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PERIODIC MINIMAL SURFACES IN SEMIDIRECT PRODUCTS

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Abstract

In this paper we prove the existence of complete minimal surfaces in some metric semidirect products. These surfaces are similar to the doubly and singly periodic Scherk minimal surfaces in $\mathbb{R}^3$. In particular, we obtain these surfaces in the Heisenberg space with its canonical metric, and in Sol_3 with a one-parameter family of nonisometric metrics.

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1. Introduction

In this paper we construct examples of periodic minimal surfaces in some semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, depending on the matrix A. By 'periodic surface' we mean a properly embedded surface invariant with respect to a nontrivial discrete group of isometries.

One of the simplest examples of a semidirect product is $\mathbb{H}^2 \times \mathbb{R} = \mathbb{R}^2 \rtimes_A \mathbb{R}$, when we take $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In this space, Mazet *et al.* [2] proved some results about periodic constant mean curvature surfaces and constructed examples of such surfaces. One of their methods is to solve a Plateau problem for a certain contour. In [5], using a similar technique, Rosenberg constructed examples of complete minimal surfaces in $M^2 \times \mathbb{R}$, where M is either the 2-sphere or a complete Riemannian surface with nonnegative curvature or the hyperbolic plane.

Meeks *et al.* [3] have proved results concerning the geometry of solutions to Plateau type problems in metric semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution (see Theorem 2.5).

The first example that we construct is a complete periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$. It is invariant with respect to two

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translations that commute and is a four-punctured sphere in the quotient of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by the group of isometries generated by the two translations. In the final section we obtain a complete periodic minimal surface analogous to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.

These surfaces are obtained by solving the Plateau problem for a geodesic polygonal contour Γ (it uses a result by Meeks *et al.* [3] about the geometry of solutions to the Plateau problem in semidirect products), and letting some sides of Γ tend to infinity in length, so that the associated Plateau solutions all pass through a fixed compact region (this will be assured by the existence of minimal annuli playing the role of barriers). Then a subsequence of the Plateau solutions will converge to a minimal surface bounded by a geodesic polygon with edges of infinite length. We complete this surface by symmetry across the edges. The whole construction requires precise geometric control and uses curvature estimates for stable minimal surfaces.

These results are obtained for semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. For example, we obtain periodic minimal surfaces in the Heisenberg space, when $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and in Sol_3, when $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with their well-known Riemannian metrics. When we consider the one-parameter family of matrices $A(c) = \left(\begin{smallmatrix} 0 & c \\ 1/c & 0 \end{smallmatrix}\right)$, $c \geq 1$, we get a one-parameter family of metrics in Sol_3 which are not isometric.

2. Preliminary results

Generalizing direct products, a semidirect product is a particular way in which a group can be constructed from two subgroups, one of which is a normal subgroup. As a set, it is the cartesian product of the two subgroups but with a particular multiplication operation.

In our case, the normal subgroup is $\mathbb{R}^2$ and the other subgroup is $\mathbb{R}$. Given a matrix $A \in \mathcal{M}_2(\mathbb{R})$, we can consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where the group operation is given by

$$(p_1, z_1) * (p_2, z_2) = (p_1 + e^{z_1 A} p_2, z_1 + z_2), \quad p_1, p_2 \in \mathbb{R}^2, \quad z_1, z_2 \in \mathbb{R}, \tag{2.1}$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

We choose coordinates $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$. Then $\partial_x = \partial/\partial x, \partial_y, \partial_z$ is a parallelization of $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$. Taking derivatives at $t = 0$ in (2.1) of the left multiplication by $(t, 0, 0) \in G$ (respectively by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{F_1, F_2, F_3\}$ of the right invariant vector fields on G:

$$F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3 = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z.$$

Analogously, if we take derivatives at $t = 0$ in (2.1) of the right multiplication by $(t, 0, 0) \in G$ (respectively by $(0, t, 0), (0, 0, t)$), we obtain the following basis $\{E_1, E_2, E_3\}$ of the Lie algebra of G:

$$E_1 = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2 = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z,$$

where we have denoted

$$e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

We define the *canonical left invariant metric* on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, denoted by $\langle , \rangle$, to be that one for which the left invariant basis $\{E_1, E_2, E_3\}$ is orthonormal.

The Riemannian connection ∇ for the canonical left invariant metric of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ in this frame is expressed as:

$$\begin{aligned}
&\nabla_{E_1} E_1 = aE_3, \quad \nabla_{E_1} E_2 = \frac{b+c}{2} E_3, \quad \nabla_{E_1} E_3 = -aE_1 - \frac{b+c}{2} E_2, \\
&\nabla_{E_2} E_1 = \frac{b+c}{2} E_3, \quad \nabla_{E_2} E_2 = dE_3, \quad \nabla_{E_2} E_3 = -\frac{b+c}{2} E_1 - dE_2, \\
&\nabla_{E_3} E_1 = \frac{c-b}{2} E_2, \quad \nabla_{E_3} E_2 = \frac{b-c}{2} E_1, \quad \nabla_{E_3} E_3 = 0.
\end{aligned}$$

In particular, for every $(x_0, y_0) \in \mathbb{R}^2$, $\gamma(z) = (x_0, y_0, z)$ is a geodesic in G.

Remark 2.1. Since $[E_1, E_2] = 0$, we have for all z that $\mathbb{R}^2 \rtimes_A \{z\}$ is flat and the horizontal straight lines are geodesics. Moreover, the mean curvature of $\mathbb{R}^2 \rtimes_A \{z\}$ with respect to the unit normal vector field E_3 is the constant $H = \mathrm{tr}(A)/2$.

The change from the orthonormal basis $\{E_1, E_2, E_3\}$ to the basis $\{\partial_x, \partial_y, \partial_z\}$ produces the following expression for the metric $\langle , \rangle$:

$$\begin{aligned}
\langle , \rangle_{(x,y,z)} &= [a_{11}(-z)^2 + a_{21}(-z)^2]dx^2 + [a_{12}(-z)^2 + a_{22}(-z)^2]dy^2 + dz^2 \\
&\quad + [a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)](dx \otimes dy + dy \otimes dx) \\
&= e^{-2\mathrm{tr}(A)z}\{[a_{21}(z)^2 + a_{22}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\
&\quad - e^{-2\mathrm{tr}(A)z}[a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)](dx \otimes dy + dy \otimes dx).
\end{aligned}$$

In particular, for every matrix $A \in \mathcal{M}_2(\mathbb{R})$, the rotation by angle π around the vertical geodesic $\gamma(z) = (x_0, y_0, z)$ given by the map $R(x, y, z) = (-x + 2x_0, -y + 2y_0, z)$ is an isometry of $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle)$ into itself.

Remark 2.2. As we observed, the vertical lines of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are geodesics of its canonical metric. For any line l in $\mathbb{R}^2 \rtimes_A \{0\}$ let P_l denote the vertical plane $\{(x, y, z) : (x, y, 0) \in l; z \in \mathbb{R}\}$ containing the set of vertical lines passing through l. It follows that P_l is ruled by vertical geodesics and, since rotation by angle π around any vertical line in P_l is an isometry that leaves P_l invariant, P_l has zero mean curvature.

Although the rotation by angle π around horizontal geodesics is not always an isometry, we have the following result.

Proposition 2.3. *Let $A = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \mathcal{M}_2(\mathbb{R})$ and consider the horizontal geodesic $\alpha = \{(x_0, t, 0) : t \in \mathbb{R}\}$ in $\mathbb{R}^2 \rtimes_A \{0\}$ parallel to the y-axis. Then the rotation by angle π around α is an isometry of $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle , \rangle)$ into itself. The same result is true for a horizontal geodesic parallel to the x-axis.*

Proof. The rotation by angle π around α is given by the map $\phi(x, y, z) = (-x + 2x_0, y, -z)$, so $\phi_x = -\partial_x$, $\phi_y = \partial_y$ and $\phi_z = -\partial_z$.

If $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, then

$$e^{zA} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(bc)^k z^{2k}}{(2k)!} & \sum_{k=1}^{\infty} \frac{b^k c^{k-1} z^{2k-1}}{(2k-1)!} \\ \sum_{k=1}^{\infty} \frac{c^k b^{k-1} z^{2k-1}}{(2k-1)!} & \sum_{k=0}^{\infty} \frac{(bc)^k z^{2k}}{(2k)!} \end{pmatrix}.$$

Hence, $a_{11}(z) = a_{22}(z)$ and $e^{-zA} = \begin{pmatrix} a_{11}(z) & -a_{12}(z) \\ -a_{21}(z) & a_{11}(z) \end{pmatrix}$. Then

$$\begin{aligned} \langle , \rangle_{(x,y,z)} = {} & \{[a_{21}(z)^2 + a_{11}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\ & - [a_{11}(z)a_{21}(z) + a_{12}(z)a_{11}(z)](dx \otimes dy + dy \otimes dx) \end{aligned}$$

and

$$\begin{aligned} \langle , \rangle_{\phi(x,y,z)} = {} & \{[a_{21}(z)^2 + a_{11}(z)^2]dx^2 + [a_{11}(z)^2 + a_{12}(z)^2]dy^2\} + dz^2 \\ & + [a_{11}(z)a_{21}(z) + a_{12}(z)a_{11}(z)](dx \otimes dy + dy \otimes dx). \end{aligned}$$

Therefore, $\langle \phi_x, \phi_x \rangle_{\phi(x,y,z)} = \langle \partial_x, \partial_x \rangle_{(x,y,z)}$, $\langle \phi_y, \phi_y \rangle = \langle \partial_y, \partial_y \rangle$, $\langle \phi_z, \phi_z \rangle = \langle \partial_z, \partial_z \rangle$, that is, ϕ is an isometry. Analogously, we can show that the rotation by angle π around a horizontal geodesic parallel to the x-axis is also an isometry. $\square$

Remark 2.4. When the matrix A in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have the Heisenberg space and Sol_3, respectively, with their well-known Riemannian metrics. When we consider the one-parameter family of matrices $A(c) = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$, $c \geq 1$, we get a one-parameter family of metrics in Sol_3 which are not isometric. For more details, see [4].

Meeks *et al.* [3] have proved results concerning the geometry of solutions to Plateau type problems in metric semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution. More precisely, they proved the following theorem.

Theorem 2.5 (Meeks *et al.* [3]). *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product with its canonical metric and let $\Pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ denote the projection $\Pi(x, y, z) = (x, y, 0)$. Suppose that E is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial E$ and $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that $\Pi : \Gamma \to C$ monotonically parameterizes C. Then:*

(1) *Γ is the boundary of a compact embedded disk Σ of finite least area;*
(2) *the interior of Σ is a smooth Π-graph over the interior of E.*

3. A doubly periodic Scherk minimal surface

Throughout this section, we consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the canonical left invariant metric $\langle , \rangle$, where $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. In this space, we prove the

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existence of a complete minimal surface analogous to Scherk's doubly periodic minimal surface in $\mathbb{R}^3$.

Fix $0 < c_0 < c_1$ and let a be a sufficiently small positive quantity such that

$$\begin{aligned} a < \int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{21}^2(z)} + \sqrt{a_{11}^2(z) + a_{12}^2(z)}\, dz \\ - \int_{c_0}^{c_1} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\, dz. \end{aligned} \tag{3.1}$$

Note that such positive number a exists, as

$$|\partial_x| = \sqrt{a_{11}^2(z) + a_{21}^2(z)}, \quad |\partial_y| = \sqrt{a_{11}^2(z) + a_{12}^2(z)}$$

and

$$|\partial_x + \partial_y| = \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}.$$

For each $c > 0$, consider the polygon P_c in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with sides $\alpha_1, \alpha_2, \alpha_3^c, \alpha_4^c$ and α_5^c defined by

$$\begin{aligned} \alpha_1 &= \{(t, 0, 0) : 0 \le t \le a\}, \\ \alpha_2 &= \{(0, t, 0) : 0 \le t \le a\}, \\ \alpha_3^c &= \{(a, 0, t) : 0 \le t \le c\}, \\ \alpha_4^c &= \{(0, a, t) : 0 \le t \le c\}, \\ \alpha_5^c &= \{(t, -t + a, c) : 0 \le t \le a\}, \end{aligned}$$

as illustrated in Figure 1.

We will denote $\alpha_1^0 = \{(t, 0, 0) : 0 \le t < a\}$, $\alpha_2^0 = \{(0, t, 0) : 0 \le t < a\}$, $\alpha_3 = \{(a, 0, t) : t > 0\}$ and $\alpha_4 = \{(0, a, t) : t > 0\}$, hence $P_\infty = \alpha_1^0 \cup \alpha_2^0 \cup \alpha_3 \cup \alpha_4 \cup \{(a, 0, 0), (0, a, 0)\}$.

Let $\Pi : \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ denote the projection $\Pi(x, y, z) = (x, y, 0)$. The next proposition is proved in [3, Lemma 1.2], using the maximum principle and the fact that for every line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$, the vertical plane $\Pi^{-1}(L)$ is a minimal surface.

Proposition 3.1. *Let E be a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ with boundary $C = \partial E$ and let Σ be a compact minimal surface with boundary in $\Pi^{-1}(C)$. Then every point in* $\mathrm{int}\Sigma$ *is contained in* $\mathrm{int}\Pi^{-1}(E)$.

Observe that, for each $c > 0$, the polygon P_c is transverse to the Killing field $X = \partial_x + \partial_y$ and each integral curve of X intersects P_c at at most one point. From now on, denote by P the common projection of every P_c over $\mathbb{R}^2 \rtimes_A \{0\}$, that is, $P = \Pi(P_c) = \Pi(P_d)$ for any $c, d \in \mathbb{R}$, and denote by E the disk in $\mathbb{R}^2 \rtimes_A \{0\}$ with boundary P. Let us denote by $\mathcal{R}$ the region $E \times \{z \ge 0\}$. Using Theorem 2.5, we conclude that P_c is the boundary of a compact embedded disk Σ_c of finite least area and the interior of Σ_c is a smooth Π-graph over the interior of E.

Let $\Omega_c = \{(t, -t + a, s) : 0 \le t \le a; 0 \le s \le c\}$.

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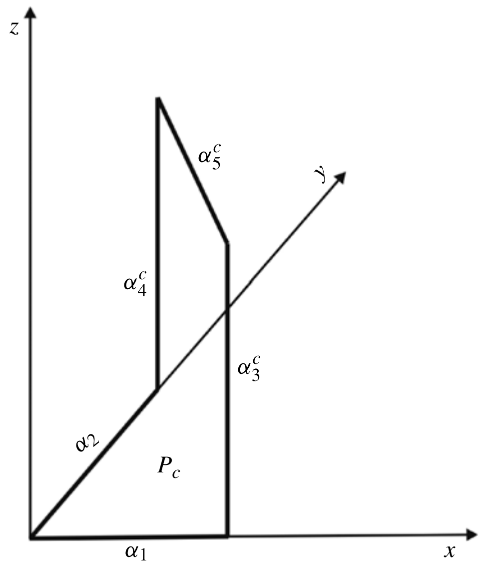

FIGURE 1. Polygon P_c.

PROPOSITION 3.2. *If S is a compact minimal surface with boundary P_c, then $S = \Sigma_c$.*

PROOF. By Proposition 3.1, $\mathrm{int}\Sigma_c, \mathrm{int}S \subset \mathrm{int}\Pi^{-1}(E)$; then, in particular, $\mathrm{int}\Sigma_c, \mathrm{int}S \subset \mathrm{int}\{\varphi_t(p) : t \in \mathbb{R}; p \in \Omega_c\}$, where φ_t is the flow of the Killing field X.

As S is compact, there exists t such that $\varphi_t(\Sigma_c) \cap S = \emptyset$. If $S \neq \Sigma_c$, then there exists $t_0 > 0$ such that $\varphi_{t_0}(\Sigma_c) \cap S \neq \emptyset$ and, for $t > t_0$, $\varphi_t(\Sigma_c) \cap S = \emptyset$. Since for all $t \neq 0$, $\varphi_t(P_c) \cap S = \emptyset$, the point of intersection is an interior point and, by the maximum principle, $\varphi_{t_0}(\Sigma_c) = S$. But that is a contradiction, since $t_0 \neq 0$. Therefore, $S = \Sigma_c$. □

The next proposition is a classical result.

PROPOSITION 3.3. *Let N^3 be a homogeneous 3-manifold. Let Σ_n be an oriented, properly embedded minimal surface in N. Suppose that there exist $c > 0$ such that for all n, $|A_{\Sigma_n}| \leq c$, and a sequence of points $\{p_n\}$ in Σ_n such that $p_n \to p \in N$. Then there exists a subsequence of Σ_n that converges to a complete minimal surface Σ with $p \in \Sigma$. Here A_{Σ_n} denotes the second fundamental form of Σ_n.*

For each $n \in \mathbb{N}$, let Σ_n be the solution to the Plateau problem with boundary P_n. By Theorem 2.5 and Proposition 3.2, Σ_n is stable and unique. We are interested in proving the existence of a subsequence of Σ_n that converges to a complete minimal surface with boundary P_∞. In order to do so, we will use a minimal annulus as a barrier (whose existence is guaranteed by the Douglas criterion (see [1, Theorem 2.1])) to show that there exist points $p_n \in \Sigma_n$, $\Pi(p_n) = q \in \mathrm{int}E$ for all n, which converge to a point $p \in \mathbb{R}^2 \rtimes_A \mathbb{R}$, and then we will use Proposition 3.3.

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Consider the parallelepiped with the faces A, B, C, D, E and F, defined by

$$\begin{aligned}
A &= \{(u, -\epsilon, v) : \epsilon \leq u \leq a+\epsilon;\ c_0 \leq v \leq c_1\},\\
B &= \{(-\epsilon, u, v) : \epsilon \leq u \leq a+\epsilon;\ c_0 \leq v \leq c_1\},\\
C &= \{(u, -u, v) : -\epsilon \leq u \leq \epsilon;\ c_0 \leq v \leq c_1\},\\
D &= \{(u, -u+a, v) : -\epsilon \leq u \leq a+\epsilon;\ c_0 \leq v \leq c_1\},\\
E &= \{(u, -u+v, c_0) : -\epsilon \leq u \leq v+\epsilon;\ 0 \leq v \leq a\},\\
F &= \{(u, -u+v, c_1) : -\epsilon \leq u \leq v+\epsilon;\ 0 \leq v \leq a\},
\end{aligned}$$

where ϵ is a positive constant that we will choose later. Observe that each one of these faces is the least area minimal surface with its boundary. Let us analyse the area of each face.

(1) In the plane $\{y = \text{constant}\}$ the induced metric is given by $g(x, z) = (a_{11}^2(z) + a_{21}^2(z))dx^2 + dz^2$. Hence,

$$\begin{aligned}
\text{area } A &= \int_{c_0}^{c_1}\int_{\epsilon}^{a+\epsilon} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\, dx\, dz\\
&= a\int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\, dz.
\end{aligned}$$

(2) In the plane $\{x = \text{constant}\}$ the induced metric is given by $g(y, z) = (a_{11}^2(z) + a_{12}^2(z))dy^2 + dz^2$. Hence,

$$\begin{aligned}
\text{area } B &= \int_{c_0}^{c_1}\int_{\epsilon}^{a+\epsilon} \sqrt{a_{11}^2(z) + a_{12}^2(z)}\, dx\, dz\\
&= a\int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{12}^2(z)}\, dz.
\end{aligned}$$

(3) The face C is contained in the plane parameterized by $\phi(u, v) = (u, -u, v)$ and the face D is contained in the plane parameterized by $\psi(u, v) = (u, -u + a, v)$. We have $\psi_u = \phi_u = \partial_x - \partial_y$, $\psi_v = \phi_v = \partial_z$. Then

$$|\psi_u \wedge \psi_v| = |\phi_u \wedge \phi_v| = \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}.$$

Hence,

$$\begin{aligned}
\text{area } C &= \int_{c_0}^{c_1}\int_{-\epsilon}^{+\epsilon} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\, du\, dv\\
&= 2\epsilon\int_{c_0}^{c_1} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z)^2}\mathrm{d}z,\\
\text{area } D &= \int_{c_0}^{c_1}\int_{-\epsilon}^{a+\epsilon} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\, du\, dv\\
&= (a + 2\epsilon)\int_{c_0}^{c_1} \sqrt{(a_{11}(z) + a_{12}(z))^2 + (a_{11}(z) + a_{21}(z))^2}\, dz.
\end{aligned}$$

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(4) As the plane $\{z = \text{constant}\}$ is flat, the induced metric is the Euclidean metric. Hence,

$$\text{area } E = \text{area } F = \int_0^a \int_{-\epsilon}^{v+\epsilon} du\, dv = \frac{a(a+4\epsilon)}{2}.$$

Therefore,

$$\text{area } C + \text{area } D + \text{area } E + \text{area } F < \text{area } A + \text{area } B$$

if, and only if,

$$(a+4\epsilon)\Big[a + \int_{c_0}^{c_1} \sqrt{(a_{11}+a_{12})^2 + (a_{11}+a_{21})^2}\, dz\Big] < a \int_{c_0}^{c_1} \sqrt{a_{11}^2 + a_{21}^2}\, dz + a \int_{c_0}^{c_1} \sqrt{a_{11}^2 + a_{12}^2}\, dz$$

if, and only if,

$$\epsilon < \frac{a}{4} \frac{\int_{c_0}^{c_1} \sqrt{a_{11}^2(z) + a_{21}^2(z)} + \sqrt{a_{11}^2(z) + a_{21}^2(z)}\, dz}{a + \int_{c_0}^{c_1} \sqrt{(a_{11}(z)+a_{12}(z))^2 + (a_{11}(z)+a_{21}(z))^2}\, dz} - \frac{a}{4}. \tag{3.2}$$

As we chose a satisfying (3.1), the factor on the right-hand side of (3.2) is a positive number, and we can choose $\epsilon > 0$ such that the Douglas criterion is satisfied [1]. Hence we obtain a minimal annulus $\mathcal{A}$ with boundary $\partial A \cup \partial B$ such that its projection $\Pi(\mathcal{A})$ contains points of int$E$, where E is the disk in $\mathbb{R} \rtimes_A \{0\}$ with boundary P (see Figure 2).

As $\mathbb{R}^2 \rtimes_A \{z\}$ is a minimal surface, the maximum principle implies that, for each c, Σ_c is contained in the slab bounded by the planes $\{z = 0\}$ and $\{z = c\}$. Then for $c < c_0$, $\Sigma_c \cap \mathcal{A} = \emptyset$. As Σ_c is unique, Σ_c varies continuously with c, and when c increases the boundary $\partial\Sigma_c = P_c$ does not touch $\partial\mathcal{A}$. Therefore, using the maximum principle, $\Sigma_c \cap \mathcal{A} = \emptyset$ for all c, and Σ_c is under the annulus $\mathcal{A}$, which means that over any vertical line that intersects $\mathcal{A}$ and Σ_c, the points of Σ_c are under the points of $\mathcal{A}$.

Consider the flow φ_t of the Killing field $X = \partial_x + \partial_y$. Observe that $\{\varphi_t(\mathcal{A})\}_{t<0}$ forms a barrier for all points $p_n \in \Sigma_n$ such that $\Pi(p_n)$ is contained in a neighborhood $\mathcal{U} \subset E$ of the origin $o = (0, 0, 0)$. Moreover, for any $c_2 < c_3$ we can use the flow φ_t of the Killing field X and the maximum principle to conclude that Σ_{c_2} is under Σ_{c_3} in the same sense as before.

As, by Theorem 2.5, each Σ_n is a vertical graph in the interior, then $\Sigma_n \cap \Pi^{-1}(q)$ is only one point p_n, for every point $q \in$ int$E$. Moreover, by the previous paragraph, the sequence $p_n = \Sigma_n \cap \Pi^{-1}(q)$ is monotone. Then, since we have a barrier, the sequence $\{p_n = \Sigma_n \cap \Pi^{-1}(q)\}$ converges to a point $p \in \Pi^{-1}(q)$, for all $q \in \mathcal{U}$.

In order to understand the convergence of the surfaces Σ_n we need to observe some properties of these surfaces.

First, notice that rotation by angle π around α_3, which we will denote by R_{α_3}, is an isometry. By the Schwarz reflection, we obtain a minimal surface $\widetilde{\Sigma}_n = \Sigma_n \cup R_{\alpha_3}(\Sigma_n)$ that has intα_3 in its interior. Note that the boundary of $\widetilde{\Sigma}_n$ is transverse to the Killing

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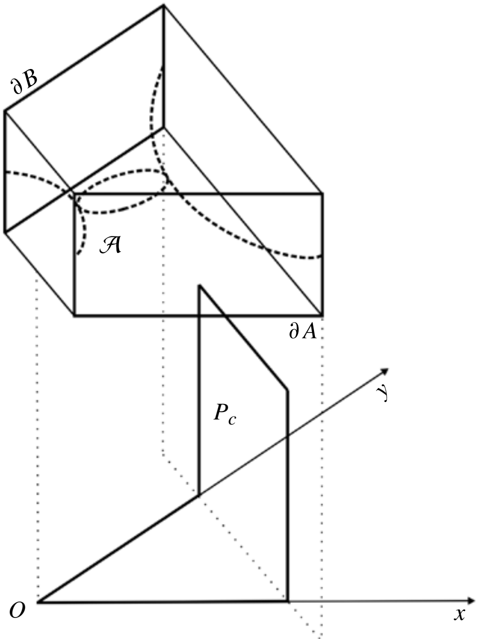

Figure 2. Annulus $\mathcal{A}$.

field $X = \partial_x + \partial_y$, and if φ_t denotes the flow of X, we have that $\varphi_t(\partial\widetilde{\Sigma}_n) \cap \widetilde{\Sigma}_n = \emptyset$ for all $t \neq 0$, hence, using the same arguments of the proof of Proposition 3.2, we can show that the minimal surface $\widetilde{\Sigma}_n$ is the unique minimal surface with its boundary. In particular, it is area-minimizing, and then it is stable. Hence, by main theorem in [6], we have uniform curvature estimates for points far from the boundary of $\widetilde{\Sigma}_n$. In particular, we get uniform curvature estimates for Σ_n in a neighborhood of α_3. Analogously, we have uniform curvature estimates for Σ_n in a neighborhood of α_4.

Hence, for every compact contained in $\{z > 0\} \cap \mathcal{R}$, there exists a subsequence of Σ_n that converges to a minimal surface. Taking exhaustion by compact sets and using a diagonal process, we conclude that there exists a subsequence of Σ_n that converges to a minimal surface Σ that has $\alpha_3 \cup \alpha_4$ in its boundary. From now on, we will use the notation Σ_n for this subsequence.

It remains to prove that in fact Σ is a minimal surface with boundary P_∞. In order to do so, we will use the fact that the interior of each Σ_n is a vertical graph over the interior of E. Let us denote by u_n the function defined in int$E$ such that $\Sigma_n = \mathrm{Graph}(u_n)$. We already know that $u_{n-1} < u_n$ in int$E$ for all n.

Claim 3.4. *There are uniform gradient estimates for $\{u_n\}$ for points in $\alpha_1^0 \cup \alpha_2^0$.*

Proof. For $x_0 < 0$ and $\delta > 0$ consider the vertical strip bounded by $\beta_1 = \{(x_0, y, c_1) : -\delta \leq y \leq 0\}$, $\beta_2 = \{(x_0, t, -(c_1/a)t + c_1) : 0 \leq t \leq a\}$, $\beta_3 = \{(x_0, t - \delta, -(c_1/a)t + c_1) : 0 \leq t \leq a\}$ and $\beta_4 = \{(x_0, y, 0) : a - \delta \leq y \leq a\}$. This is a minimal surface transversal

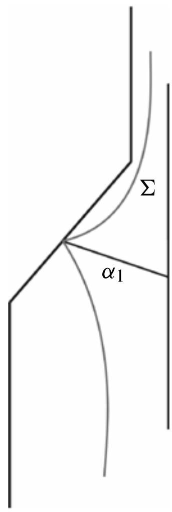

FIGURE 3. Rotation by angle π around α_1 of Σ.

to the Killing field ∂_x, hence any small perturbation of its boundary gives a minimal surface with that perturbed boundary. Thus, if we consider a small perturbation of the boundary of this vertical strip by just slightly perturbing β_1 by a curve contained in $\{x \geq x_0\}$ joining the points $(x_0, -\delta, c_1)$ and $(x_0, 0, c_1)$, we will get a minimal surface S with this perturbed boundary. This minimal surface S will have the property that the tangent planes at the interior of β_4 are not vertical, by the maximum principle with boundary.

Applying translations along the x-axis and y-axis, we can use the translates of S to show that Σ_n is under S in a neighborhood of α_2^0, and then we have uniform gradient estimates for points in α_2^0. Analogously, constructing similar barriers, we can prove that we have uniform gradient estimates in a neighborhood of α_1^0. □

Observe that besides the gradient estimates, the translates of the minimal surface S form a barrier for points in a neighborhood of $\alpha_1^0 \cup \alpha_2^0$.

We have that Σ_n is a monotone increasing sequence of minimal graphs with uniform gradient estimates in $\alpha_1^0 \cup \alpha_2^0$, and it is a bounded graph for points in a neighborhood $\mathcal{U}$ of the origin (because of the barrier given by the annulus $\mathcal{A}$). Therefore, there exists a subsequence of Σ_n that converges to a minimal surface $\widetilde{\Sigma}$ with $\alpha_1^0 \cup \alpha_2^0$ in its boundary. As we already know that Σ_n converges to the minimal surface Σ, we conclude that in fact $\Sigma = \widetilde{\Sigma}$, and then Σ is a minimal surface with $\alpha_1^0 \cup \alpha_2^0 \cup \alpha_3 \cup \alpha_4$ in its boundary. Notice that we can assume that Σ has P_∞ as its boundary, with Σ being of class C^1 up to $P_\infty \setminus \{(a, 0, 0), (0, a, 0)\}$ and continuous up to P_∞.

Now considering the rotation by angle π around α_1 of Σ, we obtain the surface illustrated in Figure 3.

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Continuing to rotate by angle π around the y-axis, the resulting surface will be a minimal surface with four vertical lines as its boundary: $\{(a, 0, t) : t \in \mathbb{R}\}$, $\{(0, a, t) : t \in \mathbb{R}\}$, $\{(-a, 0, t) : t \in \mathbb{R}\}$, $\{(0, -a, t) : t \in \mathbb{R}\}$.

Now we can use the rotations by angle π around the vertical lines to get a complete minimal surface that is analogous to the doubly periodic minimal Scherk surface in $\mathbb{R}^3$. It is invariant with respect to two translations that commute and it is a four-punctured sphere in the quotient of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ by the group of isometries generated by the two translations.

Theorem 3.5. *In any semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$, there exists a periodic minimal surface similar to the doubly periodic Scherk minimal surface in $\mathbb{R}^3$.*

4. A singly periodic Scherk minimal surface

Throughout this section, we consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the canonical left invariant metric $\langle , \rangle$, where $A = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. In this space, we construct a complete minimal surface similar to the singly periodic Scherk minimal surface in $\mathbb{R}^3$.

Fix $c_0 > 0$ and take $0 < \epsilon < a$ sufficiently small so that

$$a + 2\epsilon < \int_0^{c_0} \sqrt{a_{11}^2(z) + a_{21}^2(z)}\, dz.$$

For each $c > 0$, consider the polygon P_c in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with the six sides defined by

$$\begin{aligned}
\alpha_1^c &= \{(t, 0, 0) : 0 \le t \le c\},\\
\alpha_2^c &= \{(c, t, 0) : 0 \le t \le a\},\\
\alpha_3^c &= \{(t, a, 0) : 0 \le t \le c\},\\
\alpha_4^c &= \{(0, a, t) : 0 \le t \le c\},\\
\alpha_5^c &= \{(0, t, c) : 0 \le t \le a\},\\
\alpha_6^c &= \{(0, 0, t) : 0 \le t \le c\},
\end{aligned}$$

and for each $\delta > 0$ with $\delta < a/2$, consider the polygon P_c^δ with the six sides

$$\begin{aligned}
\alpha_1^{\delta,c} &= \left\{\left(t, \frac{\delta}{c}t, 0\right) : 0 \le t \le c\right\}\\
\alpha_2^{\delta,c} &= \{(c, t, 0) : \delta \le t \le a - \delta\}\\
\alpha_3^{\delta,c} &= \left\{\left(t, \frac{ac - \delta t}{c}, 0\right) : 0 \le t \le c\right\},
\end{aligned}$$

$\alpha_4^c, \alpha_5^c, \alpha_6^c$, as illustrated in Figure 4.

Denote by $\Omega(\delta, c)$ the region in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by $\alpha_1^{\delta,c}, \alpha_2^{\delta,c}, \alpha_3^{\delta,c}$ and the segment $\{(0, t, 0) : 0 \le t \le a\}$. For each c and δ, we have compact minimal surfaces Σ_c and Σ_c^δ with boundary P_c and P_c^δ, respectively, which are solutions to the Plateau problem. By Theorem 2.5, we know that Σ_c and Σ_c^δ are stable and smooth Π-graphs over the interior

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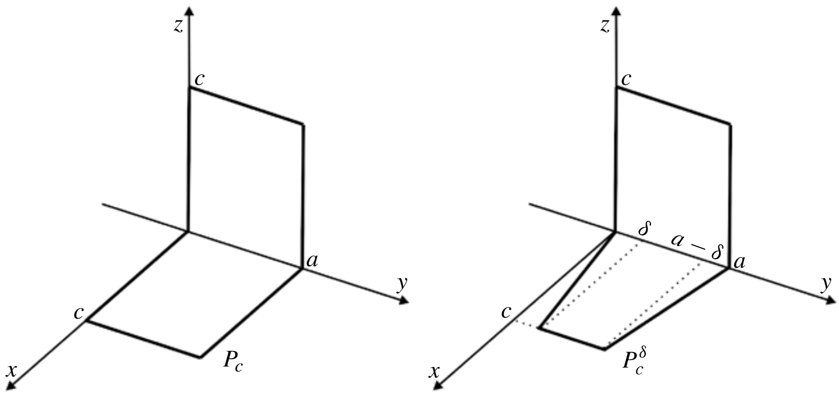

FIGURE 4. Polygons P_c and P_c^δ.

of $\Omega(0, c)$, $\Omega(\delta, c)$, respectively. We will show that Σ_c is the unique compact minimal surface with boundary P_c.

Fix c. For each $0 < \delta < a/2$, P_c^δ is a polygon transverse to the Killing field ∂_x and each integral curve of ∂_x intersects P_c^δ at at most one point. Thus we can prove, as we did in Proposition 3.2, that Σ_c^δ is the unique compact minimal surface with boundary P_c^δ.

Denote by u_c^δ, v_c the functions defined in the interior of $\Omega(\delta, c)$, $\Omega(0, c)$, whose Π-graphs are Σ_c^δ, Σ_c, respectively. Then, as ∂_x is a Killing field and each P_c^δ is transversal to ∂_x, we can use the flow of ∂_x and the maximum principle to prove that for $\delta' < \delta$ we have $0 \le u_c^\delta \le u_c^{\delta'} \le v_c$ in $\mathrm{int}\Omega(\delta, c)$, hence v_c is a barrier for our sequence u_c^δ. Because of the monotonicity and the barrier, the family u_c^δ converges to a function u_c defined in $\mathrm{int}\Omega(0, c)$ whose graph is a compact minimal surface with boundary P_c, and we still have $u_c \le v_c$ on $\Omega(0, c)$.

We will now find another compact minimal surface with boundary P_c, whose interior is the graph of a function w_c defined in $\mathrm{int}\Omega(0, c)$ such that $v_c \le w_c$ and we will show that $u_c = w_c$. In order to do so, for each $0 < \delta < a/2$, consider the polygon $\widetilde{P}_c^\delta$ with the six sides defined by

$$\begin{aligned}
\widetilde{\alpha}_1^{\delta,c} &= \left\{\left(t, \frac{\delta t - \delta c}{c}, 0\right) : 0 \le t \le c\right\},\\
\alpha_2^{c} &= \{(c, t, 0) : 0 \le t \le a\},\\
\widetilde{\alpha}_3^{\delta,c} &= \left\{\left(t, \frac{(a+\delta)c - \delta t}{c}, 0\right) : 0 \le t \le c\right\},\\
\widetilde{\alpha}_4^{\delta,c} &= \{(0, a+\delta, t) : 0 \le t \le c\},\\
\widetilde{\alpha}_5^{\delta,c} &= \{(0, t, c) : -\delta \le t \le a+\delta\},\\
\widetilde{\alpha}_6^{\delta,c} &= \{(0, -\delta, t) : 0 \le t \le c\},
\end{aligned}$$

as illustrated in Figure 5.

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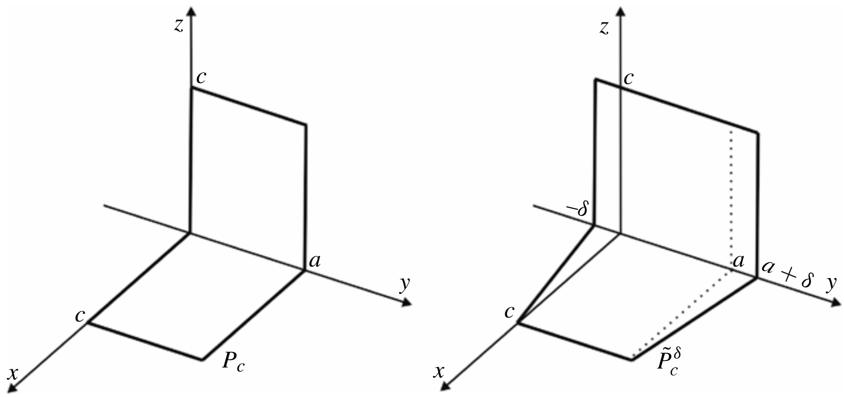

FIGURE 5. Polygons P_c and $\widetilde{P}_c^\delta$.

Denote by $\widetilde{\Omega}(\delta, c)$ the region in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by $\widetilde{\alpha}_1^{\delta,c}$, α_2^c, $\widetilde{\alpha}_3^{\delta,c}$ and the segment $\{(0, t, 0) : -\delta \leq t \leq a + \delta\}$. For each δ, we have a compact minimal disk $\widetilde{\Sigma}_c^\delta$ with boundary $\widetilde{P}_c^\delta$ and $\widetilde{\Sigma}_c^\delta$ is a smooth Π-graph over the interior of $\widetilde{\Omega}(\delta, c)$. As $\widetilde{P}_c^\delta$ is transversal to the Killing field ∂_x, we can prove that $\widetilde{\Sigma}_c^\delta$ is the unique compact minimal surface with boundary $\widetilde{P}_c^\delta$.

Denote by w_c^δ the function defined in $\mathrm{int}\widetilde{\Omega}(\delta, c)$ whose graph is $\widetilde{\Sigma}_c^\delta$. Using the flow of ∂_x and the maximum principle, we can prove that for $\delta' < \delta$ we have $w_c^{\delta'} \leq w_c^\delta$ in $\mathrm{int}\widetilde{\Omega}(\delta', c)$ and for all δ, $v_c \leq w_c^\delta$ in $\mathrm{int}\Omega(0, c)$. Because of the monotonicity and the barrier, the family w_c^δ converges to a function w_c defined in $\mathrm{int}\widetilde{\Omega}(0, c) = \mathrm{int}\Omega(0, c)$ whose graph is a compact minimal surface with boundary P_c, and we still have $v_c \leq w_c$ in $\mathrm{int}\Omega(0, c)$.

Let us call Σ_1, Σ_2 the graphs of u_c, w_c, respectively. We will now prove that $\Sigma_1 = \Sigma_2$. Denote by ν_i the conormal to Σ_i along P_c, $i = 1, 2$. (See Figure 6.)

Suppose that $u_c \neq w_c$; then in fact we have $u_c < w_c$ in $\mathrm{int}\Omega(0, c)$. As ∂_x is tangent to α_1^c and α_3^c, we have that $\langle \nu_i, \partial_x \rangle = 0$, $i = 1, 2$, in α_1^c and α_3^c. On the other sides of P_c we have $\langle \nu_1, \partial_x \rangle < \langle \nu_2, \partial_x \rangle$. Therefore,

$$\int_{P_c} \langle \nu_1, \partial_x \rangle < \int_{P_c} \langle \nu_2, \partial_x \rangle.$$

But, using the flux formula for Σ_1 and Σ_2 with respect to the Killing field ∂_x,

$$\int_{P_c} \langle \nu_1, \partial_x \rangle = 0 = \int_{P_c} \langle \nu_2, \partial_x \rangle.$$

Then $u_c = w_c$, and therefore $\Sigma_c = \Sigma_1 = \Sigma_2$. In particular, Σ_c is the unique compact minimal surface with boundary P_c.

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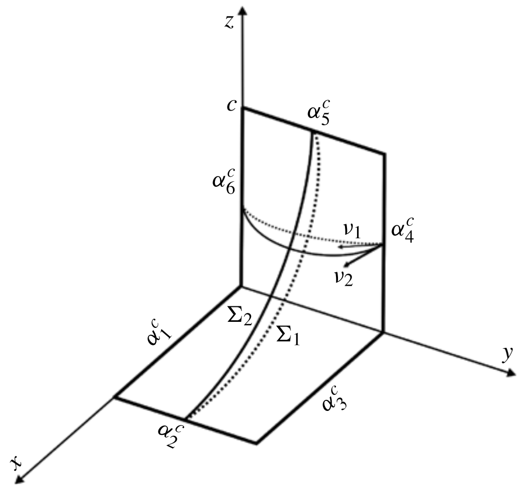

FIGURE 6. Σ_1 and Σ_2.

Denote by $\Omega(\infty)$ the infinite strip $\{(x, y, 0) : x \geq 0, 0 \leq y \leq a\}$, and by $\mathcal{R}$ the region $\{(x, y, z) : x \geq 0, 0 \leq y \leq a, z \geq 0\}$. Moreover, denote $\alpha_1 = \{(x, 0, 0) : x > 0\}$, $\alpha_3 = \{(x, a, 0) : x > 0\}$, $\alpha_4 = \{(0, a, z) : z > 0\}$ and $\alpha_6 = \{(0, 0, z) : z > 0\}$, hence $P_\infty = \alpha_1 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6 \cup \{(0, 0, 0), (0, a, 0)\}$.

For each $n \in \mathbb{N}$, let Σ_n be the unique compact minimal surface with boundary P_n. We are interested in proving the existence of a subsequence of Σ_n that converges to a complete minimal surface with boundary P_∞. Using the existence of a minimal annulus, guaranteed by the Douglas criterion, we will show that there exist points $p_n \in \Sigma_n$, $\Pi(p_n) = q \in \operatorname{int} \Omega(\infty)$ for all n, which converge to a point $p \in \mathbb{R}^2 \rtimes_A \mathbb{R}$, and then we will use Proposition 3.3.

Consider the parallelepiped with faces A, B, C, D, E and F, defined by

$$
\begin{aligned}
A &= \{(u, -\epsilon, v) : \epsilon \leq u \leq d; 0 \leq v \leq c_0\},\\
B &= \{(u, a + \epsilon, v) : \epsilon \leq u \leq d; 0 \leq v \leq c_0\},\\
C &= \{(u, v, 0) : \epsilon \leq u \leq d; -\epsilon \leq v \leq a + \epsilon\},\\
D &= \{(u, v, c_0) : \epsilon \leq u \leq d; -\epsilon \leq v \leq a + \epsilon\},\\
E &= \{(\epsilon, u, v) : -\epsilon \leq u \leq a + \epsilon; 0 \leq v \leq c_0\},\\
F &= \{(d, u, v) : -\epsilon \leq u \leq a + \epsilon; 0 \leq v \leq c_0\},
\end{aligned}
$$

where $d > \epsilon$ is a constant that we will choose later.

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As we did in the previous section, we can calculate the area of each one of these faces and we obtain:

$$\begin{aligned}
\text{area } A = \text{area } B &= (d-\epsilon)\int_0^{c_0}\sqrt{a_{11}^2(z)+a_{21}^2(z)}\,dz,\\
\text{area } C = \text{area } D &= (d-\epsilon)(a+2\epsilon),\\
\text{area } E = \text{area } F &= (a+2\epsilon)\int_0^{c_0}\sqrt{a_{11}^2(z)+a_{12}^2(z)}\,dz.
\end{aligned}$$

Hence,

$$\text{area } C + \text{area } D + \text{area } E + \text{area } F < \text{area } A + \text{area } B$$

if, and only if,

$$(d-\epsilon)(a+2\epsilon)+(a+2\epsilon)\int_0^{c_0}\sqrt{a_{11}^2+a_{12}^2}\,dz < (d-\epsilon)\int_0^{c_0}\sqrt{a_{11}^2+a_{21}^2}\,dz$$

if, and only if,

$$(d-\epsilon)\Big[(a+2\epsilon)-\int_0^{c_0}\sqrt{a_{11}^2+a_{21}^2}\,dz\Big] < -(a+2\epsilon)\int_0^{c_0}\sqrt{a_{11}^2+a_{12}^2}\,dz$$

if, and only if,

$$d > \epsilon - \frac{(a+2\epsilon)\displaystyle\int_0^{c_0}\sqrt{a_{11}^2(z)+a_{12}^2(z)}\,dz}{(a+2\epsilon)-\displaystyle\int_0^{c_0}\sqrt{a_{11}^2(z)+a_{21}^2(z)}\,dz}.$$

As we chose $a+2\epsilon < \int_0^{c_0}\sqrt{a_{11}^2(z)+a_{21}^2(z)}\,dz$, we can choose $d > \epsilon$ so that the Douglas criterion is satisfied [1]. Thus, there exists a minimal annulus $\mathcal{A}$ with boundary $\partial A \cup \partial B$ such that its projection $\Pi(\mathcal{A})$ contains points of $\mathrm{int}\Omega(\infty)$. (See Figure 7.)

We know that, for each $c < \epsilon$, $\Sigma_c \cap \mathcal{A} = \emptyset$. When c increases P_c does not intersect $\partial\mathcal{A}$; then, using the maximum principle, $\Sigma_c \cap \mathcal{A} = \emptyset$ for all c, and Σ_c is under the annulus $\mathcal{A}$. Thus, there exists a point $q \in \mathrm{int}\Omega(\infty)$ such that $p_n = \Sigma_n \cap \Pi^{-1}(q)$ has a subsequence that converges to a point $p \in \Pi^{-1}(q)$. Observe that, applying the flow of the Killing field ∂_x to the annulus $\mathcal{A}$, we can conclude that, in the region $\{x \geq d\}$, the surfaces Σ_n are bounded above by, for example, the plane $\{z = c_0\}$.

In order to understand the convergence of the surfaces Σ_n we need to prove some properties of these surfaces.

Claim 4.1. *The surfaces Σ_n are transversal to the Killing field ∂_x in the interior.*

Proof. Fix n. Suppose that at some point $p \in \mathrm{int}\Sigma_n$ the tangent plane $T_p\Sigma_n$ contains the vector ∂_x. As the planes that contain the direction ∂_x are minimal surfaces, we

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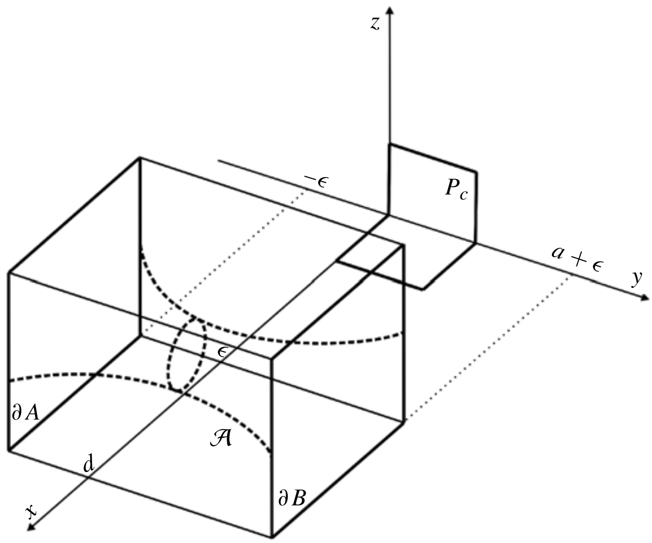

FIGURE 7. Annulus $\mathcal{A}$.

have that Σ_n and $T_p\Sigma_n$ are minimal surfaces tangent at p, and then the intersection between them is formed by $2k$ curves, $k \geq 1$, passing through p making equal angles at p. By the shape of P_n (the boundary of Σ_n), we know that $T_p\Sigma_n$ intersects P_n either at only two points or at one point and a segment of straight line (α_1^n or α_3^n). Therefore, we will necessarily have a closed curve contained in the intersection. As Σ_n is simply connected this curve bounds a disk in Σ_n, but as the planes parallel to $T_p\Sigma_n$ are minimal surfaces, we can use the maximum principle to prove that this disk is contained in the plane $T_p\Sigma_n$ and then they coincide, which is impossible. Thus, the vector ∂_x is transversal to Σ at points $p \in \operatorname{int}\Sigma_n$. □

Observe that, besides the interior points, the surfaces Σ_n are also transversal to ∂_x at the points in α_4 and α_6, by the maximum principle with boundary. Thus rotation by angle π around α_4 (respectively α_6) gives a minimal surface which is also transversal to the Killing field ∂_x in the interior, extends the surface Σ_n and has α_4^n (respectively α_6^n) in the interior. Therefore, we have uniform curvature estimates for Σ_n up to $\alpha_4 \cup \alpha_6$.

Hence, for every compact contained in $\{z > 0\} \cap \mathcal{R}$, there exists a subsequence of Σ_n that converges to a minimal surface. Taking exhaustion by compact sets and using a diagonal process, we conclude that there exists a subsequence of Σ_n that converges to a minimal surface Σ that has $\alpha_4 \cup \alpha_6$ in its boundary. From now on we will use the notation Σ_n for this subsequence.

It remains to prove that in fact Σ is a minimal surface with boundary P_∞. In order to do so, we will use the fact that each Σ_n is a vertical graph in the interior.

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Let us denote by u_n the function defined in $\mathrm{int}\Omega(n)$ such that $\Sigma_n = \mathrm{Graph}(u_n)$, where $\Omega(n) = \{(x, y, 0) : 0 \le x \le n; 0 \le y \le a\}$.

Claim 4.2. $u_{n-1} < u_n$ *in* $\mathrm{int}\Omega(n-1)$.

Proof. Recall that each Σ_n is the limit of a sequence of minimal graphs $\widetilde{\Sigma}_n^\delta = \mathrm{Graph}(w_n^\delta)$ whose boundary is transversal to the Killing field ∂_x. Using the flow of the Killing field ∂_x, we can prove that each $\widetilde{\Sigma}_n^\delta$ is above Σ_{n-1}, and then the limit surface Σ_n has to be above Σ_{n-1}. In fact, Σ_n is strictly above Σ_{n-1} in the interior, because as Σ_n and Σ_{n-1} are minimal surfaces, if they intersect at an interior point, there will be points of Σ_n under Σ_{n-1}, and we already know that, by the property of $\widetilde{\Sigma}_n^\delta$, this is not possible. □

Claim 4.3. *There are uniform gradient estimates for* $\{u_n\}$ *for points in* $\alpha_1 \cup \alpha_3$.

Proof. We will use the same idea as in Claim 3.4. For $y_0 > a$ and $\delta > 0$ consider the vertical strip bounded by $\beta_1 = \{(x, y_0, c_0) : d \le x \le d + \delta\}$, $\beta_2 = \{(t, y_0, c_0/dt) : 0 \le t \le d\}$, $\beta_3 = \{(t + \delta, y_0, c_0/dt) : 0 \le t \le d\}$ and $\beta_4 = \{(x, y_0, 0) : 0 \le x \le \delta\}$. This is a minimal surface transversal to the Killing field ∂_y, hence any small perturbation of its boundary gives a minimal surface with that perturbed boundary. Thus, if we consider a small perturbation of the boundary of this vertical strip by just slightly perturbing β_1 by a curve contained in $\{y \le y_0\}$ joining the points (d, y_0, c_0) and $(d + \delta, y_0, c_0)$, we will get a minimal surface S with this perturbed boundary. This minimal surface S will have the property that the tangent planes at the interior points of β_4 are not vertical, by the maximum principle with boundary.

Applying translations along the x-axis and y-axis, we can use the translates of S to show that Σ_n is under S in a neighborhood of α_3, and then we have uniform gradient estimates for points in α_3. Analogously, constructing similar barriers, we can prove that we have uniform gradient estimates in a neighborhood of α_1. □

Observe that besides the gradient estimates, the translates of the minimal surface S form a barrier for points in a neighborhood of $\alpha_1 \cup \alpha_3$.

We have that Σ_n is a monotone increasing sequence of minimal graphs with uniform gradient estimates in $\alpha_1 \cup \alpha_3$, and it is a bounded graph for points in $\{x \ge d\}$ (because of the barrier given by the annulus $\mathcal{A}$). Therefore, there exists a subsequence of Σ_n that converges to a minimal surface $\widetilde{\Sigma}$ with $\alpha_1 \cup \alpha_3$ in its boundary. As we already know that Σ_n converges to the minimal surface Σ, we conclude that in fact $\Sigma = \widetilde{\Sigma}$, and then Σ is a minimal surface with $\alpha_1 \cup \alpha_3 \cup \alpha_4 \cup \alpha_6$ in its boundary. Notice that we can assume that Σ has P_∞ as its boundary, with Σ being of class C^1 up to $P_\infty \setminus \{(0, 0, 0), (0, a, 0)\}$ and continuous up to P_∞. The expected 'singly periodic Scherk minimal surface' is obtained by recursively rotating Σ by an angle π about the vertical and horizontal geodesics in its boundary.

Theorem 4.4. *In any semidirect product* $\mathbb{R}^2 \rtimes_A \mathbb{R}$, *where* $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, *there exists a periodic minimal surface similar to the singly periodic Scherk minimal surface in* $\mathbb{R}^3$.

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