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ALMOST POLAR-DENSE LATTICES

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ABSTRACT—We introduce almost polar-dense lattices and prove that the generalized interval topology of an almost polar-dense, modular lattice is equivalent to its interval topology. Furthermore, for totally ordered sets, the converse holds: if the generalized interval topology is the interval topology, then the set is almost polar-dense.

1. **Introduction.** We suggested in [6] that, instead of intervals in an arbitrary partially ordered set, one should consider generalized intervals (defined below). Replacing intervals by generalized intervals in the definition of the interval topology [4] proved to give a topology which was preserved by cardinal products of dually directed sets. However, this generalized interval topology did not necessarily contain the usual interval topology. By adjusting the definition of the generalized interval topology, we defined the generalized star-interval topology, which, for dually directed sets, was precisely the topology generated by the generalized interval topology and the interval topology.

In [6], we specified a condition (having trivial polars) which implied that the generalized interval and star-interval topologies were equivalent for dually directed sets. In this note, we define a condition (being almost polar-dense) which is weaker than the previous condition and which implies the equivalence of the two topologies on modular lattices. Furthermore, a totally ordered set satisfies this condition if and only if the two topologies are equivalent.

Terminology left undefined may be found in [1], [2], or [8]. If $\{P_\alpha \mid \alpha \in A\}$ is a collection of partially ordered sets, then we denote the *cardinal product* of the P_α by $|\Pi|\{P_\alpha \mid \alpha \in A\}$; thus, $|\Pi|\{P_\alpha \mid \alpha \in A\}$ is the product of the P_α ordered pointwise, i.e. by: $f \leq g$ if and only if $\alpha f \leq \alpha g$ for all $\alpha \in A$. If A is finite, say $A=\{1, 2, \ldots, n\}$, then we denote the cardinal product by $P_1 |\times| P_2 |\times| \cdots |\times| P_n$. We use N to denote the natural numbers, Z the integers, and R the real numbers. We let N, Z, and R have their usual order, and we let $R^+=\{r \in R \mid r \geq 0\}$, $Z^+=\{i \in Z \mid i \geq 0\}$, and $Z^-=\{i \in Z \mid i \leq 0\}$.

Generalized intervals were designed to incorporate into the idea of "interval" the "relatively perpendicular" elements which may exist in non-totally ordered sets. Our method of achieving this depends on the idea of polar, which is borrowed

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from the theory of l-groups (see [3], [7]), where generalized intervals were originally defined [5].

Let $(L, \leq)$ be a lattice. Let $r, s, t \in L$ be such that $r \leq s \leq t$. The *upper polar of* t *with respect to* s is the set

$$(s, t)^{\perp} = \{l \in L \mid l \wedge t = s\};$$

the *lower polar of* r *with respect to* s is the set

$$(s, r)_{\top} = \{l \in L \mid l \vee r = s\}.$$

Star-polars are defined as follows: $*(s, t)^{\perp} = (s, t)^{\perp}$ if $s < t$, and $*(s, t)^{\perp} = \{s\}$ if $s = t$; $*(r, s)_{\top} = (r, s)_{\top}$ if $r < s$, and $*(r, s)_{\top} = \{s\}$ if $r = s$.

A *generalized initial segment* of L is a set of the form

$$(-\infty, s, t] = \{x \in L \mid x \leq t \vee b \text{ for some } b \in (s, t)^{\perp}\},$$

where $s, t \in L$ are such that $s \leq t$. A *generalized final segment* of L is a set of the form

$$[r, s, \infty) = \{x \in L \mid x \geq r \wedge b \text{ for some } b \in (s, r)_{\top}\},$$

where $r, s \in L$ are such that $r \leq s$. *Generalized final* and *initial star-segments* are defined analogously, with the polars replaced by the corresponding star-polars.

The *generalized interval topology* (or *gi-topology*) on L, denoted by $\mathscr{G}(L)$, takes as a subbase for its closed sets, L, ϕ, and all the generalized final and initial segments. Similarly, the *generalized star-interval topology* (or *gi-*topology*) on L, denoted by $\mathscr{G}^*(L)$, takes as a subbase for its closed sets, L, ϕ, and all the generalized final and initial star-segments. Recall that the *interval topology* [4] on L, denoted by $\mathscr{I}(L)$, takes as a subbase for its closed sets, L, ϕ, and all *initial* and *final segments*, i.e., all sets of the forms

$$(-\infty, l] = \{x \in L \mid x \leq l\},$$
$$[l, \infty) = \{x \in L \mid x \geq l\},$$

for all $l \in L$.

In [6], we proved that $\mathscr{G}(L)$ and $\mathscr{G}^*(L)$ are intrinsic topologies, and that $\mathscr{G}^*(L) \supseteq \mathscr{I}(L)$.

Intervals, of course, are sets of the form $[r, t] = [r, \infty) \cap (-\infty, t]$. Analogously, we may define *generalized intervals* and *generalized star-intervals* to be sets of the forms

$$[r, s, t] = [r, s, \infty) \cap (-\infty, s, t],$$
$$*[r, s, t] = *[r, s, \infty) \cap *(-\infty, s, t],$$

respectively. If we consider all segments to be intervals as well as the sets defined above, then $\mathscr{I}(L)$ is the coarsest topology whose closed sets contain the intervals. We may characterize $\mathscr{G}(L)$ and $\mathscr{G}^*(L)$ similarly.

We will use only the first part of the following result; we include the second part for its intrinsic interest, i.e., to show how well-behaved distributive lattices are with regard to generalized segments.

PROPOSITION 1.1. *Let* $(L, \leq)$ *be a lattice. Let* $r, t \in L$ *be such that* $r \leq t$. *If* L *is modular, then*

(i) *for all* $x \in (-\infty, r, t] \cap [t, \infty)$, *there exists* $b \in (r, t)^{\perp}$ *such that* $x = t \vee b$.

If L *is distributive, then*

(ii) *for all* $x \in (-\infty, r, t] \cap [r, \infty)$, *there exists* $b \in (r, t)^{\perp}$ *such that* $x = (x \wedge t) \vee b$.

Proof. (i) Since $x \in (-\infty, r, t]$, there exists $d \in (r, t)^{\perp}$ such that $x \leq t \vee d$. Let $b = x \wedge d$. Since $x \in [t, \infty)$, $r \leq x$, and hence

$$r = d \wedge t \geq b \wedge t = x \wedge d \wedge t \geq r,$$

i.e. $b \in (r, t)^{\perp}$. Furthermore, since L is modular, since $t \leq x$, and since $b \leq x$, we have that

$$\begin{aligned} t \vee b &= (x \wedge t) \vee b \\ &= x \wedge (t \vee b) \\ &= x \wedge (t \vee (d \wedge x)) \\ &= x \wedge (t \vee d) \wedge x \\ &= x. \end{aligned}$$

(ii) As in (i), we may find $b \in (r, t)^{\perp}$ such that $b = x \wedge d$ for some $d \in (r, t)^{\perp}$ with $x \leq t \vee d$. Since L is distributive,

$$(x \wedge t) \vee b = (x \wedge t) \vee (x \wedge d) = x \wedge (t \vee d) = x.$$

2. **Almost polar-dense lattices.** A lattice $(L, \leq)$ is said to *have trivial polars* [6] if for all $l \in L$, there exist $r, t \in L$ such that $r < l < t$, $(r, l)^{\perp} = \{r\}$, and $(t, l)_{\top} = \{t\}$. We proved in [6] that if $(L, \leq)$ is a lattice which has trivial polars, then $\mathscr{G}(L) = \mathscr{G}^*(L)$.

However, it is not difficult to see (cf. the proof of Proposition 2.5) that, since R^+ is dense-in-itself and totally ordered, $\mathscr{G}(R^+) = \mathscr{I}(R^+) = \mathscr{G}^*(R^+)$, and it is clear that R^+ does not have trivial polars. Thus, in view of Proposition 1.1, one might consider polar-dense lattices defined as follows: a lattice $(L, \leq)$ is *polar-dense* if for all $x, y \in L$ with $x < y$, there exists $d \in L$ such that $x < d < y$ and $y \notin d \vee (x, d)^{\perp}$. However, a polar-dense lattice must clearly be dense-in-itself, and thus Z is not polar-dense. But clearly, Z has trivial polars, and hence $\mathscr{G}(Z) = \mathscr{G}^*(Z)$.

Therefore, we are lead to weaken the notion of polar-density as follows: A lattice $(L, \leq)$ is *almost polar-dense* if, whenever $x, y \in L$ are such that $x < y$ and for all $x < d < y$, $y \in d \vee (x, d)^{\perp}$, there exist $c, e \in L$ such that $c < x < y < e$, $y \notin x \vee (c, x)^{\perp}$, and $x \notin y \wedge (e, y)_{\top}$.

(We note that clearly, replacing polars by star-polars in the above definitions gives equivalent conditions.)

We first show that we have in fact generalized lattices with trivial polars.

PROPOSITION 2.1. *Let L be a lattice. If L has trivial polars, then L is almost polar-dense.*

Proof. Let $x<y$, and suppose that for all $x<d<y$, $y\in d \vee (x, d)^{\perp}$. Since L has trivial polars, there exist $c<x<y<d$ such that $(c, s)^{\perp}=\{c\}$ and $(d, y)_{\top}=\{d\}$. Then

$$y \notin \{x\} = x \vee (c, x)^{\perp},$$
$$x \notin \{y\} = y \wedge (d, y)_{\top}.$$

Thus, L is almost polar-dense.

Our next result shows that being polar-dense and being almost polar-dense are self-dual properties, and indicates several other, equivalent ways to define these two conditions.

PROPOSITION 2.2. *Let $(L, \leq)$ be a lattice and suppose that $x, y, z \in L$ are such that $x<y<z$. Then the following two statements are equivalent:*

(i) $z \in y \vee (x, y)^{\perp}$;
(ii) $x \in y \wedge (z, y)_{\top}$.

Proof. Clearly, (i) and (ii) are both equivalent to: (iii) there exists $b \in L$ such that $z=y \vee b$ and $x=y \wedge b$.

The following examples show that the distinctions made in the above definitions are not trivial.

EXAMPLE 2.3. We give an example of an almost polar-dense lattice with elements x, y such that $x<y$ and for all $d<x$, $y \in x \vee (d, x)^{\perp}$.

Let $L=R^{+} \,|\times|\, R$. In L, $(0, 0)<(1, 0)$, and if $(a, b)<(0, 0)$, then $a=0$ and $b<0$. Thus

$$(a, b) = (0, b) = (1, b) \wedge (0, 0),$$

i.e. $(1, b) \in ((a, b), (0, 0))^{\perp}$. Since $(1, 0)=(1, b) \vee (0, 0)$, $(1, 0) \in (0, 0) \vee ((a, b), (0, 0))^{\perp}$. Clearly, however, L is almost polar-dense.

EXAMPLE 2.4. We give an example of an almost polar-dense lattice with elements x, y such that $x<y$, the cardinality of $[x, y]$ is infinite, and for all $x<e<y$, $y \in e \vee (x, e)^{\perp}$.

For each $n \in N$, let $P_n=Z$. Let $L=|\Pi| \{P_n \mid n \in N\}$. Let $x, y \in L$ be the constant functions $nx=1$ and $ny=2$ for all $n \in N$. Then $x<y$. Suppose that $x<e<y$. Let $e' \in L$ be defined by

$$ne' = \begin{cases} 1 & \text{if } ne = 2 \\ 2 & \text{if } ne = 1. \end{cases}$$

Clearly, $e \wedge e'=x$ and $e \vee e'=y$, and hence $y \in e \vee (x, e)^{\perp}$. Clearly, the cardinality of $[x, y]$ is infinite, and clearly, L is almost polar-dense.

We now show that, at least for modular lattices, being almost polar-dense retains sufficient power from having trivial polars to force the *gi*-topology to be equivalent to the *gi*-*topology.

PROPOSITION 2.5. *Let $(L, \leq)$ be an almost polar-dense, modular lattice. Then $\mathscr{G}(L)=\mathscr{G}^*(L)$.*

Proof. Let $x \in L$. Clearly it suffices to show that

$$(1) \qquad [x, \infty) = (\bigcap \{[x, r, \infty) \mid x < r\}) \cap (\bigcap \{[s, x, \infty) \mid s < x\}),$$

$$(2) \qquad (-\infty, x] = (\bigcap \{(-\infty, r, x] \mid r < x\}) \cap (\bigcap \{(-\infty, x, s] \mid x < s\}).$$

It is easy to see (e.g. [6; Proposition 3.2]) that

$$(-\infty, x] \subseteq (\bigcap \{(-\infty, r, x] \mid r < x\}) \cap (\bigcap \{(-\infty, x, s] \mid x < s\}).$$

Conversely, suppose that $y \notin (-\infty, x]$. Then $y \vee x > x$. Suppose that there exists $x<d<y \vee x$ such that $y \vee x \notin d \vee (x, d)^{\perp}$. Since L is modular, then $y \vee x \notin (-\infty, x, d]$ by Proposition 1.1 (i). Clearly, if $y \in (-\infty, x, d]$, then $y \vee x \in (-\infty, x, d]$, and hence $y \notin (-\infty, x, d]$. Suppose that no such d exists. Then, since L is almost polar-dense, there exists $c \in L$ such that $c<x<y \vee x$ and $y \vee x \notin x \vee (c, x)^{\perp}$. Using Proposition 1.1(i) again, we can see that $y \vee x \notin (-\infty, c, x]$, and hence that $y \notin (-\infty, c, x]$. Therefore,

$$(-\infty, x] \supseteq (\bigcap \{(-\infty, r, x,] \mid r < x\}) \cap (\bigcap \{(-\infty, x, s] \mid x < s\}),$$

and hence (2) holds. Similarly, (1) holds.

Since, as we noted above, R^+ is polar-dense but does not have trivial polars and Z has trivial polars but is not polar-dense, neither of these conditions is necessary for the equivalence of the generalized interval topology and the interval topology on totally ordered sets. However, as we show next, a totally ordered set whose interval and generalized interval topologies are equivalent must be almost polar-dense.

PROPOSITION 2.6. *Let T be a totally ordered set. Then the following are equivalent:*

(i) $\mathscr{G}(T) = \mathscr{G}^*(T)$;

(ii) $\mathscr{G}(T) = \mathscr{I}(T)$;

(iii) T is almost polar-dense.

Proof. Since T is totally ordered, $\mathscr{G}^*(T)=\mathscr{I}(T)$ by [6; Corollary 3.4]. Thus (i) is equivalent to (ii). By Proposition 2.5, (iii) implies (i). Thus, it suffices to show that (ii) implies (iii).

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Hence, suppose that $\mathscr{G}(T)=\mathscr{I}(T)$, and let $x, y \in T$ be such that $x<y$. Clearly, if there exists $d \in T$ with $x<d<y$, then

$$y \notin \{d\} = d \vee (x, d)^{\perp}.$$

Thus, suppose that $[x, y]=\{x, y\}$. Suppose further that $(-\infty, x]=\{x\}$. Let $s, t \in T$ be such that $s \leq t$. If $s=t$, then

$$y \in T = [s, t, \infty) = (-\infty, s, t].$$

Suppose that $s \neq t$. Since T is a lattice, and since $(-\infty, x]=\{x\}$, then $x \leq s < t$. Thus, since T is totally ordered, and since $[x, y]=\{x, y\}$, $y \leq t$. Hence $y \in (-\infty, s, t]$. If $x \in [s, t, \infty)$, then clearly $y \in [s, t, \infty)$. We conclude that y is a member of the closure of $\{x\}$ with respect to $\mathscr{G}(T)$, and hence that $\mathscr{G}(T)$ is not T_1. Since $\mathscr{I}(T)$ is T_1, this contradicts our hypothesis that $\mathscr{G}(T)=\mathscr{I}(T)$. Therefore, $(-\infty, x] \neq \{x\}$, i.e., there exists $c \in T$ such that $c<x$. Then clearly

$$y \notin \{x\} = x \vee (c, s)^{\perp}.$$

Similarly, there exists $d \in T$ with $y<d$ such that

$$x \notin \{y\} = y \wedge (d, y)_{\top}.$$

Therefore, T is almost polar-dense.

We conclude this note by showing how to construct many natural examples of almost polar-dense lattices.

Let $\{L_\gamma \mid \gamma \in \Gamma\}$ be a collection of lattices. Let $\lambda \in \Gamma$, $l \in L_\lambda$, and $f \in |\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$. Define $l^f \in |\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$ by

$$\gamma l^f = \begin{cases} l & \text{if } \gamma = \lambda \\ \gamma f & \text{otherwise.} \end{cases}$$

PROPOSITION 2.7. *Let $\{L_\gamma \mid \gamma \in \Gamma\}$ be a collection of lattices. If at least one L_γ is almost polar-dense, then $|\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$ is almost polar-dense.*

Proof. (A) Suppose first that $f, g \in |\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$ are such that $f<g$ and for some $\lambda \in \Gamma$ there exists $l \in L_\lambda$ such that $\lambda f < l < \lambda g$ and $\lambda g \notin l \vee (\lambda f, l)^{\perp}$. Then $f < l^f < g$. If $x \in (f, l^f)^{\perp}$, then $\lambda x \in (\lambda f, l)^{\perp}$, and hence

$$\lambda g \neq l \vee (\lambda x) = (\lambda l^f) \vee (\lambda x) = \lambda(l^f \vee x),$$

i.e., $g \neq l^f \vee x$. Thus $g \notin l^f \vee (f, l^f)^{\perp}$.

(B) Now suppose that $f, g \in |\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$ are such that $f<g$ and for all $f<h<g$, $g \in h \vee (f, h)^{\perp}$. Then by (A), for all $\gamma \in \Gamma$, for all $\gamma f < l < \gamma g$, $\gamma g \in l \vee (\gamma f, l)^{\perp}$. Let $\lambda \in \Gamma$ be such that L_λ is almost polar-dense. Then there exist $c, e \in L_\lambda$ such that $c < \lambda f < \lambda g < e$, $\lambda g \notin \lambda f \vee (c, \lambda f)^{\perp}$, and $\lambda f \notin \lambda g \wedge (e, \lambda g)_{\top}$. Clearly, $c^f < f < g < e^g$, and by (A) and its dual, $g \notin f \vee (c^f, f)^{\perp}$ and $f \notin g \wedge (e^g, g)_{\top}$.

Therefore, $|\Pi| \{L_\gamma \mid \gamma \in \Gamma\}$ is almost polar-dense.

A partially ordered set $(P, \leq)$ is *unbounded* if for all $p \in P$, there exist $r, t \in P$ such that $r<p<t$.

COROLLARY 2.8. *Let* $\{T_\gamma \mid \gamma \in \Gamma\}$ *be a collection of totally ordered sets. If at least one* T_γ *is unbounded, then* $|\Pi| \{T_\gamma \mid \gamma \in \Gamma\}$ *is an almost polar-dense, distributive lattice.*

Proof. Since at least one T_γ is unbounded, at least one T_γ has trivial polars. Thus, by Proposition 2.5, at least one T_γ is almost polar-dense. The result follows from Proposition 2.7.

We note that neither Z^+ nor Z^- is almost polar-dense. However, $Z^+ |\times| Z^-$ is easily seen to be almost polar-dense, and thus the converses of Proposition 2.7 and Corollary 2.8 fail to hold. Furthermore, $Z^+ |\times| Z^+$ is an example of a cardinal product that is not almost polar-dense.

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