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The Brauer group of cubic surfaces

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1. Let V be a non-singular rational surface defined over an algebraic number field k. There is a standard conjecture that the only obstructions to the Hasse principle and to weak approximation on V are the Brauer–Manin obstructions. A prerequisite for calculating these is a knowledge of the Brauer group of V; indeed there is one such obstruction, which may however be trivial, corresponding to each element of $\mathrm{Br}\, V/\mathrm{Br}\, k$. Because k is an algebraic number field, the natural injection

$$\mathrm{Br}\, V/\mathrm{Br}\, k \rightarrow H^1(k, \mathrm{Pic}\, \bar{V})$$

is an isomorphism; so the first step in calculating the Brauer–Manin obstruction is to calculate the finite group $H^1(k, \mathrm{Pic}\, \bar{V})$.

But although finding $H^1(k, \mathrm{Pic}\, \bar{V})$ in any particular case is a finite calculation, the standard method of doing it is frequently tedious; and it would be convenient to have it done once for all. In the case when V is fibred by conics, this has been done by Iskovskih [**4**]. In the case of Del Pezzo surfaces of degree $d > 4$, it is known that H^1 is trivial. The main purpose of this paper is to settle the case $d = 3$, when V is a non-singular cubic surface; from this the case $d = 4$, when V is the intersection of two quadrics in $\mathbb{P}^4$, follows easily. The cases $d = 2$ and $d = 1$ could presumably be dealt with by the same methods, but they have not so far aroused sufficient interest to justify the labour involved. The basic idea is to turn the problem back-to-front, asking what are the conditions on V for a given expression to represent a non-trivial element of $H^1(k, \mathrm{Pic}\, \bar{V})$.

If V is a non-singular cubic surface, it turns out that $H^1(k, \mathrm{Pic}\, \bar{V})$ must be trivial or isomorphic to one of $C_2, C_2 \times C_2, C_3$ or $C_3 \times C_3$, where C_n denotes the cyclic group of order n; which one of these possibilities holds in any particular case can be read off from Lemmas 1, 2, 4 and 5. If V_1, defined over another algebraic number field k_1, is the non-singular intersection of two quadrics in $\mathbb{P}^4$, it turns out that $H^1(k_1, \mathrm{Pic}\, \bar{V}_1)$ must be trivial or isomorphic to C_2 or $C_2 \times C_2$; which one of these possibilities holds in any particular case can be read off from Lemmas 10 and 11. It is not hard to verify, both for V and for V_1, that all these possibilities actually arise.

Some particular cases of these results can already be found in the literature.

Diagonal cubic surfaces are treated in [1]. Proposition 3·18 of [2] gives a necessary condition for $H^1(k, \operatorname{Pic} \bar{V})$ to be non-trivial when V is an intersection of two quadrics. I am indebted to J.-L. Colliot-Thélène for showing me that the proof of that Proposition can be extended to show that $H^1(k, \operatorname{Pic} \bar{V})$ is killed by 2 when V is an intersection of two quadrics, and by 54 when V is a cubic surface; this gives alternative proofs of Lemma 9 and of a crucial remark in Section 7. Manin [5] has tabulated $H^1(k, \operatorname{Pic} \bar{V})$ in the case when V is a cubic surface and all 27 lines are defined over a cyclic extension of k; and under the same cyclicity condition Urabe [8] has extended the table to the cases when V is a Del Pezzo surface of degree 2 or 1. He has also corrected two errors in Manin's table 1 (pp. 176–7), where the entries $\mathbb{Z}_2$ in column 7 should both be 0.

2. From here to the end of Section 6 we assume that V is a non-singular cubic surface. Let K be the least extension of k which is a field of definition for each of the 27 lines on $\bar{V}$, and denote by G the Galois group of K/k. If G_0 denotes the group of order $2^7 \cdot 3^4 \cdot 5 = 51840$ whose elements are the incidence-preserving permutations of the 27 lines, then G is canonically embedded in G_0; in particular n divides 51840, where n is the order of G.

Write for convenience $P = \operatorname{Pic} V$ and $\bar{P} = \operatorname{Pic} \bar{V}$. In what follows, we shall usually not distinguish between a divisor and its class. It is known that $\bar{P}$ is a free $\mathbb{Z}$-module of rank 7, and that we can choose a base $\lambda_1, \ldots, \lambda_6, \mu$ for it which is orthogonal with respect to intersection number, where the λ_i are mutually skew lines and μ is a twisted cubic which does not meet any λ_i. (If we map $\bar{V}$ to $\mathbb{P}^2$ by blowing down the λ_i, then μ becomes a straight line.) We denote the class of a plane section of V by

$$\pi = 3\mu - \lambda_1 - \ldots - \lambda_6;$$

π is in P. The classes of the 27 lines on $\bar{V}$ are the six λ_i, the fifteen $\mu - \lambda_i - \lambda_j$ and the six $\pi + \lambda_i - \mu$.

The cohomology group we are concerned with is

$$H^1(k, \bar{P}) = \text{Cocycles/Coboundaries},$$

where a cocycle is a function $\phi: G \to \bar{P}$ which satisfies

$$\phi(gg') = g\phi(g') + \phi(g) \tag{1}$$

and a coboundary is a function ϕ which has the special form

$$\phi(g) = \alpha - g\alpha$$

for some fixed α in $\bar{P}$. For any cocycle ϕ we can sum (1) over all values of g' and obtain

$$n\phi(g) = \alpha_0 - g\alpha_0 \quad \text{where} \quad \alpha_0 = \sum \phi(g').$$

It follows that

$$\phi(g) = \beta - g\beta \tag{2}$$

for some β in $\bar{P} \otimes \mathbb{Q}$ and we can even choose β so that $n\beta$ is in $\bar{P}$; moreover this equation determines β up to an arbitrary element of $P \otimes \mathbb{Q}$. It follows that $H^1(k, \bar{P})$ is killed by n, as is well known; so if it is non-trivial it must contain an element of order p where $p = 2$, 3 or 5. It is such elements that we primarily seek.

Let ϕ be an element of $H^1(k, \bar{P})$ of order precisely m, where we take $m > 1$, and let β in $\bar{P} \otimes \mathbb{Q}$ satisfy (2) with $m\beta$ in $\bar{P}$. By hypothesis $m'\beta$ cannot be in $\bar{P}$ for any m' with

$0 < m' < m$. Changing β by an element of $\bar{P}$ changes ϕ by a coboundary, which is harmless; hence for fixed m we need only consider a finite number of candidates for β, and the number of equivalence classes of these under G_0 turns out to be quite small. This leads us to ask how the choice of β constrains the allowable values of G. It does so in two ways. The first, which stops P being too large and hence implicitly stops G being too small, is that $m'\beta$ must not be in $P \otimes \mathbb{Q} + \bar{P}$ for any m' with $0 < m' < m$; and here we can clearly confine ourselves to those m' which divide m. The second constraint, which arises from (2), is that

$$g\beta \equiv \beta \bmod \bar{P} \quad \text{for all } g \text{ in } G; \tag{3}$$

this implies that G cannot be too large.

We can put this second constraint into a more convenient form as follows. For any two elements γ_1, γ_2 of $\bar{P} \otimes \mathbb{Q}$, we denote their intersection number by $(\gamma_1 \cdot \gamma_2)$. Now let λ be any one of the 27 lines on $\bar{V}$; then (3) implies

$$(\lambda \cdot \beta) \equiv (\lambda \cdot g\beta) = (g^{-1}\lambda \cdot \beta) \quad \bmod 1. \tag{4}$$

For $0 \leqslant r < m$ let S_r be the set of lines λ which satisfy

$$(\lambda \cdot \beta) \equiv r/m \quad \bmod 1; \tag{5}$$

then (4) implies that G fixes each S_r. Conversely, if G fixes each S_r then (4) follows for all λ; and this implies (3) because if $(\beta - g\beta)$ has integral intersection number with each λ then it lies in $\bar{P}$.

In what follows, we shall show that the cases $m = 4$, 5, 6 and 9 are all impossible, and for each β of order 2 or 3 in $(\bar{P} \otimes \mathbb{Q})/\bar{P}$ we shall determine the two constraints above on G in terms of the associated rationality properties over k of the 27 lines. All the necessary information about the incidence properties of the 27 lines can be found, for example, in Segre [**6**].

3. In this section we consider the case $m = 2$. Recall that a double-six on $\bar{V}$ consists of two sets of six skew lines such that each line of one set meets five of the six lines of the other set. Any set of six skew lines is contained in a unique double-six.

LEMMA 1. *The order of $H^1(k, \bar{P})$ is even if and only if there is a double-six S on $\bar{V}$ with the following properties*:

(i) *each element of G maps the set S to itself*;

(ii) *if β_0 denotes half the sum of one set of skew lines in S, then β_0 is not in $P \otimes \mathbb{Q} + \bar{P}$.*

Given G and S such that (i) holds, it is easy to check whether (ii) also holds; and this depends only on the partitioning of S into conjugacy classes over k. Moreover, if we choose $\lambda_1, \ldots, \lambda_6$ to be one of the two sets of skew lines in S then the other one consists of the lines $\pi + \lambda_i - \mu$. Hence the two candidates for β_0 in (ii) are $\frac{1}{2}(3\mu - \pi)$ and $\frac{1}{2}(5\pi - 3\mu)$; so it does not matter which one we choose for the test.

Proof. In the recipe of Section 2 we can take β to be of the form

$$\beta = c_0 \mu + c_1 \lambda_1 + \ldots + c_6 \lambda_6, \tag{6}$$

where every c_i is congruent to 0 or $\frac{1}{2}$ mod 1. We can force c_0 to be 0, because if it is $\frac{1}{2}$ we can replace β by $\beta - \frac{1}{2}\pi$, since π is in P. Since the permutation group on the λ_i is a subgroup of G_0, we are only really concerned with q, the number of c_i with

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$1 \leqslant i \leqslant 6$ which are not integers. Since $q = 0$ is ruled out, there are apparently six cases to consider; but we shall show that they actually reduce to only two.

Suppose first that $q = 1$; without loss of generality we can take $\beta = \frac{1}{2}\lambda_1$. In the notation of (5), S_1 consists of λ_1 and the ten other lines that meet it. The property of meeting all the other lines in S_1 picks out λ_1 uniquely within S_1; hence not only is S_1 fixed by G but the same is true of λ_1. It follows that λ_1 is in P and β gives only the trivial element of H^1.

Next suppose that $q = 2$ and take $\beta = \frac{1}{2}(\lambda_1 + \lambda_2)$. Now S_1 consists of λ_1 and λ_2, $\pi + \lambda_1 - \mu$ and $\pi + \lambda_2 - \mu$, and the eight lines like $\mu - \lambda_1 - \lambda_3$. It is easy to verify that these form a double-six, one set of skew lines consisting of λ_1, $\pi + \lambda_1 - \mu$ and the $\mu - \lambda_2 - \lambda_i$ for $i = 3, 4, 5, 6$. Hence G must preserve this double-six. But in this notation $\beta_0 = \pi + \frac{3}{2}(\lambda_1 - \lambda_2)$, and if this were in $P \otimes \mathbb{Q} + \bar{P}$ the same would be true of β – in which case β would induce the trivial element of H^1.

To reduce the cases $q = 3, 4, 5, 6$ to those already considered, we use the alternative base for $\bar{P}$ given by

$$\left.\begin{aligned} \mu' = 2\mu - \lambda_4 - \lambda_5 - \lambda_6, \quad \lambda_1' = \lambda_1, \quad \lambda_2' = \lambda_2, \quad \lambda_3' = \lambda_3, \\ \lambda_4' = \mu - \lambda_5 - \lambda_6, \quad \lambda_5' = \mu - \lambda_4 - \lambda_6, \quad \lambda_6' = \mu - \lambda_4 - \lambda_5, \end{aligned}\right\} \tag{7}$$

which has exactly the same properties as the old one. We have

$$\begin{aligned} \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) &\equiv \tfrac{1}{2}\lambda_4', \\ \tfrac{1}{2}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) &\equiv \tfrac{1}{2}(\lambda_1' + \lambda_4' + \lambda_5' + \lambda_6'), \\ \tfrac{1}{2}(\lambda_2 + \lambda_3 + \lambda_6) &\equiv (\lambda_1' + \lambda_6'), \\ \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) &\equiv \tfrac{1}{2}(\lambda_4' + \lambda_5' + \lambda_6'), \end{aligned}$$

all taken mod $(P \otimes \mathbb{Q} + \bar{P})$. The first two show that the case $q = 4$ reduces to $q = 1$, and the case $q = 5$ directly to $q = 4$ and thence to $q = 1$. The latter two show that the case $q = 3$ reduces to $q = 2$, and the case $q = 6$ directly to $q = 3$ and thence to $q = 2$. It follows from what we have already proved that if the order of H^1 is even then there is a double-six S such that (i) and (ii) hold. Moreover, given an element of order 2 in H^1 there is a canonical construction of the corresponding S; for the element of H^1 determines $\beta \bmod (\frac{1}{2}P + \bar{P})$ and S is then either the S_0 or the S_1 corresponding to that β. (We cannot hope to be more specific than this, because adding $\frac{1}{2}\pi$ to β interchanges S_0 and S_1.)

Conversely, if S satisfies the conditions of the Lemma then β_0 generates an element of H^1 of order 2; and the double-six which we recover from β_0 is just S itself.

COROLLARY 1. *If the order of $H^1(k, \bar{P})$ is even then V contains points defined over k if it contains points over every local field $k_\mathfrak{b}$.*

Proof. Let G_1 be the subgroup of G which fixes each of the two sets of skew lines in S, and let $k_1 \subset K$ be the fixed field of G_1. Clearly $[G : G_1] = 2$ and therefore $[k_1 : k] = 2$. But V contains a set of six skew lines defined over k_1 and therefore satisfies the Hasse principle over k_1; under the conditions of the Corollary V contains points defined over k_1 and therefore points defined over k. ∎

We now turn to the question of how many elements of order 2 there can be in H^1, the answer plainly being $2^q - 1$ for some q. In fact we can only have $q = 0$, 1 or 2 and criteria for distinguishing these three cases are given by Lemma 1 above (which tells when $q \geqslant 1$) and Lemma 2 below (which tells when $q = 2$).

LEMMA 2. *$H^1(k, \bar{P})$ cannot contain more than three elements of order 2, and it contains as many as three if and only if there are three disjoint sets S^1, S^2 and S^3, each consisting of three skew lines and their three transversals, with the following properties:*

(i) *the union of any two of the sets S^i is a double-six;*

(ii) *each of the S^i is fixed under G;*

(iii) *if β^* denotes half the sum of one set of three skew lines in some S^i then no such β^* is in $P \otimes \mathbb{Q} + \bar{P}$.*

Proof. We can assume that H^1 contains at least three elements of order 2, generated by β_1, β_2 and $\beta_1 - \beta_2 = \beta_3$ say. In the light of the proof of Lemma 1, we can assume that $\beta = \frac{1}{2}(\lambda_1 + \ldots + \lambda_6)$ and that β_2 has the form (6) with $c_0 = 0$ and just q of the c_i non-integral. From the proof of Lemma 1 we see that $q = 2$ or 3, and by considering β_3 we find that $6 - q = 2$ or 3 also. Hence $q = 3$. Reordering the λ_i if necessary, we can take

$$\beta_2 = \tfrac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \quad \beta_3 = \tfrac{1}{2}(\lambda_4 + \lambda_5 + \lambda_6).$$

It follows that H^1 cannot contain more than three elements of order 2; for if there were a further one and β_4 were the associated β, then β_4 and $\beta_2 + \beta_4$ could not both have $q = 3$.

Now let $S_r^{(i)}$ be the set defined by (5) with $\beta = \beta_i$. By Lemma 1, each $S_r^{(i)}$ is fixed under G. But now $S_1^{(1)} \cap S_0^{(2)}$ consists of the three lines $\lambda_1, \lambda_2, \lambda_3$ and their three transversals, $S_1^{(1)} \cap S_1^{(2)}$ consists of the three lines $\lambda_4, \lambda_5, \lambda_6$ and their three transversals, and $S_0^{(1)} \cap S_0^{(2)}$ consists of the three lines $\mu - \lambda_1 - \lambda_2$, $\mu - \lambda_1 - \lambda_3$, $\mu - \lambda_2 - \lambda_3$ and their three transversals. Call these sets S^1, S^2 and S^3; then (i) and (ii) are trivial, the double-sixes being those which correspond to the β_i according to Lemma 1. Moreover, each of the β^* is congruent to some $\beta_i \bmod P \otimes \mathbb{Q} + \bar{P}$, and hence (iii) follows from (ii) of Lemma 1.

Conversely, suppose we have three sets S^i satisfying the conditions of the Lemma. We can choose our standard base for $\bar{P}$ so that S^1 consists of $\lambda_1, \lambda_2, \lambda_3$ and their three transversals. Since S^1 can be extended to a double-six in just two ways, S^2 and S^3 must be as in the previous paragraph; and with that notation it follows easily that the β_i generate elements of order 2 in H^1. ∎

4. LEMMA 3. *$H^1(k, \bar{P})$ contains no elements of order 4.*

Proof. Suppose that β corresponds to an element of order 4; then 2β corresponds to an element of order 2 and, as in the proof of Lemma 2, we can assume the canonical base for $\bar{P}$ so chosen that

$$2\beta \equiv \tfrac{1}{2}(\lambda_1 + \ldots + \lambda_6) \mod P \otimes \mathbb{Q} + \bar{P}.$$

We can add $\frac{1}{2}\pi$ to β, permute the λ_i, subtract any element of $\bar{P}$ and replace β by $-\beta$; so we can confine ourselves to the cases

$$\beta = \tfrac{1}{4}(\lambda_1 + \ldots + \lambda_q) - \tfrac{1}{4}(\lambda_{q+1} + \ldots + \lambda_6)$$

for $q = 3, 4, 5, 6$. In each case we derive a contradiction by noting that the sum of the elements of S_r is in P, where S_r is defined by (5); it will follow at once that 2β is in $P \otimes \mathbb{Q} + \bar{P}$, contrary to the hypothesis that 2β has order 2.

If $q = 3$ then S_3 consists of λ_i and $\pi + \lambda_i - \mu$ for $i = 1, 2, 3$; and the sum of these six elements is $2\pi + 4\beta$. If $q = 4$ then S_3 consists of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $\pi + \lambda_5 - \mu$ and $\pi + \lambda_6 - \mu$;

the sum of these six elements is $4\mu-2(\lambda_5+\lambda_6)-4\beta$. If $q=5$ then S_1 consists of λ_6 and $\pi+\lambda_6-\mu$; the sum of these two elements is $2\mu-4\beta$. If $q=6$ then S_3 consists of $\lambda_1, \ldots, \lambda_6$; the sum of these six elements is 4β. In each case it follows at once that 2β is in $P \otimes \mathbb{Q}+\bar{P}$. ∎

5. In this section we consider the case $m=3$. We can again take β to be given by (6), where this time each c_i is 0 or $\pm\frac{1}{3}$ and not all of them are 0. We denote by the symbol $[c, s, t]$ the case when $c_0=c/3$, s of the c_i with $i \neq 0$ are equal to $\frac{1}{3}$ and t of them are equal to $-\frac{1}{3}$. Subtracting $\pi/3$ from β takes $[c, s, t]$ to $[c, 6-s-t, s]$, and replacing β by $-\beta$ takes $[c, s, t]$ to $[-c, t, s]$; so it is enough to consider the 16 cases

$$[c,1,0],[c,2,0] \quad \text{and} \quad [c,2,1] \quad \text{for} \quad c=0,1,2,$$

$$[c,1,1],[c,3,0] \quad \text{and} \quad [c,2,2] \quad \text{for} \quad c=0,1, \quad \text{and} \quad [1,0,0].$$

Now consider the effect of the transformation (7). Taking all congruences mod $\bar{P}$, we have

$$\tfrac{1}{3}\lambda_6' \equiv -\tfrac{1}{3}(2\mu+\lambda_4+\lambda_5),$$
$$\tfrac{1}{3}(2\mu'+\lambda_1') \equiv -\tfrac{1}{3}(2\mu+\lambda_2+\lambda_3)-\tfrac{1}{3}\pi,$$

so that $[0,1,0],[2,1,0]$ and $[2,2,0]$ are equivalent;

$$\tfrac{1}{3}(\mu'+\lambda_1') \equiv \tfrac{1}{3}\pi+\tfrac{1}{3}(2\mu+\lambda_2+\lambda_3-\lambda_1),$$
$$\tfrac{1}{3}(\mu'+\lambda_4') \equiv \tfrac{1}{3}(\lambda_5+\lambda_6-\lambda_4),$$
$$\tfrac{1}{3}(\lambda_1'+\lambda_4') \equiv -\tfrac{1}{3}(2\mu+\lambda_5+\lambda_6-\lambda_1),$$
$$\tfrac{1}{3}(\mu'+\lambda_1'+\lambda_2') \equiv \tfrac{1}{3}\pi-\tfrac{1}{3}(\mu+\lambda_1+\lambda_2-\lambda_3),$$
$$\tfrac{1}{3}(\mu'+\lambda_1'+\lambda_4') \equiv -\tfrac{1}{3}\pi-\tfrac{1}{3}(\lambda_2+\lambda_3-\lambda_4),$$

so that $[1,1,0]$, $[0,2,0]$, $[1,2,0]$, $[0,2,1]$, $[1,2,1]$ and $[2,2,1]$ are all equivalent;

$$\tfrac{1}{3}(\lambda_1'-\lambda_4') \equiv -\tfrac{1}{3}\pi-\tfrac{1}{3}(\mu+\lambda_2+\lambda_3+\lambda_4),$$
$$\tfrac{1}{3}\mu' \equiv -\tfrac{1}{3}(\mu+\lambda_4+\lambda_5+\lambda_6),$$

so that $[0,1,1]$, $[1,3,0]$ and $[1,0,0]$ are all equivalent;

$$\tfrac{1}{3}(\mu'-\lambda_1'+\lambda_4') \equiv \tfrac{1}{3}(-\lambda_1-\lambda_4+\lambda_5+\lambda_6),$$

so that $[1,1,1]$ and $[0,2,2]$ are equivalent; and

$$\tfrac{1}{3}(\lambda_1'+\lambda_2'+\lambda_4') \equiv \tfrac{1}{3}(\mu+\lambda_1+\lambda_2-\lambda_5-\lambda_6),$$

so that $[0,3,0]$ and $[1,2,2]$ are equivalent. We are therefore left with five cases to consider, of which it will turn out that only the last is productive.

In order to state the next Lemma, we need to recall the concept of a *triple-nine*. This is extensively investigated in Henderson [3], though without ever being given a name; but it seems to have been hardly noticed since. A 'nine' consists of three skew lines on $\bar{V}$ together with the six lines each of which meets two of the first three. It is a highly symmetric object, though that is not evident from this description; a 'nine' can be split into three triples of skew lines in two ways, and the same 'nine' is generated by this construction, whichever triple one starts from. Given a 'nine', the remaining 18 lines can be partitioned into two further 'nines' in just one way, and the resulting three sets of lines are called a triple-nine.

LEMMA 4. *The order of $H^1(k, \bar{P})$ is divisible by 3 if and only if there is a triple-nine on $\bar{V}$ with the following properties:*

(i) *each nine is mapped to itself by every element of* G;

(ii) *if* β_0 *denotes one third the sum of three skew lines within a nine, then* β_0 *is not in* $P \otimes \mathbb{Q} + \bar{P}$.

There are 18 candidates for β_0 but it does not matter which one we choose; for if β_0 is any one of them, each of the others is congruent mod $\bar{P}$ to β_0 or $\frac{1}{3}\pi - \beta_0$.

Proof. For the case [0, 1, 0], or one of those equivalent to it, we can without loss of generality take $\beta = \frac{1}{3}\lambda_6$. In the notation of (5), S_2 consists of the single line λ_6, whose class must therefore be in P; so β is in $P \otimes \mathbb{Q}$ and induces the trivial element of H^1.

For the case [1, 1, 0], or one of those equivalent to it, we can without loss of generality take $\beta = \frac{1}{3}(\mu + \lambda_6)$. Now S_2 consists of $\lambda_6, \pi + \lambda_6 - \mu$ and their five common transversals $\mu - \lambda_i - \lambda_6$. The lines λ_6 and $\pi + \lambda_6 - \mu$ are the only members of S_2 which meet five other members of S_2; so as a pair they are preserved by G. Hence their sum, which is $\pi + 3\lambda_6 - 3\beta$, is in P; and hence β is in $P \otimes \mathbb{Q} + \bar{P}$ and induces the trivial element of H^1.

For the case [0, 1, 1], or one of those equivalent to it, we can without loss of generality take $\beta = \frac{1}{3}(\lambda_5 - \lambda_6)$. Now S_1 consists of $\lambda_6, \pi + \lambda_6 - \mu$ and the four lines $\mu - \lambda_i - \lambda_5$ for $i = 1, 2, 3, 4$; their sum, which is $2\pi - 9\beta$, is in P and hence β is in $P \otimes \mathbb{Q}$ and induces the trivial element of H^1.

For the case [1, 1, 1] or one of those equivalent to it, we can without loss of generality take $\beta = \frac{1}{3}(\mu + \lambda_5 - \lambda_6)$. Now the lines $\pi + \lambda_6 - \mu, \mu - \lambda_5 - \lambda_6$ and λ_5 are coplanar, and each S_r consists of one of these lines and the eight lines not in this plane that meet it. It follows that each of these lines is in P, whence β is in $P \otimes \mathbb{Q}$ and induces the trivial element of H^1.

There remain the equivalent cases [0, 3, 0] and [1, 2, 2]. Without loss of generality we can take $\beta = \frac{1}{3}(\lambda_4 + \lambda_5 + \lambda_6)$. It is now easy to verify that S_0, S_1 and S_2 constitute a triple-nine and that β is equal to one of the candidates for β_0; moreover, this argument can be reversed because any two triple-nines can be taken into one another by the action of G_0.

LEMMA 5. *For* $H^1(k, \bar{P})$ *to have more than two elements of order* 3, *it is necessary and sufficient that* G *should be a group of order* 3, *each of whose orbits consist of three coplanar lines. In this case* $H^1(k, \bar{P})$ *is isomorphic to* $C_3 \times C_3$.

Proof. After the proof of Lemma 4, if H^1 contains more than two elements of order 3 we can assume that one is generated by

$$\beta_1 = \tfrac{1}{3}(\lambda_4 + \lambda_5 + \lambda_6)$$

and another is generated by some β_2 not congruent to a multiple of β_1. Thus we can assume that β_2 is of type [0, 3, 0] or [1, 2, 2], and each of $\beta_2 \pm \beta_1$ must belong to one of the equivalent types [0, 3, 0], [0, 3, 3], [0, 0, 3], [1, 2, 2] or [2, 2, 2] mod $\bar{P}$. It is easy to check that β_2 cannot be in [0, 3, 0] and if it is in [1, 2, 2] then after permuting $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4, \lambda_5, \lambda_6$ if necessary we have

$$\beta_2 = \tfrac{1}{3}(\mu + \lambda_1 - \lambda_2 + \lambda_4 - \lambda_5).$$

Now the intersection of an S_q for β_1 with an S_r for β_2 consists of three coplanar lines for each q, r, and it is easy to see that there are only three elements of G_0 which fix each of these triples.

Conversely, if G is of this form then by the tables in [7] we can assume that a generator of it is given by

$$\lambda_1' = \mu - \lambda_1 - \lambda_2, \quad \lambda_2' = \mu - \lambda_2 - \lambda_3, \quad \lambda_3' = \mu - \lambda_1 - \lambda_3,$$
$$\lambda_4' = \pi + \lambda_5 - \mu, \quad \lambda_5' = \pi + \lambda_6 - \mu, \quad \lambda_6' = \pi + \lambda_4 - \mu.$$

Since we have (3) and we know that 3β is in $\bar{P}$, we must have

$$\beta \equiv \tfrac{1}{3}\{c_0(\mu - \lambda_1 + \lambda_2 - \lambda_4 + \lambda_5) + c_1(\lambda_1 + \lambda_2 + \lambda_3) + c_2(\lambda_4 + \lambda_5 + \lambda_6)\}$$

$\bmod \bar{P}$ for some integers c_0, c_1, c_2. Only nine of these values of β are incongruent $\bmod P \otimes \mathbb{Q} + \bar{P}$, and one of them is congruent to 0; so H^1 can have at most eight elements of order 3, and it is easy to check that it does indeed have eight. ▌

In what follows, it will be useful to have further information about the G-orbits of the 27 lines.

LEMMA 6. *If the order of $H^1(k, \bar{P})$ is divisible by 3 then every G-orbit of lines on $\bar{V}$ consists of three coplanar lines, their complement in a nine which contains them, or a nine.*

Proof. Let S', S'' and S''' be the triple-nine constructed in Lemma 4, let S be a G-orbit contained in S' and suppose that S contains q lines. By Lemma 4, no set of three skew lines within a nine can be fixed by G.

If $q = 1$ then S is a single line λ, and the three lines in S'' which meet λ form a skew set fixed by G, contradicting what has just been said. If $q = 2$ we have two cases: S can consist of two coplanar lines, in which case the third line in their plane is defined over G; or it consists of two skew lines, in which case the line in S' skew to both of them is defined over G. In either case the impossibility of $q = 2$ follows from that of $q = 1$. If $q = 3$, transitivity shows that the lines of S are either coplanar or mutually skew, and the second case is forbidden. If $q = 4$, suppose each line of S meets s of the others; we have $s \leqslant 3$, and there are just $4(4-s)$ intersections of a line of S with a line of $S' \backslash S$. Since this number is not divisible by 5, $S' \backslash S$ must be a sum of more than one G-orbit, and the impossibility of this case follows from those of $q = 1$ and 2. Finally, the cases with $q > 4$ are constrained by the fact that $S' \backslash S$ must be a sum of G-orbits, and is therefore either empty or a sum of three coplanar lines. ▌

6. In this section we deal with the remaining cases $m = 5$, 6 and 9, none of which turn out to be possible.

LEMMA 7. *$H^1(k, \bar{P})$ contains no elements of order* 9.

Proof. Suppose that there is such an element, and that it is generated by β. In view of the proof of Lemma 4, we can choose the canonical base for $\bar{P}$ so that

$$3\beta \equiv \tfrac{1}{3}(\lambda_4 + \lambda_5 + \lambda_6) \mod \bar{P};$$

hence if we take $\qquad \beta \equiv c_0 \mu + c_1 \lambda_1 + \ldots + c_6 \lambda_6 \mod \bar{P}$

then $c_0, \ldots, c_3$ are 0, $\frac{1}{3}$ or $\frac{2}{3}$ and c_4, c_5 and c_6 are $\frac{1}{9}$, $\frac{4}{9}$ or $\frac{7}{9}$. Interchanging λ_5 and λ_6 if necessary, we can deduce from Lemma 6 that

$$\lambda_4, \mu - \lambda_4 - \lambda_5, \pi + \lambda_5 - \mu \quad \text{are in the same } S_r,$$

together with the two statements derived from this by cyclic permutation of the subscripts 4, 5, 6. This is equivalent to

$$-c_4 \equiv c_0 + c_4 + c_5 \equiv 2c_0 + c_1 + c_2 + c_3 + c_4 + c_6 \mod 1$$

and two similar statements. Replacing β by 4β or 7β if necessary in order to force $c_4 = \frac{1}{9}$, we deduce

$$c_1 + c_2 + c_3 \equiv \tfrac{1}{3}, \quad c_5 \equiv c_0 + \tfrac{1}{9}, \quad c_6 \equiv 2c_0 + \tfrac{1}{9}. \tag{8}$$

But, interchanging λ_2 and λ_3 if necessary, we also deduce from Lemma 6 that

$$\mu - \lambda_1 - \lambda_4, \mu - \lambda_2 - \lambda_5, \mu - \lambda_3 - \lambda_6 \quad \text{are in the same } S_r,$$

together with the two statements derived from this by cyclic permutation of the subscripts 1, 2, 3. This is equivalent to

$$c_0 + c_1 + c_4 \equiv c_0 + c_2 + c_5 \equiv c_0 + c_3 + c_6 \mod 1,$$

and in view of the last two congruences in (8) this implies

$$c_1 \equiv c_2 + c_0 \equiv c_3 + 2c_0 \mod 1$$

which cannot be reconciled with the first congruence (8).

Lemma 8. *$H^1(k, \bar{P})$ contains no elements of order* 6.

Proof. If there were such an element, then it would follow from Lemmas 1 and 6 that a double-six could be written as the sum of four triples of coplanar lines. Since a double-six contains six mutually skew lines, this is absurd.

Lemma 9. *$H^1(k, \bar{P})$ contains no elements of order* 5.

Proof. Suppose otherwise. Since H^1 is killed by n, the order of G, we must have $5 \mid n$. But G_0 only contains one conjugacy class of elements of order 5, so without loss of generality we can assume that G contains one preassigned member g of this class; we choose g to be the element which fixes λ_1 and cyclicly permutes $\lambda_2, \ldots, \lambda_6$. It follows from (3) that

$$\beta = \tfrac{1}{5}\{d_0\mu + d_1\lambda_1 + d_2(\lambda_2 + \ldots + \lambda_6)\}$$

where the d_i are integers which can be taken mod 5. By subtracting a multiple of $\frac{1}{5}\pi$ we can force d_2 to be 0; and by multiplying β by 2, 3 or 4 if necessary we can reduce to the six cases $d_0 = 1$, $d_1 = 0$ or $0 \leqslant d_0 < 5$, $d_1 = 1$.

If $\beta = \frac{1}{5}\mu$ then S_0 consists of $\lambda_1, \ldots, \lambda_6$ whose sum is $15\beta - \pi$. If $\beta = \frac{1}{5}\lambda_1$, then S_4 consists of the single line $\lambda_1 = 5\beta$. If $\beta = \frac{1}{5}(\mu + \lambda_1)$ then S_0 consists of the five lines $\lambda_2, \ldots, \lambda_6$ whose sum is $5\beta + \pi - 2(\pi + \lambda_1 - \mu)$; and $\pi + \lambda_1 - \mu$ is in P because it is the unique transversal to the five lines in S_0. If $\beta = \frac{1}{5}(2\mu + \lambda_1)$ then S_4 consists of λ_1 and $\pi + \lambda_1 - \mu$ whose sum is $\pi - \frac{5}{2}(\beta - \lambda_1)$. If $\beta = \frac{1}{5}(3\mu + \lambda_1)$ then S_0 consists of the five lines $\lambda_2, \ldots, \lambda_6$ whose sum is $\frac{1}{2}\pi + \frac{5}{2}\beta - \frac{3}{2}(\pi + \lambda_1 - \mu)$; and $\pi + \lambda_1 - \mu$ is in P because it is the unique transversal to the five lines in S_0. If $\beta = \frac{1}{5}(4\mu + \lambda_1)$ then S_2 consists of the single line $\pi + \lambda_1 - \mu = \pi + 5(\beta - \mu)$. Hence in every case β is in $P \otimes \mathbb{Q} + \bar{P}$ and therefore induces the trivial element of H^1. ∎

7. In this last section, we assume that V_1 is defined over k_1 and is the non-singular intersection of two quadrics in $\mathbb{P}^4$, and we give $P_1, \bar{P}_1$ the obvious meanings. We could

investigate $H^1(k_1, \bar{P}_1)$ in this case by methods very similar to those which we used for non-singular cubic surfaces; but there is a short-cut available. Let A be generic on V_1 over k_1 and let V, defined over $k = k_1(A)$, be the cubic surface obtained from V_1 by blowing up the point A. We have canonical isomorphisms

$$H^1(k_1, \bar{P}_1) \simeq H^1(k, \bar{P}_1) \simeq H^1(k, \bar{P}) \tag{9}$$

and we can apply all our previous Lemmas to the expression on the right because we only used the assumption that k is an algebraic number field in the opening paragraph of the paper and in the Corollary to Lemma 1. All we need to do, therefore, is to specialize these Lemmas to the case when one of the 27 lines on $\bar{V}$ is defined over k and then translate back from the 27 lines on $\bar{V}$ to the 16 lines on $\bar{V}_1$. The only questions of interest relate to the elements of order 2 in H^1, for it follows from Lemma 6 that there can be no element of order 3.

Since the geometry of $\bar{V}_1$ is not as well known as that of $\bar{V}$, and I do not know of a convenient reference, it may be helpful to state some of it here. Two skew lines have two common transversals, and the four lines constitute a hyperplane section of $\bar{V}_1$. A *double-four* consists of four skew lines on $\bar{V}_1$ having no common transversal, together with the four other lines which meet three of them; a double-four can be partitioned into two sets of co-hyperplanar lines in three distinct ways. Conversely a set of four co-hyperplanar lines can be extended to a double-four in three ways; in each case the four lines added are co-hyperplanar, and in this way we get a partition of the 16 lines into four co-hyperplanar sets. The complement of a double-four is another double-four.

LEMMA 10. *The order of $H^1(k_1, \bar{P}_1)$ is even if and only if there is a double-four T with the following properties:*

(i) *T is fixed under the action of G;*

(ii) *if γ denotes half the sum of a line in T and the three others in T that meet it, then γ is not in $P_1 \otimes \mathbb{Q} + \bar{P}_1$.*

As with Lemma 1, it does not matter which of the eight possible values of γ we use for the test in (ii). It will turn out that the relationship between T here and S in Lemma 1 is straightforward, but that is by no means so for the relationship between γ and β.

Proof. In specializing Lemma 1 we appear to have two cases to consider, according as the blow-up of A is in S or not. But in fact the first case is impossible, because that set of six skew lines in S which contains the blow-up would then be defined over k and (ii) of Lemma 1 could not be satisfied.

Suppose that the order of the groups (9) is even and that S is chosen to satisfy the conditions of Lemma 1. We choose the canonical basis of $\bar{P}$ so that S consists of the lines λ_i and $\pi + \lambda_i - \mu$, and the blow-up of A is $\mu - \lambda_1 - \lambda_4$; and we take for T the eight lines in $\bar{V}_1$ whose images are in S – that is, the λ_i and $\pi + \lambda_i - \mu$ for $i = 2, 3, 5, 6$. Clearly T is a double-four fixed under the action of G. Moreover $\beta = \frac{1}{2}(\lambda_1 + \ldots + \lambda_6)$ and the image of γ in $\bar{V}$ is

$$\tfrac{1}{2}\{(\pi + \lambda_6 - \mu) + \lambda_2 + \lambda_3 + \lambda_5\} = \beta - \mu + \tfrac{1}{2}\pi + \tfrac{1}{2}(\mu - \lambda_1 - \lambda_4)$$

so γ is in $P_1 \otimes \mathbb{Q} + \bar{P}_1$ if and only if β is in $P \otimes \mathbb{Q} + \bar{P}$. If conversely we assume the existence of T satisfying the conditions of Lemma 10, then we can take S to be the unique double-six which contains the image of T in $\bar{V}$. Taking the same basis for $\bar{P}$ as before, we see that S satisfies the conditions of Lemma 1 and therefore the order of the groups (9) is even.

LEMMA 11. *$H^1(k_1, \bar{P}_1)$ cannot contain more than three elements of order 2, and it contains as many as three if and only if the lines on $\bar{V}_1$ can be partitioned into four disjoint cohyperplanar sets T^i with the following properties:*

(i) *the union of any two of the sets T^i is a double-four;*

(ii) *each of the T^i is fixed under G;*

(iii) *if γ^* is half the sum of a line λ in some T^i, the two lines in the same T^i that meet λ, and one other line that meets λ, then no such γ^* is in $P_1 \otimes \mathbb{Q} + \bar{P}_1$.*

Here it does not matter which of the 16 lines λ we choose for our test, but we do have to apply the test to all the γ^* arising from a given λ.

Proof. Suppose that each of the groups (9) contains three elements of order 2. As in the proof of Lemma 2, we can choose the canonical base of $\bar{P}$ so that S_1 consists of

$$\lambda_1, \lambda_2, \lambda_3, \pi+\lambda_4-\mu, \pi+\lambda_5-\mu, \pi+\lambda_6-\mu;$$

then S_2 and S_3 necessarily are

$$\lambda_4, \lambda_5, \lambda_6, \pi+\lambda_1-\mu, \pi+\lambda_2-\mu, \pi+\lambda_3-\mu;$$

$$\mu-\lambda_1-\lambda_2, \mu-\lambda_1-\lambda_3, \mu-\lambda_2-\lambda_3, \mu-\lambda_4-\lambda_5, \mu-\lambda_4-\lambda_6, \mu-\lambda_5-\lambda_6.$$

As in the proof of Lemma 10, the blow-up of A cannot be in any S^i and so we can take it to be $\mu-\lambda_1-\lambda_4$. We take T^i to consist of those lines in $\bar{V}_1$ which are images of lines in S^i for $i = 1, 2, 3$, and T^4 to consist of the remaining four lines; hence these sets are the images of

$$\lambda_2, \lambda_3, \pi+\lambda_5-\mu, \pi+\lambda_6-\mu; \tag{10}$$

$$\lambda_5, \lambda_6, \pi+\lambda_2-\mu, \pi+\lambda_3-\mu; \tag{11}$$

$$\mu-\lambda_1-\lambda_2, \mu-\lambda_1-\lambda_3, \mu-\lambda_4-\lambda_5, \mu-\lambda_4-\lambda_6; \tag{12}$$

$$\mu-\lambda_1-\lambda_5, \mu-\lambda_1-\lambda_6, \mu-\lambda_2-\lambda_4, \mu-\lambda_3-\lambda_4. \tag{13}$$

These clearly satisfy (i) and (ii) of Lemma 11. As for (iii), the three values of β^* are

$$\tfrac{1}{2}(\lambda_1+\lambda_2+\lambda_3), \tfrac{1}{2}(\lambda_4+\lambda_5+\lambda_6), \tfrac{1}{2}(3\mu-2\lambda_1-2\lambda_2-2\lambda_3)$$

and if we choose $\lambda = \pi+\lambda_5-\mu$ the three values of γ^* lifted to $\bar{V}$ are

$$\tfrac{1}{2}(\pi+\lambda_2+\lambda_3-\lambda_1), \tfrac{1}{2}(\pi+\lambda_2+\lambda_3-\lambda_4), \tfrac{1}{2}(2\mu-\lambda_1-\lambda_4)$$

and (iii) follows easily.

If conversely we assume the existence of the T^i satisfying the conditions of Lemma 11, then we can choose a basis for $\bar{P}$ so that the image of T^4 in $\bar{V}$ is given by (13) and the blow-up of A is $\mu-\lambda_1-\lambda_4$. The images of the other T^i are then necessarily given by (10), (11) and (12) and we have to show how to augment these last three to suitable sets S^i. There are just three ways of augmenting the image of a T^i in $\bar{V}$ to a set of three skew lines and their three transversals; we must add one of the three pairs λ_1 and $\pi+\lambda_4-\mu$, λ_4 and $\pi+\lambda_1-\mu$, or $\mu-\lambda_2-\lambda_3$ and $\mu-\lambda_5-\lambda_6$. But once we have decided to drop a particular T^i, say T^4, there is only one way of assigning one pair to each remaining T^i so that the resulting S^i satisfy (i) of Lemma 2. It follows from this uniqueness, together with (ii) of Lemma 11, that each S^i is fixed under G, and the rest of the proof is now trivial. ∎

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