *we have*

$$\sum_{k=n}^{[\mu n]} |\Delta c_k| \log k = O\Big(\max_{[\lambda^{-1}n]\leq k\leq[\lambda n]} |c_k| \log k \Big), \quad n \to \infty,$$

where the implicit constant depends only on the sequence $\{c_n\}$ *and* λ.

For sufficiently large n, the lemma can be derived directly from the conditions that $1 < \mu < 2$ and $\{c_n\} \in MVBVS$.

Lemma 2.4 *Let* $\{\hat{f}(n)\} \in K(\theta_0)$ *for some* $\theta_0 \in [0, \pi/2)$, *then*

$$\sum_{k=1}^{n} \frac{1}{k} \big| \hat{f}(n+k) \big| = O\left(\|f - S_n(f)\|_L \right)$$

for all $n = 1, 2, \ldots$, *where the implicit constant depends only on* θ_0.

Proof Write

$$\phi_{\pm n}(x) := \sum_{k=1}^{n} \frac{1}{k} \left(e^{i(k\mp n)x} - e^{-i(k\pm n)x} \right).$$

It follows from a well-known inequality (*e.g.*, see [6, Theorem 2.5])

$$\sup_{n\geq 1} \Big| \sum_{k=1}^{n} \frac{\sin kx}{k} \Big| \leq 3\sqrt{\pi}$$

that $|\phi_{\pm n}(x)| \leq 6\sqrt{\pi}$. Hence

$$\frac{1}{6\sqrt{\pi}} \left| \int_{-\pi}^{\pi} (f(x) - S_n(f,x))\phi_{\pm n}(x)dx \right| \leq \|f - S_n(f)\|_L,$$

and therefore

$$\Big| \sum_{k=1}^{n} \frac{1}{k} \hat{f}(n+k) \Big| = O(\|f - S_n(f)\|_L).$$

Now as $\{\hat{f}(n)\} \in K(\theta_0)$ for some $\theta_0 \in [0, \pi/2)$ and for all $n \geq 1$, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k} \big| \hat{f}(n+k) \big| &\leq C(\theta_0) \sum_{k=1}^{n} \frac{1}{k} \mathrm{Re}\hat{f}(n+k) \leq C(\theta_0) \Big| \sum_{k=1}^{n} \frac{1}{k} \hat{f}(n+k) \Big| \\ &= O(\|f - S_n(f)\|_L). \end{aligned}$$

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Lemma 2.5 ([5]) *Write*

$$D_k(x) := \frac{\sin((2k+1)x/2)}{2\sin(x/2)},$$

$$D_k^*(x) := \begin{cases} \dfrac{\cos(x/2) - \cos((2k+1)x/2)}{2\sin(x/2)} & |x| \le 1/n, \\ -\dfrac{\cos((2k+1)x/2)}{2\sin(x/2)} & 1/n \le |x| \le \pi, \end{cases}$$

$$E_k(x) := D_k(x) + iD_k^*(x).$$

Then for $k = n, n+1, \ldots, 2n$, *we have*

$$E_k(\pm x) - E_{k-1}(\pm x) = e^{\pm ikx}, \tag{2.2}$$

$$E_k(x) + E_k(-x) = 2D_k(x), \tag{2.3}$$

$$\|E_k\|_L + \|D_k\|_L = O(\log k). \tag{2.4}$$

Lemma 2.6 *Let* $\{\hat{f}(n)\} \in MVBVS$. *If* $\lim_{n\to\infty} \|f - S_n(f)\|_L = 0$, *then*

$$\lim_{n\to\infty} \hat{f}(n)\log n = 0.$$

Proof By the definition of *MVBVS*, we derive that for $k = n, n+1, \ldots, 2n$,

$$|\hat{f}(2n)| \le \sum_{j=k}^{2n-1} |\Delta\hat{f}(j)| + |\hat{f}(k)| \le \sum_{j=k}^{2k} |\Delta\hat{f}(j)| + |\hat{f}(k)|$$

$$= O\Big(\frac{1}{n}\sum_{j=[\lambda^{-1}k]}^{[\lambda k]} |\hat{f}(j)|\Big) + |\hat{f}(k)|.$$

Therefore, it follows that from the fact that

$$\log n \le C(\lambda) \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j},$$

we have

(2.5)

$$\begin{aligned} |\hat{f}(2n)|\log n &\le C(\lambda)|\hat{f}(2n)| \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \\ &\le C(\lambda) \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j}\Big(\frac{1}{n}\sum_{k=[\lambda^{-1}(n+j)]}^{[\lambda(n+j)]} |\hat{f}(k)| + |\hat{f}(n+j)|\Big) \\ &= \frac{C(\lambda)}{n} \sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]} \frac{1}{j} \sum_{k=[\lambda^{-1}(n+j)]}^{[\lambda(n+j)]} |\hat{f}(k)| + C(\lambda) \sum_{j=1}^{[(\lambda+1)^{-2}n]} \frac{1}{j}|\hat{f}(n+j)| \\ &=: I_1 + I_2. \end{aligned}$$

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By applying Lemma 2.4, we see that

$$I_2 \le C(\lambda,\theta_0)\|f - S_n(f)\|_L. \tag{2.6}$$

We calculate I_1 as follows (note that we may add more repeated terms in the right hand side of every inequality below):

$$\begin{aligned}
I_1 &\le \frac{C(\lambda)}{n}\sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]}\frac{1}{j}\sum_{k=[\lambda^{-1}n]+[\lambda^{-1}j]}^{[\lambda n]+[\lambda j]+1}|\hat f(k)| \\
&\le \frac{C(\lambda)}{n}\sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]}\sum_{m=1}^{[(\lambda+1)^2]}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]+1}\frac{\left|\hat f\Big(m[\lambda^{-1}j]+k\Big)\right|}{j} \\
&\le \frac{C(\lambda)}{n}\sum_{m=1}^{[(\lambda+1)^2]}\sum_{j=[\lambda]+1}^{[(\lambda+1)^{-2}n]}\sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1}\frac{\left|\hat f\Big([\lambda^{-1}n]+m[\lambda^{-1}j]+k\Big)\right|}{j} \\
&\le \frac{C(\lambda)}{n}\sum_{m=1}^{[(\lambda+1)^2]}\sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1}\sum_{j=1}^{m[(\lambda(\lambda+1)^2)^{-1}n]}\frac{\left|\hat f\Big([\lambda^{-1}n]+k+j\Big)\right|}{j} \\
&\le \frac{C(\lambda)}{n}\sum_{m=1}^{[(\lambda+1)^2]}\sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1}\left\|f - S_{[\lambda^{-1}n]+k}(f)\right\|_L \text{ (by Lemma 2.4)} \\
&\le \frac{C(\lambda)}{n}\sum_{k=0}^{[\lambda n]-[\lambda^{-1}n]+1}\left\|f - S_{[\lambda^{-1}n]+k}(f)\right\|_L.
\end{aligned} \tag{2.7}$$

Finally, by combining (2.5)–(2.7) with the condition $\lim_{n\to\infty}\|f - S_n(f)\|_L = 0$, we get $\lim_{n\to\infty}\hat f(2n)\log n = 0$. A similar argument shows $\lim_{n\to\infty}|\hat f(2n+1)|\log n = 0$. This proves Lemma 2.6. ∎

We now come to the proof of Theorem 2.2.

Proof of Theorem 2.2 *Sufficiency.* Given $\varepsilon > 0$, by (2.1), there is a $1 < \mu < 2$ such that

$$\sum_{k=n}^{[\mu n]}\left|\Delta\hat f(k) - \Delta\hat f(-k)\right|\log k \le \varepsilon \tag{2.8}$$

holds for sufficiently large $n > 0$. Let

$$\tau_{\mu n,n}(f,x) := \frac{1}{[\mu n]-n}\sum_{k=n}^{[\mu n]-1}S_k(f,x)$$

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be the Vallée Poussin sum of order n of f. Then we have

$$\lim_{n\to\infty} \|f - \tau_{\mu n,n}(f)\|_L = 0. \tag{2.9}$$

By (2.2), (2.3), and applying Abel transformation, we get

(2.10)
$$\begin{aligned}
&\tau_{\mu n,n}(f,x) - S_n(f,x)\\
&\quad= \frac{1}{[\mu n]-n}\sum_{k=n+1}^{[\mu n]}([\mu n]-k)\big(\hat f(k)e^{ikx} + \hat f(-k)e^{-ikx}\big)\\
&\quad= \frac{1}{[\mu n]-n}\sum_{k=n}^{[\mu n]}([\mu n]-k)\big(2\Delta\hat f(k)D_k(x) - (\Delta\hat f(k) - \Delta\hat f(-k))E_k(-x)\big)\\
&\qquad+ \frac{1}{[\mu n]-n}\sum_{k=n}^{[\mu n]-1}\big(\hat f(k+1)E_k(x) - \hat f(-k-1)E_k(-x)\big)\\
&\qquad- \big(\hat f(n)E_n(x) + \hat f(-n)E_n(-x)\big).
\end{aligned}$$

Thus, by (2.4) and Lemma 2.3, we have

(2.11)
$$\begin{aligned}
\|f - S_n(f)\|_L &\le \|f - \tau_{\mu n,n}(f)\|_L + \|\tau_{\mu n,n}(f) - S_n(f)\|_L\\
&= \|f - \tau_{\mu n,n}(f)\|_L + O\Big(\sum_{k=n}^{[\mu n]}\big|\Delta\hat f(k)\big|\log k\Big)\\
&\quad+ O\Big(\sum_{k=n}^{[\mu n]}\big|\Delta\hat f(k) - \Delta\hat f(-k)\big|\log k\Big) + O\Big(\max_{n\le|k|\le[\mu n]}|\hat f(k)|\log|k|\Big)\\
&= \|f - \tau_{\mu n,n}(f)\|_L + O\Big(\max_{[\lambda^{-1}n]\le|k|\le[\lambda n]}|\hat f(k)|\log|k|\Big)\\
&\quad+ O\Big(\sum_{k=n}^{[\mu n]}\big|\Delta\hat f(k) - \Delta\hat f(-k)\big|\log k\Big),
\end{aligned}$$

then $\limsup_{n\to\infty}\|f - S_n(f)\|_L \le \varepsilon$ follows from (2.8), (2.9), and the condition that $\lim_{n\to\infty}\hat f(n)\log|n| = 0$. This implies that $\lim_{n\to\infty}\|f - S_n(f)\|_L = 0$.

Necessity. Since $\{\hat f(n)\} \in MVBVS$, by applying Lemma 2.6, we have

$$\lim_{n\to\infty}\hat f(n)\log n = 0. \tag{2.12}$$

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In order to prove $\lim_{n\to-\infty}\hat{f}(n)\log|n|=0$, by applying (2.10) and (2.4), we see that for any given μ, $1<\mu<2$,

$$\begin{aligned}\|\hat{f}(-n)E_n(-x)\|_L &\le \|\tau_{\mu n,n}(f)-S_n(f)\|_L\\ &\quad+\frac{1}{[\mu n]-n}\Big\|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)E_k(-x)\Big\|_L\\ &\quad+O\Big(\sum_{k=n}^{[\mu n]}\Big(|\Delta\hat{f}(k)-\Delta\hat{f}(-k)|\log k+|\Delta\hat{f}(k)|\log k\Big)\Big)\\ &\quad+O\Big(\max_{n\le k\le[\mu n]}|\hat{f}(k)|\log k\Big). \qquad (2.13)\end{aligned}$$

It is not difficult to see that

$$\Big\|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)E_k(-x)\Big\|_L = I+O\Big(n\max_{n<k\le[\mu n]}|\hat{f}(-k)|\Big),$$

where

$$I:=\int_{n^{-1}\le|x|\le\pi}\Big|\frac{1}{2\sin(x/2)}\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)e^{\frac{i(2k+1)x}{2}}\Big|\,dx.$$

Since the trigonometric function system is orthonormal, we have

$$\begin{aligned}I&\le\Big(\int_{n^{-1}\le|x|\le\pi}\Big|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)e^{\frac{i(2k+1)x}{2}}\Big|^2dx\Big)^{1/2}\times\Big(\int_{n^{-1}}^{\pi}\frac{1}{\sin^2(x/2)}dx\Big)^{1/2}\\ &=O\Big(\sqrt{n}\Big(\sum_{k=n+1}^{[\mu n]}|\hat{f}(-k)|^2\Big)^{1/2}\Big)=O\Big(n\max_{n\le k\le[\mu n]}|\hat{f}(-k)|\Big),\end{aligned}$$

which yields that

$$\frac{1}{[\mu n]-n}\Big\|\sum_{k=n}^{[\mu n]-1}\hat{f}(-k-1)E_k(-x)\Big\|_L=O\Big(\max_{n<k\le[\mu n]}|\hat{f}(-k)|\Big). \qquad (2.14)$$

By combining (2.9) and (2.12)–(2.14) with Lemma 2.3 and using the condition $\lim_{n\to\infty}\|f-S_n(f)\|_L=0$ and the fact (since $f\in L_{2\pi}$) that $\lim_{n\to\infty}\hat{f}(-n)=0$, we have for $n\to\infty$

(2.15)

$$\begin{aligned}\|\hat{f}(-n)E_n(-x)\|_L &\le \sum_{k=n}^{[\mu n]}|\Delta\hat{f}(k)-\Delta\hat{f}(-k)|\log k+\|\tau_{\mu n,n}(f)-S_n(f)\|_L\\ &\quad+O\Big(\max_{[\lambda^{-1}n]\le k\le[\lambda n]}|\hat{f}(k)|\log k\Big)+O\Big(\max_{n<k\le[\mu n]}|\hat{f}(-k)|\Big)\\ &=\sum_{k=n}^{[\mu n]}|\Delta\hat{f}(k)-\Delta\hat{f}(-k)|\log k+o(1).\end{aligned}$$

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On the other hand, we have

$$\|\hat{f}(-n)E_n(-x)\|_L \geq |\hat{f}(-n)|\|D_n(x)\|_L \geq \frac{1}{\pi}|\hat{f}(-n)|\log n. \tag{2.16}$$

Hence, from (2.15), (2.16), and (2.8), we have that

$$|\hat{f}(-n)|\log n \leq C\sum_{k=n}^{[\mu n]}|\Delta\hat{f}(k) - \Delta\hat{f}(-k)|\log k \leq \varepsilon$$

holds for sufficiently large n, which, together with (2.12), completes the proof of necessity. ∎

In view of Lemma 2.3, we can see that condition (2.1) in Theorem 2.2 can be replaced by the following condition

$$\lim_{\mu\to 1^+}\limsup_{n\to\infty}\sum_{k=n}^{[\mu n]}|\Delta\hat{f}(-k)|\log k = 0,$$

and the proof of the result is easier. Therefore we have a corollary to Theorem 2.2.

Corollary 2.7 *Let $f(x) \in L_{2\pi}$ be a complex valued function. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in$ MVBVS and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in$ MVBVS, then*

$$\lim_{n\to\infty}\|f - S_n(f)\|_L = 0 \text{ if and only if } \lim_{n\to\infty}\hat{f}(n)\log|n| = 0.$$

If $f(x)$ is a real-valued function, then its Fourier coefficients $\hat{f}(n)$ and $\hat{f}(-n)$ are a pair of conjugate complex numbers. Consequently, $\{\hat{f}(n)\}_{n=0}^{+\infty} \in$ *MVBVS* if and only if $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in$ *MVBVS*. Thus, we have the following generalization of the classical result (cf. Result Two in the introduction).

Corollary 2.8 *Let $f(x) \in L_{2\pi}$ be a real valued even function and* (1.1) *its Fourier series. If $\mathbf{A} = \{a_n\}_{n=0}^{+\infty} \in$ MVBVS in the real sense,* i.e., *$\{a_n\}$ is a nonnegative sequence, and there is a number $\lambda \geq 2$ such that*

$$\sum_{k=m}^{2m}|\Delta a_k| \leq C(\mathbf{A})\frac{1}{m}\sum_{k=[\lambda^{-1}m]}^{[\lambda m]}a_k$$

for all $n = 1, 2, \ldots$, then

$$\lim_{n\to\infty}\|f - S_n(f)\|_L = 0 \text{ if and only if } \lim_{n\to\infty}a_n\log n = 0.$$

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3 L^1 Approximation

Let $E_n(f)_L$ be the best approximation of a complex valued function $f \in L_{2\pi}$ by trigonometric polynomials of degree n in L^1 norm, that is,

$$E_n(f)_L := \inf_{c_k} \left\| f - \sum_{k=-n}^{n} c_k e^{ikx} \right\|_L .$$

We establish the corresponding L^1-approximation theorem in a similar way to Theorem 2.2.

Theorem 3.1 *Let $f(x) \in L_{2\pi}$ be a complex valued function and $\{\psi_n\}$ a decreasing sequence tending to zero with*

$$\psi_n \sim \psi_{2n}, \tag{3.1}$$

i.e., *there exist positive constants C_1 and C_2, such that $C_1\psi_n \le \psi_{2n} \le C_2\psi_n$. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in MVBVS$ and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$, then*

$$\|f - S_n(f)\|_L = O(\psi_n) \tag{3.2}$$

if and only if

$$E_n(f)_L = O(\psi_n) \text{ and } \hat{f}(n)\log|n| = O(\psi_{|n|}). \tag{3.3}$$

Proof Under the conditions of Theorem 3.1, we see from (2.11) in the proof of Theorem 2.2 that

$$\begin{aligned} \|f - S_n(f)\|_L &\le \|f - \tau_{\mu n,n}(f)\|_L + O\Big(\max_{[\lambda^{-1}n]\le|k|\le[\lambda n]} |\hat{f}(k)|\log|k|\Big) \\ &\le C(\mu)E_n(f) + O\Big(\max_{[\lambda^{-1}n]\le|k|\le[\lambda n]} |\hat{f}(k)|\log|k|\Big), \end{aligned}$$

thus (3.2) holds if (3.1) and (3.3) hold. Now if (3.2) holds, then $E_n(f)_L = O(\psi_n)$ and $\|f-\tau_{\mu n,n}(f)\|_L = O(\psi_n)$. From (2.5)–(2.7) in the proof of Lemma 2.6 and condition (3.1), we have

$$\begin{aligned} |\hat{f}(n)|\log n &\le \frac{C(\lambda)}{n} \sum_{j=1}^{[\lambda n]-[\lambda^{-1}n]+1} \left\|f - S_{[\lambda^{-1}n]+j}(f)\right\|_L + C(\lambda)\|f - S_n(f)\|_L \\ &= O(\psi_n). \end{aligned}$$

Since $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in MVBVS$, we also have $|\hat{f}(-n)|\log n = O(\psi_n)$ by a similar argument to (3). This completes the proof of Theorem 3.1. ∎

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In particular, if we take

$$\psi_n := \frac{1}{(n+1)^r}\omega\Big(f^{(r)}, \frac{1}{n+1}\Big)_L,$$

where r is a positive integer, and $\omega(f,t)_L$ is the modulus of continuity of f in L^1 norm, *i.e.,*

$$\omega(f,t)_L := \max_{0\le h\le t} \|f(x+h) - f(x)\|_L.$$

By Theorem 3.1 and the Jackson theorem (e.g. see [6] or [9]) in L^1-space, we immediately have the following.

Corollary 3.2 *Let $f(x) \in L_{2\pi}$ be a complex valued function. If both $\{\hat{f}(n)\}_{n=0}^{+\infty} \in$ MVBVS and $\{\hat{f}(-n)\}_{n=0}^{+\infty} \in$ MVBVS hold, then*

$$\|f - S_n(f)\|_L = O\Big(\frac{1}{(n+1)^r}\omega\Big(f^{(r)}, \frac{1}{n+1}\Big)_L\Big)$$

if and only if

$$\hat{f}(n)\log|n| = O\Big(\frac{1}{(n+1)^r}\omega\Big(f^{(r)}, \frac{1}{n+1}\Big)_L\Big).$$

This corollary generalizes the corresponding results in [5] and [2].

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https://doi.org/10.4153/CMB-2009-061-6 Published online by Cambridge University Press