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THE CONJUGATE FUNCTION ON THE FINITE DIMENSIONAL TORUS

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ABSTRACT. We consider the group $\mathbf{T}^a$, its group of characters $\mathbf{Z}^a$, and an arbitrary order P on $\mathbf{Z}^a$. For $\chi \in \mathbf{Z}^a$, let $\mathrm{sgn}_P\,\chi$ be 1, -1, or 0 according as $\chi \in P\backslash\{0\}$, $\chi \in (-P)\backslash\{0\}$, or $\chi = 0$. For f in $\mathcal{L}_p(\mathbf{T}^a)$, $1 < p < \infty$, it is known that there is a function $\tilde{f}$ in $\mathcal{L}_p(\mathbf{T}^a)$ such that $\hat{\tilde{f}}(\chi) = -i\,\mathrm{sgn}_P(\chi)\hat{f}(\chi)$ for all χ in $\mathbf{Z}^a$. Summability methods for $\tilde{f}$ are also available. In this paper, we obtain summability methods for $\tilde{f}$ that apply for f in $\mathcal{L}_1(\mathbf{T}^a)$, and we show how various properties of $\tilde{f}$ can be derived from our construction.

1. **Notation.** Throughout the paper, a will denote an arbitrary but fixed positive integer; the symbol $\mathbf{T}$ will denote the circle group parametrized as $\mathbf{T} = \{\exp(it) \in \mathbf{C} : -\pi < t \leqq \pi\}$. The symbol $\mathbf{T}^a$ will denote the product of a copies of $\mathbf{T}$. The character group of $\mathbf{T}^a$ is the group $\mathbf{Z}^a$. The symbol P will denote an arbitrary order on $\mathbf{Z}^a$. That is, P is a subset of $\mathbf{Z}^a$ with the following properties:

$$P \cap (-P) = \{0\};$$
$$P \cup (-P) = \mathbf{Z}^a;$$
$$P + P = P.$$

Haar measure on $\mathbf{T}^a$ will be denoted by μ. Lebesgue measure on the real line $\mathbf{R}$ will be denoted by λ. For a real number $p \geqq 1$, we write $\mathcal{L}_p(\mathbf{T}^a)$ for the space of all complex-valued Haar-measurable functions g on $\mathbf{T}^a$ for which the norm

$$\|g\|_p = \left[\int_{\mathbf{T}^a} |g(t)|^p d\mu(t)\right]^{1/p}$$

is finite. The space $\mathcal{L}_p(\mathbf{R})$ is defined similarly. We denote by $C(\mathbf{T}^a)$ the linear space of continuous functions on $\mathbf{T}^a$, and by $M(\mathbf{T}^a)$ the set of complex-valued regular Borel measures on $\mathbf{T}^a$. Let X be any set, and let A and B be subsets of X. The symbol $A\backslash B$ denotes the subset of X such that $A\backslash B = \{x : x \in A, x \notin B\}$. The symbol 1_A will denote the function defined on X by

$$1_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \in X\backslash A. \end{cases}$$

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2. **Orders on $\mathbf{Z}^a$.** Our construction of $\tilde{f}$ depends on a peculiar property of orders on $\mathbf{Z}^a$ that we derive in this section. The following result about orders on $\mathbf{Z}^a$ will be needed in the sequel; its proof can be found in Asmar and Hewitt [1], Theorem (4.9.i).

THEOREM (2.1.) *Let P be any order on $\mathbf{Z}^a$. There exists a nonzero homomorphism L from $\mathbf{Z}^a$ into $\mathbf{R}$ such that*

(i) $L^{-1}(]0,\infty[) \subsetneqq P \subset L^{-1}([0,\infty[)$.

(2.2) *Definitions and Remarks.* (a) Notions of independent subsets of $\mathbf{Z}^a$, of basis for subgroups of $\mathbf{Z}^a$, and of dimensions of subgroups of $\mathbf{Z}^a$ have the same meanings as in Hewitt and Ross [6], pp. 441–442, (A.10). (b) It is easy to construct proper subgroups of $\mathbf{Z}^a$ with dimension a. For example, in $\mathbf{Z}^2$, the subgroup generated by $x = (2,2)$ and $y = (-2,2)$ clearly has dimension 2, since the equality $mx + ny = 0$ holds for no integers m and n with $m \neq 0$ and $n \neq 0$. However, the following result is an immediate consequence of Hewitt and Ross [6], pp. 450–451, Theorem (A.26). (c) Let B_1 and B_2 be two nonzero subgroups of $\mathbf{Z}^a$ such that $B_1 \subset B_2$ and

(i) whenever $kx \in B_j$ $(j = 1,2)$ for some positive integer k, then $x \in B_j$ $(j = 1,2)$. Then

(ii) there is a basis $\{x_1, x_2, \ldots, x_l, x_{l+1}, \ldots, x_m\}$ for B_2 such that $\{x_1, x_2, \ldots, x_l\}$ is a basis for B_1;

(iii) every basis for B_1 extends to a basis for B_2;

(iv) $\dim B_1 \leqq \dim B_2$

with equality holding only when $B_1 = B_2$. (d) For any real-valued homomorphism L of $\mathbf{Z}^a$, the set $\{x : Lx = 0\}$ has property (c, i).

We now present two simple Lemmas.

LEMMA (2.3.) *Suppose that B is a proper nonzero subgroup of $\mathbf{Z}^a$ such that* (2.2.c.i) *holds. Let r be the dimension of B. Then there is a homomorphism τ of $\mathbf{Z}^a$ onto $\mathbf{Z}^r$ such that τ is an isomorphism of B onto $\mathbf{Z}^r$.*

PROOF. Let $\{x_1, x_2, \ldots, x_r\}$ be a basis for B. Extend this basis to a basis $\{x_1, x_2, \ldots, x_r, \ldots, x_a\}$ of $\mathbf{Z}^a$ (2.2.c.iii). Let $e_1, e_2, \ldots, e_a$ be the standard basis for $\mathbf{Z}^a$. Define τ on $\mathbf{Z}^a$ by: $\tau(x_j) = e_j$ for $j = 1, 2, \ldots, r$; and $\tau(x_j) = 0$ for $j = r+1, \ldots, a$. □

LEMMA (2.4.) *Let B be as in* (2.3). *Suppose that P is an order on $\mathbf{Z}^a$. Then there is a nonzero homomorphism L_B of $\mathbf{Z}^a$ into $\mathbf{R}$ such that*

(i) $\emptyset \subsetneqq B \cap L_B^{-1}(]0,\infty[) \subsetneqq P \cap B \subset L_B^{-1}([0,\infty[)$

PROOF. Let $r(< a)$ be the dimension of B. Let τ be the isomorphism of B onto $\mathbf{Z}^r$ as in (2.3). The set $P' = \tau(P \cap B)$ is clearly an order on $\mathbf{Z}^r$. Apply Theorem (2.1) and obtain a nonzero homomorphism L of $\mathbf{Z}^r$ into $\mathbf{R}$ such that $L^{-1}(]0,\infty[) \subsetneqq P' \subset L^{-1}([0,\infty[)$. Define the homomorphism L_B by $L_B = L \circ \tau$. Clearly, L_B satisfies (i). □

We now prove a property of orders on $\mathbf{Z}^a$ that is crucial for our construction of $\tilde{f}$.

THEOREM (2.5.) *Let P be an order on $\mathbf{Z}^a$. There are a sequence of subgroups $B_0, B_1, \ldots, B_k$ of $\mathbf{Z}^a$, and a sequence of real-valued nonzero homomorphisms $L_1, L_2, \ldots, L_k$ on $\mathbf{Z}^a$ such that*

(i) $\{0\} = B_k \subsetneqq B_{k-1} \subsetneqq \cdots \subsetneqq B_1 \subsetneqq B_0 = \mathbf{Z}^a$;

(ii) $L_j(B_j) = 0$ *for* $j = 1, 2, \ldots, k$;

(iii) $L_j\left(P \cap (B_{j-1} \backslash B_j)\right) > 0$, *and*

(iv) $L_j\left((-P) \cap (B_{j-1} \backslash B_j)\right) < 0$ *for* $j = 1, 2, \ldots, k$.

PROOF. Apply (2.1.i) to obtain L_1 such that $L_1^{-1}(]0, \infty[) \subsetneqq P \subset L_1^{-1}([0, \infty[)$. Let $B_1 = \{x \in \mathbf{Z}^a : L_1(x) = 0\}$. If $B_1 = \{0\}$, then the proof is complete. If $B_1 \neq \{0\}$, then because B_1 has the property (2.2.c.i) we can apply Lemma (2.4) and obtain a homomorphism L_2 such that $\{\phi\} \subsetneqq B_1 \cap L_2^{-1}(]0, \infty[) \subsetneqq P \cap B_1 \subset L_2^{-1}([0, \infty[)$. Let $B_2 = B_1 \cap \{x : L_2(x) = 0\}$. Note that, since $B_1 \neq B_2$, it follows from (2.2.c.iv) that $\dim B_2 < \dim B_1$. Clearly (i)–(iv) hold with L_1, L_2, B_0, B_1, and B_2. If $B_2 = \{0\}$, then the proof is complete. If $B_2 \neq \{0\}$, we proceed inductively as follows. Suppose that $B_1, B_2, \ldots, B_{j-1}, L_1, L_2, \ldots, L_{j-1}$ have been defined and satisfy (i)–(iv). If $B_{j-1} = \{0\}$, then the proof is complete. If $B_{j-1} \neq \{0\}$, apply Lemma (2.4) to obtain L_j such that $\{\phi\} \subsetneqq B_{j-1} \cap L_j^{-1}(]0, \infty[) \subsetneqq P \cap B_{j-1} \subset L_j^{-1}([0, \infty[)$. Since $\dim B_j < \dim B_{j-1} \leqq a$, the process must stop after at most a steps. It is clear that the sequences of subgroups and homomorphisms thus constructed satisfy (i)–(iv). □

3. **Construction of a measure.** In this section, we set forth the background that is needed for our construction of $\tilde{f}$.

(3.1) Suppose that τ is a nonzero homomorphism of $\mathbf{Z}^a$ into $\mathbf{R}$. The adjoint homomorphism of τ, denoted by φ, is the homomorphism of $\mathbf{R}$ into $\mathbf{T}^a$ satisfying the identity

(i) $\chi \circ \varphi(r) = \exp(i\tau(\chi)r)$

for all r in $\mathbf{R}$ and all χ in $\mathbf{Z}^a$. The mapping φ is continuous and its adjoint homomorphism is τ (see Hewitt and Ross [6], p. 287).

Let k denote a function in $\mathcal{L}_1(\mathbf{R})$ with compact support. Let f be a continuous complex-valued function on $\mathbf{T}^a$. The function $f \circ \varphi$ is plainly a continuous function on $\mathbf{R}$. The integral

$$\int_{\mathbf{R}} f \circ \varphi(t) k(t) d\lambda(t)$$

exists and defines a measure ν in $M(\mathbf{T}^a)$ such that the equality

$$\text{(ii)} \qquad \int_{\mathbf{T}^a} f d\nu = \int_{\mathbf{R}} f \circ \varphi(t) k(t) d\lambda(t)$$

holds for all f in $C(\mathbf{T}^a)$.

The next theorem deals with functions of the form $(x, t) \longrightarrow f\left(x - \varphi(t)\right)$ where (x, t) is in $\mathbf{T}^a \times \mathbf{R}$, φ is as above, and f is in $\mathcal{L}_1(\mathbf{T}^a)$. To see that such functions are $\mu \times \lambda$-measurable, apply Lemma (20.6) of Hewitt and Ross [6], p. 287.

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THEOREM (3.2.) *Notation is as in* (3.1). *Let f be in $\mathcal{L}_1(\mathbf{T}^a)$. The equality*

$$\text{(i)} \qquad \nu * f(x) = \int_{\mathbf{R}} f\left(x - \varphi(t)\right) k(t) d\lambda(t)$$

holds for μ-almost all x in $\mathbf{T}^a$.

PROOF. We will show that the functions in (i) have the same Fourier transforms. For χ in $\mathbf{Z}^a$, we have

$$\begin{aligned}\text{(1)} \qquad & \int_{\mathbf{T}^a} \overline{\chi}(x) \int_{\mathbf{R}} f\left(x - \varphi(t)\right) k(t) d\lambda(t) d\mu(x) \\ &= \int_{\mathbf{R}} k(t) \int_{\mathbf{T}^a} f\left(x - \varphi(t)\right) \overline{\chi}(x) d\mu(x) d\lambda(t) \\ &= \int_{\mathbf{R}} k(t) \int_{\mathbf{T}^a} f(x) \overline{\chi}\left(x + \varphi(t)\right) d\mu(x) d\lambda(t) \\ &= \hat{f}(\chi) \int_{\mathbf{R}} k(t) \overline{\chi}\left(\varphi(t)\right) d\lambda(t).\end{aligned}$$

Using (3.1.ii) we see that the last expression in (1) is equal to

$$\begin{aligned}\text{(2)} \qquad \hat{f}(\chi) \int_{\mathbf{T}^a} \overline{\chi}(y) d\nu(y) &= \hat{f}(\chi)\hat{\nu}(\chi) \\ &= (f * \nu)\hat{\ }(\chi).\end{aligned}$$

The relations (1) and (2), and the uniqueness of the Fourier transform show that (i) holds. □

4. **The kernel and its properties.**

(4.1) *Preliminaries.* Throughout this section P will denote an arbitrary order on $\mathbf{Z}^a$. We apply Theorem (2.5) and obtain a sequence of subgroups $(B_j)_{j=0}^k$ of $\mathbf{Z}^a$, and a sequence of real-valued homomorphisms $(L_j)_{j=1}^k$ satisfying (2.5, i–iv). For $j = 1, \ldots, k$, we let N_{j-1} denote the annihilator of B_{j-1} in $\mathbf{T}^a$; $N_{j-1} = \{x \in \mathbf{T}^a : \chi(x) = 1 \forall \chi \in N_{j-1}\}$. Haar-measure on N_{j-1} is denoted by μ_{j-1}. For all χ in $\mathbf{Z}^a$, we have

$$\text{(i)} \qquad \hat{\mu}_{j-1}(\chi) = \begin{cases} 1 & \text{if } \chi \in B_{j-1}, \\ 0 & \text{otherwise.} \end{cases}$$

(This follows at once from Lemma (23.19) of Hewitt and Ross [6], p. 363). Let f be a function in $\mathcal{L}_p(\mathbf{T}^a)$ $1 \leqq p < \infty$. The function $f * \mu_{j-1}$ is in $\mathcal{L}_p(\mathbf{T}^a)$ and satisfies the inequalities

$$\begin{aligned}\text{(ii)} \qquad \|f * \mu_{j-1}\|_p &\leqq \|f\|_p \|\mu_{j-1}\| \\ &= \|f\|_p\end{aligned}$$

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(see Hewitt and Ross [6], Theorem (20.12), p. 292). For $j = 1, \ldots, k$, let φ_j denote the adjoint homomorphism of L_j. In (3.1) and (3.2), take:

$$k(t) = \pi^{-1} 1_{[-n,-(1/n)]\cup[(1/n),n]}(t)t^{-1},$$

and $\varphi = \varphi_j$. Denote by $\nu_{n,j}$ the measure corresponding to φ_j and k in (3.1.ii). We have from (3.2.i)

$$\text{(iii)} \qquad f * \mu_{j-1} * \nu_{n,j}(\chi) = \pi^{-1} \int_{(1/n)\leqq|t|\leqq n} f * \mu_{j-1}\left(x - \varphi_j(t)\right) t^{-1} d\lambda(t)$$

for μ-almost all x in $\mathbf{T}^a$ and all f in $\mathcal{L}_1(\mathbf{T}^a)$. Consider the one-parameter group of transformations U_j^t acting on $\mathbf{T}^a$ by translation by $\varphi_j(t)$. That is, for $x \in \mathbf{T}^a$, $U_j^t(x) = x + \varphi_j(t)$. Apply Theorem 1 of Calderón [3] and use the well-known properties of the Hilbert transform on $\mathbf{R}$ to obtain the following theorem. (See the remarks concerning the Hilbert transform following the statements of Theorem 1 and Theorem 2 of Calderón [3].)

THEOREM (4.2.) *For $j = 1, \ldots, k$ and for every f in $\mathcal{L}_p(\mathbf{T}^a)$, $1 < p < \infty$, the inequality*

$$(i) \qquad \|f * \mu_{j-1} * \nu_{n,j}\| \leqq A_p \|f * \mu_{j-1}\|_p$$

obtains, where A_p depends only on p. For f in $\mathcal{L}_p(\mathbf{T}^a)$ $(1 \leqq p < \infty)$ there is a function $H^j f$ such that

$$(ii) \qquad \lim_{n\to\infty} f * \mu_{j-1} * \nu_{n,j}(x) = H^j f(x)$$

for μ-almost all x in $\mathbf{T}^a$. For $1 < p < \infty$, the functions $f * \mu_{j-1} * \nu_{n,j}$ converge to $H^j f$ in the $\mathcal{L}_p(\mathbf{T}^a)$-norm; we have

$$(iii) \qquad \|H^j f\|_p \leqq A_p \|f * \mu_{j-1}\|_p.$$

For $1 \leqq p < \infty$, and for f in $\mathcal{L}_p(\mathbf{T}^a)$ we have

$$(iv) \qquad \mu(\{x : |H^j f(x)| > y\}) \leqq \frac{B_p^p}{y^p} \|f\|_p^p$$

where B_p is a constant depending only on p.

Definitions. (4.3.) *For f in $\mathcal{L}_1(\mathbf{T}^a)$ we let*

$$(i) \qquad H_n f = \sum_{j=1}^{k} f * \mu_{j-1} * \nu_{n,j};$$

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and

(*ii*) $$Hf = \sum_{j=1}^{k} H^j f$$

where $H^j f$ is the function given by (4.2.ii).

The following theorem is an immediate consequence of (4.2.iii) and (4.2.iv).

THEOREM (4.4.) *For every f in $\mathcal{L}_p(\mathbf{T}^a)$ $(1 < p < \infty)$, we have*

(*i*) $$\|Hf\|_p \leqq aA_p\|f\|_p$$

where A_p is as in (4.2.iii). *For $1 \leqq p < \infty$ and f in $\mathcal{L}_p(\mathbf{T}^a)$, we have*

(*ii*) $$\mu(\{x : |Hf(x)| > y\}) \leqq \frac{C_p}{y^p}\|f\|_p^p$$

where C_p depends only on p.

PROOF. From (4.3.ii), (4.2.iii) and (4.1.ii) we have

$$\begin{aligned}\|Hf\|_p &\leqq \sum_{j=1}^{k}\|H^j f\|_p \\ &\leqq kA_p\|f\|_p \\ &\leqq aA_p\|f\|_p.\end{aligned}$$

To prove (ii) we use (4.3.ii), (4.2.iv) and note that

$$\begin{aligned}\{x : |Hf(x)| > y\} &= \left\{x : \left|\sum_{j=1}^{k} H^j f(x)\right| > y\right\} \\ &\subseteq \left\{x : \sum_{j=1}^{k}|H^j f(x)| > y\right\} \\ &\subset \bigcup_{j=1}^{k}\left\{x : |H^j f(x)| > \frac{y}{k}\right\};\end{aligned}$$

hence

$$\begin{aligned}\mu(\{x : |Hf(x)| > y\}) &\leqq k\frac{k^p}{y^p}B_p^p\|f\|_p \\ &\leqq a^{p+1}B_p^p\frac{1}{y^p}\|f\|_p.\end{aligned}$$ □

It remains to show that Hf is *the* conjugate function of f. We do this by showing that for f in $\mathcal{L}_p(\mathbf{T}^a)$ $1 < p < \infty$, the equality $(Hf)\hat{}(\chi) = -i\,\mathrm{sgn}_P(\chi)\hat{f}(\chi)$ holds for all χ in $\mathbf{Z}^a$.

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LEMMA (4.5.) *For f in* $\mathcal{L}_p(\mathbf{T}^a)$ $(1 < p < \infty)$ *and all* χ *in* $\mathbf{Z}^a$, *we have*

$$(i) \qquad (H^j f)\hat{}(\chi) = \begin{cases} -i \operatorname{sgn}\left(L_j(\chi)\right) \hat{f}(\chi) & \textit{if } \chi \in B_{j-1}; \\ 0 & \textit{if } \chi \in \mathbf{Z}^a \backslash B_{j-1}. \end{cases}$$

PROOF. Because H^j is a bounded linear operator on $\mathcal{L}_p(\mathbf{T}^a)$ and because the trigonometric polynominals are dense in $\mathcal{L}_p(\mathbf{T}^a)$, it is enough to prove (i) for the characters of $\mathbf{T}^a$. Thus, we need to show that

$$(1) \qquad H^j \chi = -i \operatorname{sgn} L_j(\chi)(\chi)$$

if χ is in B_{j-1}, and

$$(2) \qquad H^j \chi = 0$$

if χ is in $\mathbf{Z}^a \backslash B_{j-1}$. We use the definition (4.2.ii) of $H^j \chi$ and (4.1.iii) and write

$$(3) \qquad \begin{aligned} H^j \chi(x) &= \lim_{n \to \infty} \chi * \mu_{j-1} * \nu_{n,j}(x) \\ &= \lim_{n \to \infty} \pi^{-1} \int_{(1/n) \leqq |t| \leqq n} \chi * \mu_{j-1} \left(x - \varphi_j(t)\right) t^{-1} d\lambda(t). \end{aligned}$$

Note that

$$(4) \qquad \begin{aligned} \chi * \mu_{j-1}(x) &= \int_{\mathbf{T}^a} \chi(x-y) d\mu_{j-1}(y) \\ &= \chi(x) \int_{N_{j-1}} \overline{\chi}(y) d\mu_{j-1}(y) \\ &= \begin{cases} \chi(x) & \text{if } \chi \in B_{j-1}, \\ 0 & \text{if } \chi \in \mathbf{Z}^a \backslash B_{j-1}. \end{cases} \end{aligned}$$

The relations (3) and (4) show that (2) holds. For $\chi \in B_{j-1}$, we use (3), (4) and (3.1.i):

$$\begin{aligned} H^j \chi(x) &= \lim_{n \to \infty} \pi^{-1} \int_{(1/n) \leqq |t| \leqq n} \chi\left(x - \varphi_j(t)\right) t^{-1} d\lambda(t) \\ &= \chi(x) \lim_{n \to \infty} \pi^{-1} \int_{(1/n) \leqq |t| \leqq n} \exp\left(-i L_j(\chi) t\right) t^{-1} d\lambda(t) \\ &= \chi(x) \lim_{n \to \infty} \pi^{-1} \int_{(1/n) \leqq |t| < n} (-i) \sin(L_j(\chi) t) t^{-1} d\lambda(t) \\ &= -i \operatorname{sgn}\left(L_j(\chi)\right) \chi(x). \end{aligned}$$

□

THEOREM (4.6.) *For f in* $\mathcal{L}_p(\mathbf{T}^a)$ $(1 < p < \infty)$ *and all* χ *in* $\mathbf{Z}^a$, *we have* $(Hf)\hat{}(\chi) = -i \operatorname{sgn}_P(\chi) \hat{f}(\chi)$.

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PROOF. From (4.3.ii) and (4.5.i), we have

$$(1)\qquad (Hf)\hat{}(\chi) = \sum_{j=1}^{k} (H^j f)\hat{}(\chi) = \sum_{j=1}^{k} -i \operatorname{sgn}\left(L_j(\chi)\right) 1_{B_{j-1}}(\chi)\hat{f}(\chi).$$

From Theorem (2.5), it is clear that there is exactly one integer j_0 in $\{1, \ldots, k\}$ such that

$$\chi \in B_{j_0-1} \backslash B_{j_0},$$

$$\operatorname{sgn}\left(L_{j_0}(\chi)\right) = \operatorname{sgn}_P(\chi),$$

and $L_{j_0}(B_{j_0}) = 0$. For $1 \leqq j_0 < j \leqq k$ we have $L_j(\chi) = 0$; and for $0 \leqq j < j_0$ we have $1_{B_{j-1}}(\chi) = 0$. Putting this in (1) we find that

$$(Hf)\hat{}(\chi) = -i \operatorname{sgn}\left(L_{j_0}(\chi)\right) 1_{B_{j_0-1}}(\chi)\hat{f}(\chi) = -i \operatorname{sgn}_P(\chi)\hat{f}(\chi). \qquad \square$$

REMARKS (4.7.) The questions of summability of $\tilde{f}$ were raised in the early studies of the conjugate function on groups other than **T**. (See Helson [4]). It wasn't until 1983 that the first positive result in this direction was published by Hewitt and Ritter [5]. They succeeded in constructing $\tilde{f}$ explicitly from f, where f is in $\mathcal{L} \log^+ \mathcal{L}(G)$, and G is the character group of any noncyclic subgroup of the additive group of the rationals **Q**. It is still not known whether summability methods for $\tilde{f}, f \in \mathcal{L}_1(G)$ exist on an arbitrary locally compact Abelian group G with ordered dual group. Asmar and Hewitt [1] succeeded in constructing such summability methods for f in $\mathcal{L}_P(G)$ $1 < p < \infty$, and G any locally compact Abelian group with ordered dual group. Because of the complexity of the structure of Haar-measurable orders, and because of the obvious dependence of the construction of $\tilde{f}$ on the order, it seems unlikely that a general summability method applies on all locally compact Abelian groups. Also we point out that, while the general methods of Asmar and Hewitt involve several iterated limits, the construction of the present paper involves one single limit! Sharper estimates for the $\mathcal{L}_P$-norm $(1 < p < \infty)$ of the conjugate function operator have been obtained by Berkson and Gillespie [2] for compact connected Abelian groups , and by Asmar and Hewitt [1] for locally compact Abelian groups. Contrary to what the inequality (4.4.i) may suggest, the $\mathcal{L}_P$-norm of the conjugate function operator does not depend on the dimension a.

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