Canad. Math. Bull. Vol. **33** (3), 1990

POWERS OF GENUS TWO IN FREE GROUPS

BY

CHARLES C. EDMUNDS AND GERHARD ROSENBERGER

ABSTRACT. The general problem is to express an element of the derived group of a free group as a product of a minimal number of commutators. An old conjecture is settled in the negative, and a new conjecture and a number of related questions are posed.

1. In the mid 1950's, A. Tarski asked if the elementary theory of free groups were decidable. As a test problem for the decidability of universally quantified implications, R. L. Vaught made the following (unpublished) conjecture for any free group F:

(1.1) $\forall x,\ y,\ z \in F\ (x^2y^2 = z^2 \Rightarrow xy = yx)$. (The Vaught Conjecture)

The surprising difficulty of establishing this conjecture was the impetus for much of the work done in the late 50's and early 60's on equations in free groups. The first solution of the Vaught conjecture was given by Roger Lyndon [4] in 1959 when he proved:

(1.2) If $x,\ y,\ z \in F$ with $x^2y^2 = z^2$, then $\langle x,\ y,\ z\rangle$ is a cyclic subgroup of F (where $\langle x,\ y,\ z\rangle$ denotes the subgroup of F generated by x, y, and z).

In 1975, C. C. Edmunds [3] gave a very short combinatorial proof of Lyndon's result, based on two facts which are easily established.

(1.3) The equation $x^2y^2 = z^2$ is equivalent, under automorphisms of F, to $[x,\ y] = z^2$ (where $[x,\ y]$ denotes the commutator $xyx^{-1}y^{-1}$).

(1.4) If $x,\ y,\ z \in F$ and $[x,\ y] = z^n$ for some integer $n > 1$, then $z = 1$.

It appears that Max Dehn is responsible for (1.3), a non-trivial step of the standard method used to put compact surfaces into normal form (see [6]). Result (1.4) is due to M. P. Schützenberger [8] and is easily derived, as in [3], from the following result of M. J. Wicks [9].

(1.5) If w is a commutator in F, then there exist $x,\ y,\ z \in F$ such that some conjugate of w is identically equal to $xyzx^{-1}y^{-1}z^{-1}$.

In 1974, in an attempt to test the power of the methods then available for solving equations in free groups, Edmunds (quoted in Lyndon [5]) asked the following

Received by the editors May 23, 1989
AMS (1980) Subject Classification:

https://doi.org/10.4153/CMB-1990-057-3 Published online by Cambridge University Press

questions.

(1.6) If $[v, w][x, y] = z^2(\neq 1)$ in F, does follow that z is a commutator?

(1.7) If $[v, w][x, y] = z^2(\neq 1)$ in F and z *is* a commutator, does it follow that $[v, w] = [x, y]$?

A word w in F', the derived group of F, has genus $g(\geqq 0)$, denoted genus $(w) = g$, if w is a product of g, but not fewer, commutators. Note that $z \in F'$ if and only if $z^n \in F'$ for some $n \geqq 2$. Thus (1.4) says that if $1 \neq z \in F$ and $n \geqq 2$, then genus $(z^n) \geqq 2$. Since $[x, y][x, y] = [x, y]^2$, there are clearly squares of genus 2. It is a natural question to attempt to characterize the squares of genus 2. This leads to consideration of the equation $[v, w][x, y] = z^2$. The above questions can then be stated equivalently as:

(1.6′) If genus $(z^2) = 2$ in F, does it follow that genus $(z) = 1$?

(1.7′) If genus $(z) = 1$ and genus $(z^2) = 2$, with $[v, w][x, y] = z^2$ in F, does it follow that $[v, w] = [x, y]$?

The commutator identity $[wx, y] = [[x, y], w]^{-1}[x, y][w, y]$ is easily checked. Letting $w = x$, we have

$$[x^2, y] = [[x, y], x]^{-1}[x, y]^2,$$

thus

$$[[x, y], x][x^2, y] = [x, y]^2.$$

Clearly $[[x, y], x] \neq [x^2, y]$ in F; therefore we have a negative answer to (1.7). Recently question (1.6) has been answered negatively by J. Comerford and Y. Lee [1] using a clever computer search based on methods of D. Piollet combined with an algebraic translation of a topological representation of the generators for the mapping class groups due to J. Birman and D. Chillingworth.

2. These negative results raise more questions than they answer.

QUESTION 1. Can we characterize those $z \in F$ such that genus $(z^2) = 2$?

In particular.

QUESTION 2. What is the maximum, if any, of genus (z) with genus $(z^2) = 2$?

From [1] we know that such a maximum, if it exists, is at least 2. By analogy with Lyndon's result (1.2)

QUESTION 3. If $[v, w][x, y] = z^2$ in F, what can be said about the subgroup $\langle v, w, x, y, z\rangle$?

For instance, it follows from a result of G. Baumslag and A. Steinberg (quoted in [7]) that the rank of $\langle v, w, x, y, z\rangle$ is at most 3. By analogy with Schützenberger's result (1.4), QUESTION 4. What can be said about the genus z^n for $n \geqq 2$?

The following result of G. Rosenberger [7] (cf. [2]) sheds some light on these questions in special cases $z = [a, b]$.

(2.1) Let $F = \langle a, b : \phi \rangle$, let $g \geqq 1$, $n \geqq 2$, and let $[x_1, y_1] \dots [x_g, y_g] = [a, b]^n$ in F, where $[a, b]^n$ is not a proper power in $\langle x_1, y_1, \dots, x_g, y_g \rangle$. Then n is odd and $g \geqq (n+1)/2$.

Thus as $n \longrightarrow \infty$, genus $([a, b]^n) \longrightarrow \infty$, as is proved by topological methods in [2]. The identity $[aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2] = [a, b]^3$ shows that there are cubes of genus 2. Letting $g = 2$ in (2.1) we conclude, in $F = \langle a, b : \phi \rangle$, that if $[v, w][x, y] = [a, b]^n$, then $n = 3$ or n is even and $[a, b]^n$ is a proper power in $\langle v, w, x, y \rangle$. With this in mind, we make the following conjecture.

CONJECTURE. If $[v, w][x, y] = z^n$ has a solution in a free group F with $n \geqq 2$ and $z \neq 1$, then $n \leqq 3$.

More generally we ask:

QUESTION 5. For fixed $g(=$ genus $(z^n))$, what are the possible values of n, for $z \in F$?

QUESTION 6. For fixed n, what are the possible values of genus (z^n), for $z \in F$?

REFERENCES

1. J. Comerford and Y. Lee, *Product of two commutators as a square in a free group*, Canad. Math. Bull., **33** (1990), 190–196.
2. M. Culler, *Using surfaces to solve equations if free groups*, Topology **20** (1981), 133–267.
3. C. C. Edmunds, *A short combinatorial proof of the Vaught conjecture*, Canad. Math. Bull. **18** (1975), 607–608.
4. R. C. Lyndon, *The equation $a^2b^2 = c^2$ in free groups*, Michigan Math. J. **6** (1959), 89–95.
5. ———, *Equations in groups*, Bol. Soc. Bras. Mat. **11** (1980), 79–102.
6. W. S. Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, (1977).
7. G. Rosenberger, *Minimal generating systems for plane discontinuous groups and an equation in free groups*, Proc. of "Groups-Korea 1988", (to appear).
8. M. P. Schützenberger, *Sur l'équation $a^{2+n}b^{2+m} = c^{2+p}$ dans un group libre*, C. R. Acad. Sci. Paris **248** (1959), 2435–2436.
9. M. J. Wicks, *Commutators in free products*, J. London Math. Soc. **37** (1962), 433–444.

Department of Mathematics
Mount Saint Vincent University
Halifax, Nova Scotia
Canada B3M 2J6

Fachbereich Mathematik
Universität Dortmund
Postfach 50 05 00
4600 Dortmund 50
Fed. Rep. of Germany