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ON TOPOLOGICAL PROPERTIES OF SOME COVERINGS. AN ADDENDUM TO A PAPER OF LANTERI AND STRUPPA

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ABSTRACT. Let $\pi: X' \longrightarrow X$ be a finite surjective morphism of complex projective manifolds which can be factored by an embedding of X' into the total space of an ample line bundle $\mathcal{L}$ over X. A theorem of Lazarsfeld asserts that Betti numbers of X and X' are equal except, possibly, the middle ones. In the present paper it is proved that the middle numbers are actually non-equal if either $\mathcal{L}$ is spanned and $\deg \pi \geq \dim X$, or if X is either a hyperquadric or a projective space and π is not a double cover of an odd-dimensional projective space by a hyperquadric.

Let $\pi: X' \longrightarrow X$ be a finite surjective morphism of connected complex projective manifolds. Throughout the present paper by k we denote $\dim X = \dim X'$, by n we denote the degree of π and we assume that $n, k \geq 2$. We say that $\pi: X' \longrightarrow X$ (or simply X', or π) is a (smooth) n-section of a line bundle (invertible sheaf) $\mathcal{L}$ over X if X' embeds into L — where $L = \operatorname{Spec}(\operatorname{Sym}_X \mathcal{L}^\vee)$ is the total space of $\mathcal{L}$ — so that π is the restriction of the projection $\mathbf{p}: L \longrightarrow X$.

If $\mathcal{L}$ is an ample line bundle then a theorem of Lazarsfeld (2.1 of [La], see also (1.1) of the present paper) implies the equality of Betti numbers of X and X' except, possibly, the middle one for which we have the inequality $b_k(X') \geq b_k(X)$. In [LS] Lanteri and Struppa studied smooth n-sections of ample line bundles which satisfy

$$(0.1) \qquad b_k(X') = b_k(X).$$

(Although they restricted their study to cyclic coverings, their methods cover all smooth n-sections—compare (1.2) and (1.4) of [LS] with (1.1)(b) and (c) of the present paper). Their results seem to indicate that this equality is a rather rare phenomenon among n-sections. Their observations were based on the study of the adjoint linear system $|\mathcal{O}(K_X) \otimes \mathcal{L}^{n-1}|$ which in the case of surfaces was done by Lanteri and Palleschi in [LP] and in the case of higher dimensions by Sommese [So].

In the present paper their methods are improved and the difference $b_k(X') - b_k(X)$ is estimated in terms of dimensions of other cohomology groups $H^p(\Omega^q X \otimes \mathcal{L}^s)$, see (1.9). Using this estimate we complete the results of Section 4 of [LS] and prove that for $n \geq k$ no smooth n-section of an ample and spanned line bundle satisfies (0.1), see (2.10). We prove also that the only non-trivial smooth n-section of a line bundle over a projective space which satisfies (0.1) occurs for k odd and it is a double cover by a

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smooth hyperquadric, (2.7), whilst non-trivial smooth n-sections of line bundles over the hyperquadric $\mathbf{Q}^k$, $k \geq 3$, never satisfy (0.1), see (2.9).

The notation used in the present paper is consistent with [LS].

1. **General properties of n-sections.** We summarize here general properties of smooth n-sections

PROPOSITION 1.1. *Let $\pi: X' \longrightarrow X$ be a smooth n-section of a line bundle $\mathcal{L}$ with the total space $\mathbf{p}: L \longrightarrow X$. Then*

(a) $\mathcal{O}_L(X') \simeq \mathcal{O}_L(nX_0) \simeq \mathbf{p}^*(\mathcal{L}^n)$, where X_0 denotes a divisor of zero section in L;

(b) $\pi_*\mathcal{O}_{X'} \simeq \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-n+1}$;

(c) $\mathcal{O}(K_{X'}) \simeq \pi^*\big(\mathcal{O}(K_X) \otimes \mathcal{L}^{n-1}\big)$;

(d) If $\mathcal{L}$ is ample then the induced map of complex cohomology

$$\pi^*: H^i(X, \mathbb{C}) \longrightarrow H^i(X', \mathbb{C})$$

is isomorphism for $i \neq k$ and injection for $i = k$.

PROOF. (d) is proved in a more general set-up in [La], Theorem 2.1; we sketch another proof of it in (1.7). The remaining properties seem to be known for specialists but I was not able to find any appropriate reference for them. To prove them we compactify L by adding a section at infinity. Namely, take a projectivization $\bar{L} = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$ with the projection $\bar{\mathbf{p}}: \bar{L} \longrightarrow X$ (we understand $\mathbb{P}$ as in [Ha], Section II.7), then $\bar{L}$ has a zero section $X_0 = \mathbb{P}(\mathcal{L})$ and a section at infinity $X_\infty = \mathbb{P}(\mathcal{O}_X)$ and $L = \bar{L} - X_\infty$. The relative very ample line bundle $\mathcal{O}_{\bar{L}}(1)$ is then isomorphic to $\mathcal{O}_{\bar{L}}(X_0)$ and therefore $\mathcal{O}_{\bar{L}}(X_0)_{|X_0} \simeq \mathcal{L}$, $\mathcal{O}_{\bar{L}}(X_0)_{|X_\infty} \simeq \mathcal{O}$. Now using the formula $\operatorname{Pic} \bar{L} = \bar{\mathbf{p}}^* \operatorname{Pic} X + \mathbb{Z}\,[X_0]$ we obtain the following identities:

$$\mathcal{O}_{\bar{L}}(X_0 - X_\infty) \simeq \bar{\mathbf{p}}^*\mathcal{L} \text{ and } \mathcal{O}_{\bar{L}}(X') \simeq \mathcal{O}_{\bar{L}}(nX_0).$$

Therefore $\mathcal{O}_{\bar{L}}(X' - nX_\infty) \simeq \bar{\mathbf{p}}^*\mathcal{L}^n$ which proves (a). If we use a formula for the canonical divisor on a projectivization and the adjunction formula we obtain (c) (it will be obtained below explicitly, see (1.4)). As for (b) let us note that $\mathcal{O}_{\bar{L}}(nX_0)_{|X'} \simeq \pi^*\mathcal{L}^n$ and thus we obtain the following (divisorial) sequence of sheaves on $\bar{L}$

$$0 \longrightarrow \mathcal{O}_{\bar{L}} \longrightarrow \mathcal{O}_{\bar{L}}(nX_0) \longrightarrow i_*\pi^*\mathcal{L}^n \longrightarrow 0$$

where $i: X' \longrightarrow \bar{L}$ is the embedding. The direct image $\mathbf{p}_*$ of this sequence is

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\alpha} S^n(\mathcal{O}_X \oplus \mathcal{L}) = \mathcal{O}_X \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^n \longrightarrow \mathcal{L}^n \otimes \pi_*\mathcal{O}_{X'} \longrightarrow 0.$$

We claim that this sequence splits. Indeed, note that we have a splitting $\beta: \mathcal{O}_X \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^n \longrightarrow \mathcal{O}_X$ which comes from restricting sections of $\mathcal{O}_{\bar{L}}(nX_0)$ to X_∞. Since X' does not meet X_∞ it follows that $\beta\,\alpha \neq 0$ which implies the splitting. Now (b) follows easily.

REMARK 1.2. I do not know whether (b) is a sufficient condition for a finite morphism $\pi: X' \longrightarrow X$ to be a smooth n-section of a line bundle $\mathcal{L}$; for $n \leq 3$ it is, see [Fu] (2.1).

For a complex manifold Y by ΩY we denote the sheaf of holomorphic 1-forms on Y and for $q = 0, 1, \ldots, \dim Y$, by $\Omega^q Y$ we denote the sheaf of holomorphic q-forms $\wedge^q(\Omega Y)$. Let us recall that $h^{pq}(Y)$ denotes $\dim H^{pq}(Y) = \dim H^p(Y, \Omega^q Y)$. The following observation is the key technical tool of the present note.

LEMMA 1.3. *Let $\pi : X' \longrightarrow X$ be a smooth n-section of a line bundle $\mathcal{L}$. Then for $q = 1, \ldots, k+1$ we have the following two exact sequences of locally free $O_{X'}$-modules, the middle term of each of them is the same:*

$$(I'_q) \qquad 0 \longrightarrow \pi^*(\Omega^q X \otimes \mathcal{L}^n) \longrightarrow V_q \otimes \pi^* \mathcal{L}^n \longrightarrow \pi^*(\Omega^{q-1} X \otimes \mathcal{L}^{n-1}) \longrightarrow 0$$

$$(II'_q) \qquad 0 \longrightarrow \Omega^{q-1} X' \longrightarrow V_q \otimes \pi^* \mathcal{L}^n \longrightarrow \Omega^q X' \otimes \pi^* \mathcal{L}^n \longrightarrow 0$$

(where $V_q = \wedge^q(\Omega L_{|X'})$).

PROOF. The cokernel of the differential of $\mathbf{p}$: $\mathbf{p}^*(\Omega X) \longrightarrow \Omega L$ can be identified with 1-forms cotangent to fibers of $\mathbf{p}$ which constitute an invertible sheaf isomorphic to $\mathbf{p}^* \mathcal{L}^{-1}$. Thus we obtain an exact sequence of O_L-sheaves

$$0 \longrightarrow \mathbf{p}^* \Omega X \longrightarrow \Omega L \longrightarrow \mathbf{p}^* \mathcal{L}^{-1} \longrightarrow 0.$$

By restricting this sequence to X', then taking its q-th exterior power (see eg. [Hi], 4.1.3) and finally twisting by $\pi^* \mathcal{L}^n$, we obtain I'_q with $V_q = \wedge^q(\Omega L_{|X'})$. On the other hand since the normal bundle to X' is isomorphic to $\pi^*(\mathcal{L}^n)$ (because of (1.1)(a)) we obtain the following exact sequence of $O_{X'}$-sheaves

$$0 \longrightarrow \pi^* \mathcal{L}^{-n} \longrightarrow \Omega L_{|X'} \longrightarrow \Omega X' \longrightarrow 0$$

whose q-th exterior power twisted by $\pi^* \mathcal{L}^n$ gives II'_q.

REMARK 1.4. Combining I'_{q+1} with II'_{q+1} we get (1.1)(c).

LEMMA 1.5. *Let $\pi : X' \longrightarrow X$ be a smooth n-section of a line bundle $\mathcal{L}$, then for any $q = 1, \ldots, k+1$ we have the following two exact sequences of locally free O_X-modules*

$$(I_q) \qquad 0 \longrightarrow (\Omega^q X \otimes \mathcal{L}) \oplus \cdots \oplus (\Omega^q X \otimes \mathcal{L}^n) \longrightarrow W_q$$
$$\longrightarrow \Omega^{q-1} X \oplus \cdots \oplus (\Omega^{q-1} X \otimes \mathcal{L}^{n-1}) \longrightarrow 0$$

$$(II_q) \qquad 0 \longrightarrow \pi_*(\Omega^{q-1} X') \longrightarrow W_q \longrightarrow \pi_*(\Omega^q X' \otimes \pi^* \mathcal{L}^n) = \pi_*(\Omega^q X') \otimes \mathcal{L}^n \longrightarrow 0$$

PROOF. Take the direct image π_* of (I'_q) and (II'_q), and use (1.1)(b). Note that here (and in the subsequent proofs) we use the fact that π is finite and therefore the higher derived functors of π_* are trivial.

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LEMMA 1.6. *For X, X′ and $\mathcal{L}$ as above the following identity for the Euler-Poincaré characteristic χ holds*

$$\chi(\Omega^{q-1}X') = \chi(\Omega^{q-1}X) + \sum_{i=1}^{n-1}\chi(\Omega^{q-1}X\otimes\mathcal{L}^i)$$
$$+\sum_{j=0}^{k-q}(-1)^j\sum_{i=1}^{n-1}\Big(\chi(\Omega^{q+j}X\otimes\mathcal{L}^{nj+i}) - \chi(\Omega^{q+j}X\otimes\mathcal{L}^{n(j+1)+i})\Big)$$

PROOF. Combining (I_q) and (II_q) we obtain for $q\leq k+1$ and arbitrary s

$$\chi(\Omega^{q-1}X'\otimes\pi^*\mathcal{L}^s) = \chi(\pi_*\Omega^{q-1}X'\otimes\mathcal{L}^s)$$
$$= \chi(\Omega^{q-1}X\otimes\mathcal{L}^s) + \sum_{i=1}^{n-1}\chi(\Omega^{q-1}X\otimes\mathcal{L}^{i+s})$$
$$+\sum_{i=1}^{n}\chi(\Omega^q X'\otimes\mathcal{L}^{i+s}) - \chi(\pi_*\Omega^q X\otimes\mathcal{L}^{n+s})$$

and the lemma follows by (descending) induction on q.

From now on we assume the line bundle $\mathcal{L}$ to be ample.

SKETCH OF PROOF OF (1.1)(D) 1.7. Combining Lemma 1.5 together with Kodaira vanishing theorem as in a well known proof of Lefschetz hyperplane section theorem—see [GH] Section 1.2—we obtain a proof of (d). Namely, if $\mathcal{L}$ is ample then $H^p(X,\Omega^qX\otimes\mathcal{L}^i) = H^p(X',\Omega^qX'\otimes\pi^*\mathcal{L}^i) = 0$ if either $p+q>k$ and $i>0$ or $p+q<k$ and $i<0$. Thus I_q and II_q imply equality $h^{p,q-1}(X) = h^{p,q-1}(X')$ for $p>k-q+1$ whereas the same sequences twisted by $\mathcal{L}^{-n}$ imply $h^{p,q}(X) = h^{p,q}(X')$ for $p<k-q$ and also the desired injectivity for $p=k-q$. Thus we have the following:

Let $\pi: X'\longrightarrow X$ is an n-section of an ample line bundle $\mathcal{L}$ then $h^{pq}(X') = h^{pq}(X)$ for $p+q\neq k$ and $h^{pq}(X')\geq h^{pq}(X)$ for $p+q=k$.

COROLLARY 1.8. Under the above assumptions, for $p+q=k$

$$h^{pq}(X') - h^{pq}(X) = (-1)^p\big(\chi(\Omega^qX') - \chi(\Omega^qX)\big).$$

REMARK 1.9. Note that using Lemma 1.6 we can estimate the difference between $h^{pq}(X')$ and $h^{pq}(X)$ in terms of $\chi(\Omega^rX\otimes\mathcal{L}^s)$.

2. ***n*-sections satisfying (0.1).** We deal with the problem formulated in [LS]: find when a smooth n-section $\pi: X'\longrightarrow X$ of an ample line bundle $\mathcal{L}$ satisfies (0.1). First let us note that the answer we are looking for should depend only on the manifold X, the line bundle $\mathcal{L}$ and the number n; X' is then obtained as divisor from the system $|\mathbf{p}^*\mathcal{L}^n|$ on L, see (1.1)(a). Here are some necessary conditions for (0.1) to occur.

LEMMA 2.1. *Let $\pi: X' \longrightarrow X$ be a smooth n-section of a line bundle $\mathcal{L}$. If $\mathcal{L}$ is ample and $b_k(X) = b_k(X')$ then for $q = 1, \ldots, k+1$ each of the following equations E_q is satisfied*

$$\sum_{j=-1}^{k-q}(-1)^j \sum_{i=1}^{n-1} \chi(\Omega^{q+j}X \otimes \mathcal{L}^{n(j+1)+i}) = 0$$

PROOF. In view of (1.8) Lemma 1.6 implies

$$\sum_{j=-1}^{k-q}(-1)^j \sum_{i=1}^{n-1} \chi(\Omega^{q+j}X \otimes \mathcal{L}^{n(j+1)+i}) + \sum_{j=-1}^{k-q-1}(-1)^j \sum_{i=1}^{n-1} \chi(\Omega^{q+1+j}X \otimes \mathcal{L}^{n(j+1)+i}) = 0$$

and the lemma follows by (descending) induction on q.

REMARK 2.2. The terms $\chi(\Omega^r X \otimes \mathcal{L}^s)$ appear in the equation E_q only if $r \geqq q-1$ and $n(r-q+1) \leq s \leq n(r-q+2)-1$.

LEMMA 2.3. *Let $\pi: X' \longrightarrow X$ be a smooth n-section of a line bundle $\mathcal{L}$. If $\mathcal{L}$ is ample and $b_k(X) = b_k(X')$ then*

(a) $H^p(\Omega^q X \otimes \mathcal{L}^s) = 0$ *for* $p+q=k$ *and* $s = 1, \ldots, n-1$.

(b) $h^0(\Omega^{k-1}X \otimes \mathcal{L}) + \cdots + h^0(\Omega^{k-1}X \otimes \mathcal{L}^{n-1}) = h^0\big(\mathcal{O}(K_X) \otimes \mathcal{L}^{n+1}\big) + \cdots \cdots + h^0\big(\mathcal{O}(K_X) \otimes \mathcal{L}^{2n-1}\big)$.

PROOF. Take the cohomology sequences for (I_{q+1}) and (II_{q+1})

$$(I^p_{q+1}) \qquad \cdots \longrightarrow \sum_{s=1}^{n} H^p(\Omega^{q+1}X \otimes L^s) \longrightarrow H^p(W_{q+1}) \longrightarrow H^p(\Omega^q X) \oplus \sum_{s=1}^{n-1} H^p(\Omega^q X \otimes \mathcal{L}^s) \longrightarrow \sum_{s=1}^{n} H^{p+1}(\Omega^{q+1}X \otimes L^s) \longrightarrow \cdots$$

$$(II^p_{q+1}) \qquad \cdots \longrightarrow H^{p-1}(\pi_*\Omega^{q+1}X' \otimes \mathcal{L}^n) \longrightarrow H^p(\pi_*\Omega^q X') \longrightarrow H^p(W_{q+1}) \longrightarrow H^p(\pi_*\Omega^{q+1}X' \otimes \mathcal{L}^n) \longrightarrow \cdots$$

For $p+q=k$ the last terms in each of these two sequences vanish (Kodaira vanishing) thus we have an epimorphism

$$H^p(\pi_*\Omega^q X') \longrightarrow H^p(\Omega^q X) \oplus \sum_{s=1}^{n-1} H^p(\Omega^q X \otimes \mathcal{L}^s) \longrightarrow 0$$

which proves (a). This also implies that the middle map of (II^p_{q+1}) is an injection, therefore for $p+q=k-1$ we have

$$\cdots \longrightarrow H^{p-1}(\pi_*\Omega^{q+1}X' \otimes \mathcal{L}^n) \longrightarrow H^p(\pi_*\Omega^q X') \longrightarrow H^p(W_{q+1}) \longrightarrow H^p(\pi_*\Omega^{q+1}X' \otimes \mathcal{L}^n) \longrightarrow 0.$$

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Also for $p+q=k-1$ the last term of (I^p_{q+1}) vanishes, so

$$\cdots \longrightarrow \sum_{s=1}^{n} H^p(\Omega^{q+1}X\otimes \mathcal{L}^s)\longrightarrow H^p(W_{q+1})\longrightarrow H^p(\Omega^q X)\oplus \sum_{s=1}^{n-1} H^p(\Omega^q X\otimes \mathcal{L}^s)\longrightarrow 0.$$

For $p=0$, $q=k-1$ the above two sequences imply (b).

COROLLARY 2.4 ((4.3) IN [LS]). *For X, $\mathcal{L}$ and n as above*

$$H^0\big(\mathcal{O}(K_X)\otimes \mathcal{L}^{n-1}\big)=0.$$

2.5. Now assume that $\mathcal{L}$ is spanned and $n\geq k\geq 3$. This situation was discussed in Section 4 of [LS]. In view of (2.2) the results of Sommese ([So], (4.1)) imply that the equality $b_k(X')=b_k(X)$ is possible only in one of the following cases (cf. [LS], 4.4)

(i) $n=k, k+1$ and $X\cong \mathbf{P}^k$, $\mathcal{L}\simeq \mathcal{O}(1)$;

(ii) $n=k$, X is a smooth hyperquadric in $\mathbf{P}^{k+1}$ and $\mathcal{L}\simeq \mathcal{O}(1)_{|X}$;

(iii) $n=k$ and $(X,\mathcal{L})$ is a scroll over a smooth curve C i.e. $X\cong \mathbb{P}(\mathcal{E})$ for some rank-k vector bundle on C and $\mathcal{L}\simeq \mathcal{O}_X(1)$.

In (2.7) and (2.9) below we examine smooth n-sections of all line blundles over projective space and hyperquadric but let us note that the above cases (i) and (ii) can be eliminated by using the following simple observation: the total space of the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^k$ is isomorphic to $\mathbf{P}^{k+1}$ minus a point, the projection from the point is the projection on the base of the total space which is $\mathbf{P}^k$. Accordingly any n-section of $\mathcal{O}(1)$ over $\mathbf{P}^k$ is a divisor in $\mathbf{P}^{k+1}$ of degree n. Similarly, any n-section of $\mathcal{O}(1)$ over a hyperquadric in $\mathbf{P}^{k+1}$ is an intersection of a cone over this hyperquadric in $\mathbf{P}^{k+2}$ with a divisor in $\mathbf{P}^{k+2}$ of degree n. In both cases the cohomology of the n-section can be computed easily.

LEMMA 2.6. *The pair $(X,\mathcal{L})$ in (2.5)(iii) does not satisfy the conditions (2.3)(b).*

PROOF. We start with two sequences of sheaves on X

$$0\longrightarrow \mathcal{O}_X\longrightarrow \mathcal{L}\otimes a^*\mathcal{E}^\vee\longrightarrow T\mathcal{E}\longrightarrow 0$$
$$0\longrightarrow T\mathcal{E}\longrightarrow TX\longrightarrow a^*\mathcal{O}(-K_C)\longrightarrow 0$$

where $a\colon X=\mathbb{P}(\mathcal{E})\longrightarrow C$ is the projection and $T\mathcal{E}$ is a relative tangent sheaf and TX is the tangent sheaf. From these sequences we infer that $\mathcal{O}(K_X)\simeq \mathcal{L}^{-k}\otimes a^*\big(\mathcal{O}(K_C)\otimes \det \mathcal{E}\big)$. Now using the identity $\Omega^{k-1}X=TX\otimes \mathcal{O}(K_X)$ we twist the above sequences to get

$$0\longrightarrow \mathcal{O}(K_X)\otimes \mathcal{L}^s\longrightarrow \mathcal{L}^{s+1}\otimes a^*\mathcal{E}^\vee\otimes \mathcal{O}(K_X)\longrightarrow T\mathcal{E}\otimes \mathcal{O}(K_X)\otimes \mathcal{L}^s\longrightarrow 0$$
$$0\longrightarrow T\mathcal{E}\otimes \mathcal{O}(K_X)\otimes \mathcal{L}^s\longrightarrow \Omega^{k-1}X\otimes \mathcal{L}^s\longrightarrow a^*\mathcal{O}(-K_C)\otimes \mathcal{O}(K_X)\otimes \mathcal{L}^s\longrightarrow 0$$

If $s=1,\ldots,k-1$ then the line bundle term in each of these sequences has trivial cohomology (because it is a line bundle whose restriction to any fiber of a (which is isomorphic to $\mathbf{P}^{k-1}$) is $\mathcal{O}(s-k)$) and therefore $H^0(\Omega^{k-1}X\otimes \mathcal{L}^s)=H^0\big(\mathcal{L}^{s+1}\otimes a^*\mathcal{E}^\vee\otimes \mathcal{O}(K_X)\big)$ where the latter cohomology is clearly 0 for $s<k-1$ (by the same argument

on restricting to fibers of a). On the other hand for $s = k-1$ using the formula on $\mathcal{O}(K_X)$ and the identity $\mathcal{E}^\vee \otimes \det \mathcal{E} = \wedge^{k-1}\mathcal{E}$ we obtain

$$H^0(\Omega^{k-1}X \otimes \mathcal{L}^s) = H^0\big(\mathcal{L}^k \otimes a^*\mathcal{E}^\vee \otimes \mathcal{O}(K_X)\big) = H^0\big(\mathcal{E}^\vee \otimes \det \mathcal{E} \otimes \mathcal{O}(K_C)\big)$$
$$= H^0\big(\wedge^{k-1}\mathcal{E} \otimes \mathcal{O}(K_C)\big).$$

Let us note that $\wedge^{k-1}\mathcal{E}$ is ample and spanned (because $\mathcal{L}$ is) and from Riemann-Roch we get

$$\chi\big(\wedge^{k-1}\mathcal{E} \otimes \mathcal{O}(K_C)\big) = k\big(g(C) - 1\big) + (k-1)\deg \mathcal{E}$$

where $g(C)$ is the genus of the curve C.

To the other end, similarly using the identity on $\mathcal{O}(K_X)$, we find out that

$$H^0\big(\mathcal{O}(K_X) \otimes \mathcal{L}^{k+1}\big) = H^0\big(\mathcal{L} \otimes a^*(\det \mathcal{E} \otimes \mathcal{O}(K_C))\big) = H^0\big(\mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{O}(K_C)\big).$$

As above we note that $\mathcal{E} \otimes \det \mathcal{E}$ is ample and spanned, and

$$\dim H^0\big(\mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{O}(K_C)\big) \geq \chi\big(\mathcal{E} \otimes \det \mathcal{E} \otimes \mathcal{O}(K_C)\big) = k\big(g(C) - 1\big) + (k+1)\deg \mathcal{E}.$$

Now we use the following two facts which will be proved below.

(2.6.1) If $\mathcal{F}$ is a spanned vector bundle on C then

$$\dim H^0\big(\mathcal{F} \otimes \mathcal{O}(K_C)\big) \leq \chi\big(\mathcal{F} \otimes \mathcal{O}(K_C)\big) + \operatorname{rank} \mathcal{F} + 1.$$

(2.6.2) If $\mathcal{F}$ is an ample and spanned vector bundle on C then

$$\deg \mathcal{F} \geq \operatorname{rank} \mathcal{F}.$$

To obtain the following estimate

$$H^0\big(\mathcal{O}(K_X) \otimes \mathcal{L}^{k+1}\big) - H^0(\Omega^{k-1}X \otimes \mathcal{L}^s) \geq 2\deg \mathcal{E} - \operatorname{rank} \mathcal{E} - 1 > 0$$

which concludes the proof of the lemma.

PROOF OF (2.6.1). Let r be the rank of $\mathcal{F}$. We can choose $r+1$ sections of $\mathcal{F}$ spanning it and therefore we produce a sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^{r+1} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{K}$ is a kernel of the evaluation map. Twisting the sequence by $\mathcal{O}(K_C)$ and considering cohomology of the twisted sequence we obtain $\dim H^1\big(\mathcal{F} \otimes \mathcal{O}(K_C)\big) \leq r+1$, which proves (2.6.1).

PROOF OF (2.6.2). A direct proof of it is quite easy but let us show that (2.6.2) follows as an easy consequence of a result on coverings: namely note that the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is ample and spanned and taking a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)|$ we can produce a covering of $\mathbf{P}^{\operatorname{rank}\mathcal{F}}$ whose degree is equal to $\deg \mathcal{F}$. Since $H^2\big(\mathbb{P}(\mathcal{F}), \mathbb{C}\big) \neq \mathbb{C}$, (2.6.2) follows from the main theorem of [La].

The next result is on n-sections of line bundles over $\mathbf{P}^k$.

THEOREM 2.7. *The only smooth n-section* ($n \geq 2$) *of a line bundle over* $\mathbf{P}^k$ *which has the same complex cohomology as* $\mathbf{P}^k$ *is a double cover of* $\mathbf{P}^k$ *by a smooth hyperquadric* $\mathbf{Q}^k \subseteq \mathbf{P}^{k+1}$ *for k odd.*

PROOF. Let $\pi: Y \longrightarrow \mathbf{P}^k$ be a smooth n-section of a line bundle $O(d)$ which has the same complex cohomology as $\mathbf{P}^k$, clearly $d > 0$. In view of the digression following (2.5) we will be done if we prove that k is odd, $d = 1$ and $n = 2$. Let $\Omega^q(a)$ denote the sheaf of q-forms on $\mathbf{P}^k$ twisted by $O(a)$. Then from Bott formulas it follows that

$$\dim H^0\big(\Omega^q(a)\big) = \begin{bmatrix} a+k-q \\ a \end{bmatrix} \begin{bmatrix} a-1 \\ q \end{bmatrix} \text{ for } a > q,$$

$H^q(\Omega^q) = \mathbb{C}$, and the remaining cohomology $H^p\big(\Omega^q(a)\big)$ vanish for $a \geq 0$. Since $b_k(Y) = b_k(\mathbf{P}^k)$, in view of (2.1), k, n and d must satisfy each of the following $k+1$ equations E_q:

$$\sum_{j=-1}^{k-q} (-1)^j \sum_{i=1}^{n-1} \dim H^0\Big(\Omega^{q+j}\big((nj+n+i)d\big)\Big) = 0.$$

Note that if $\big((k-q)n+2n-1\big)\cdot d < k+1$ then all terms of this equation vanish. Because of the equation E_{k+1} (cf. (2.4)) we may assume that $(n-1)d < k+1$ and therefore, since $n \geq 2$, we may choose q such that

$$\big((k-q)n+2n-1\big)\cdot d \geq k+1$$
$$\big((k-q)n+n-1\big)\cdot d < k+1.$$

then from Bott formulas it follows that all terms of the equation E_q must vanish except the following terms (compare with (2.2)):

$$\sum_{i=1}^{n-1} \dim H^0\Big(\Omega^k\big((n(k-q+1)+i)d\big)\Big)$$

and possibly

$$\dim H^0\Big(\Omega^{k-1}\big(((k-q+1)n-1)d\big)\Big)$$

where the latter term is non-zero only if $\big((k-q+1)n-1\big)d = k$ and then it is equal to $k+1$. Since these two terms must cancel each other in E_q it follows that $d = 1$, $n = 2$ and subsequently k is odd.

COROLLARY 2.8. *Assume that Y is a projective manifold of dimension k,* $k \geq 4$, *which has the same Betti numbers as* $\mathbf{P}^k$. *If there exists a spanned line bundle* $\mathcal{L}$ *over Y whose degree d (i.e. the self-intersection* $(c_1\mathcal{L})^k$*) satisfies inequalities* $1 \leq d \leq 3$ *then either* $Y \cong \mathbf{P}^k$ *or k is odd and Y is isomorphic to a smooth hyperquadric.*

PROOF. $\mathcal{L}$ is ample and spanned and therefore a subsystem of $|\mathcal{L}|$ defines a finite map onto $\mathbf{P}^k$. The degree of this map is equal to $d \leq 3$ hence, in view of 3.2 from [La], Y is a d-section of a line bundle over $\mathbf{P}^k$, so (2.7) applies.

Similarly as (2.7) we prove

THEOREM 2.9. *For $k \geq 3$ there is no non-trivial smooth n-section of a line bundle over a smooth hyperquadric $\mathbf{Q}^k \subset \mathbf{P}^{k+1}$ which has the same Betti numbers as $\mathbf{Q}^k$.*

PROOF. We use the same notation as above in the proof of (2.7), $\mathcal{L} \simeq O(d)$, $d \geq 1$. The needed information on cohomology of twisted holomorphic forms on $\mathbf{Q}^k$ is summarized in Table 3 of [Sn]. First note that since $H^{k-q}\big(\mathbf{Q}^k, \Omega^q\mathbf{Q}^k(2q-k)\big) = \mathbb{C}$ for $q > k/2$, then the conditions (2.3)(a) imply that

— if k is odd then d is even, n is arbitrary,

— if k is even then d is odd and $n = 2$.

(In both cases $d(n-1) < k$ because of (2.4).)

Therefore the terms $\chi\big(\Omega^r\mathbf{Q}^k(m)\big)$ which appear in any equation E_q from (2.1) have m of opposite parity than k, thus (use again the Table 3 in [ibid]) the positive cohomology of such $\Omega^r\mathbf{Q}^k(m)$ vanish (the cohomology in boxes in the table can be ignored). Now we conclude the proof as (2.7): we choose the largest q such that $d\big(n(k-q+2)-1\big) > k$ and comparing the Table in [ibid] with (2.2) conclude that E_q cannot be satisfied. Indeed, since $k \geq d\big(n(k-q+1)-1\big)$, it follows that the only (possibly) non-vanishing terms in E_q are

$$\sum_{i=1}^{n-1} H^0\Big(\mathbf{Q}^k, \Omega^k\mathbf{Q}^k\Big(\big(n(k-q+1)+i\big)d\Big)\Big)$$

and because $k < d\big(n(k-q+2)-1\big)$ at least one of the components must be positive.

As a conclusion of the above discussion ((2.5), (2.6), (2.7), (2.9)) we obtain

THEOREM 2.10. *If X' is a smooth n-section of an ample and spanned line bundle over X and $n \geq k = \dim X \geq 3$ then $b_k(X') > b_k(X)$.*

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