ON THE FLAT OVERRINGS OF AN INTEGRAL DOMAIN

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(Received 15 December, 1969)

The present paper deals with relations between flat overrings and quotient rings. We are mainly concerned with Richman's results [**10**] on flat overrings and with those of Davis [**2**], Gilmer [**3**], Gilmer and Heinzer [**4**], Gilmer and Ohm [**5**], and Mott [**8**], on rings with the QR property and with the property (#) defined in Section 1. Some of their results are generalized, and it is shown that certain theorems, which at first glance seem to have nothing in common, are in fact particular cases of a single more general theorem. The main result is the following (which is proved in Section 2):

THEOREM 1. *Let R be a Krull domain and let X be the family of all minimal prime ideals of R. The following statements are then equivalent*:

(a) *The divisor class group of R is a torsion group.*
(b) $S = \bigcap_{P \in Y} R_P$ *is a quotient ring of R for each subfamily Y of X.*

We wish to thank Professor R. Gilmer for his remarks and suggestions.

1. Preliminaries. In the sequel R denotes a commutative integral domain with quotient field K, unless otherwise specified. An overring of R is any subring of K containing R, a flat overring of R being an overring which is flat as an R module. R has the QR property if every overring of R is a quotient ring of R [**5**]. If P is a prime ideal of R, then R_P denotes the localization of R at P. R has the property (#) if $\bigcap_{P \in \Delta_1} R_P \neq \bigcap_{P \in \Delta_2} R_P$ for every pair of distinct families Δ_1 and Δ_2 of maximal ideals of R [**3**]. By $\dim R$ we denote the Krull-dimension of R.

We use a result of Richman in [**10**], namely that every flat overring of R is an intersection of localizations of R at prime ideals of R. For the general properties of a Krull domain and its divisor class group the reader is referred to [**11**]. It is well known that every quotient ring of a Krull domain R is of the form $\bigcap_{P \in Y} R_P$ for a suitable subfamily Y of the family of all minimal prime ideals of R. Hence every flat overring of R is of this form if R is a Krull domain.

2. Krull domains. In this section we discuss the Krull domain case, starting with the proof of Theorem 1.

Proof of Theorem 1. $(a) \Rightarrow (b)$. Let M be the set of elements of R invertible in S. Then $S \supset R_M = \bigcap_{P \in Z} R_P$ for a suitable subfamily Z of X, and obviously $Y \subset Z$. If $Z \neq Y$, let $Q \in Z - Y$. Since the divisor class group of R is a torsion group, Q^n is equivalent to a principal ideal, say Rx, for some integer n. Q and Rx are divisorial ideals; hence $Q \supset Rx \supset Q^n$. Thus $x \in Q$ and $x \notin P$ for any $P \in Y$. Hence x is invertible in S but not in R_M. We thus have a contradiction, which implies that $Y = Z$ and $S = R_M$.

$(b) \Rightarrow (a)$. Let Q be any element of X, and let $S = \bigcap_{P \in X - Q} R_P$. S is a quotient ring of R

distinct from R, so that there exists an $x \in R$, x being invertible in S but not in R. Therefore $x \in Q$ and $x \notin P$ for any $P \in X-Q$. Rx is equivalent to some product of minimal prime ideals $\prod_{i=1}^{k} P_i^{n_i} \subset \bigcap_{i=1}^{k} P_i$. But $\bigcap_{i=1}^{k} P_i$ is a divisorial ideal; hence $Rx \subset \bigcap_{i=1}^{k} P_i$. Thus only Q occurs among the P_i, and Rx is equivalent to Q^n for some integer n. This completes the proof of Theorem 1.

REMARK. A Dedekind domain R is a Krull domain in which every ideal is divisorial and over which every overring is flat. Also, the divisor class group of R is naturally isomorphic to its ideal class group. An immediate consequence of this remark and Theorem 1 is the following result, obtained at almost the same time independently by Davis [2], Gilmer and Ohm [5], and Goldman [6]:

COROLLARY 2. *A Dedekind domain R has the QR property if and only if the ideal class group of R is a torsion group.*

We can also easily derive now a result of Gilmer and Ohm [5]:

COROLLARY 3. *A unique factorization domain R has the QR property if and only if it is a principal ideal domain.*

Proof. A unique factorization domain is a Krull domain whose divisor class group is zero. If it also has the QR property, then it is a Prüfer domain [5], hence a Dedekind domain. But R is a Dedekind domain whose ideal class group is zero if and only if it is a principal ideal domain.

Richman in [10] proved that every flat overring of a unique factorization domain R is a quotient ring of R. This result is a particular case of the following obvious corollary of Theorem 1:

COROLLARY 4. *Let R be a Krull domain whose divisor class group is a torsion group. Then every flat overring of R is a quotient ring of R.*

3. Flat overrings. In this section we discuss a more general setting. Consider the following property:

(*) For any family $\{P\}\cup\{P_\alpha\}_{\alpha\in I}$ of prime ideals of R, if $P \subset \bigcup_{\alpha\in I} P_\alpha$, then P is contained in some P_α.

The following proposition is true for an arbitrary commutative integral domain R:

PROPOSITION 5. *If R has property* (*), *then every flat overring of R is a quotient ring of R.*

Proof. Let S be a flat overring of R, and let X be the family of all prime ideals P of R such that $R_P \supset S$. Then $S = \bigcap_{P\in X} R_P$.

Let M be the multiplicative set of elements of R invertible in S. Since R_M is also a flat overring of R, then $S \supset R_M = \bigcap_{P\in Y} R_P$ for the family Y of all prime ideals P of R such that $R_P \supset R_M$ and $Y \supset X$. If $Y \neq X$, let $Q \in Y-X$. If $Q \subset P$ for some $P \in X$, then $R_Q \supset R_P \supset S$, hence also $Q \in X$. Therefore, since (*) holds, $Q \subset \bigcup_{P\in X} P$ and there exists an $x \in Q-\bigcup_{P\in X} P$. Thus x is invertible in S but not in R_M, which is impossible. Hence $X = Y$ and $S = R_M$.

https://doi.org/10.1017/S0017089500001269 Published online by Cambridge University Press

Note that property (*) does not hold in general for an arbitrary ring.

For example, any maximal ideal of a Krull ring R for which $\dim R \geqq 2$ is contained in the set theoretical sum of its minimal prime ideals. However, insofar as the set I is finite, we have:

LEMMA 6 [7, Corollary 3]. *If* $P \subset \bigcup_{i=1}^{k} P_i$, *where* P *and* P_i *are prime ideals of* R, *then* P *is contained in some* P_i.

Combining Lemma 6 with Proposition 5, we arrive at another result of Richman [10].

COROLLARY 7. *If* R *has only a finite number of prime ideals, then every flat overring of* R *is a quotient ring of* R.

4. Prüfer domains. In this section R denotes a Prüfer domain. We consider here the relations between the QR property, property (#), and property (*). We recall first some results of Gilmer and Heinzer in [3] and [4].

Let Δ denote the set of all maximal ideals of R, and let $\Delta_P = \Delta - \{P\}$ for each $P \in \Delta$.

(A) R has the property (#) if and only if $R_Q \not\supset \bigcap_{P \in \Delta_Q} R_P$ for each $Q \in \Delta$.

(B) If R has property (#) and if $\dim R = 1$, then each overring of R has property (#).

(C) If $Q \not\subset \bigcup_{P \in \Delta_Q} P$ for each $Q \in \Delta$, then R has the property (#).

(D) If R has the QR property, then $P \subset \bigcup_{\alpha \in I} P_\alpha$ if and only if $\bigcap_{\alpha \in I} R_{P_\alpha} \subset R_P$ for each family $P \cup \{P_\alpha\}_{\alpha \in I}$ of prime ideals of R.

(E) Every overring of R has property (#) if and only if, for each prime ideal P of R, there exists a finitely generated ideal $J \subset P$ such that each maximal ideal of R containing J also contains P.

(F) R has the property (#) if and only if R is uniquely representable as an intersection of a family $\{V_\alpha\}$ of valuation overrings of R such that there are no containment relations among the V_α.

Since (C) holds, property (*) implies the property (#). Moreover, from (E) it can be proved that if property (*) holds, then every overring of R has property (#).

The properties (A) and (D) are similar to the following statement: If R has the QR property, then R has property (*) if and only if every overring of R has property (#). This is actually part of the main result of this section, which we now state and prove.

PROPOSITION 8. *A Prüfer domain* R *has property* (*) *if and only if* R *has the* QR *property and every overring of* R *has property* (#).

Proof. Suppose that R has property (*). Since every overring of a Prüfer domain is a flat overring, then the QR property follows from Proposition 5. As mentioned earlier, Property (E) implies that every overring of R has the property (#).

Suppose now that R has the QR property and that every overring of R has property (#). Let $P \subset \bigcup_{\alpha \in I} P_\alpha$, where P and P_α are prime ideals of R. Suppose that P is not contained in

https://doi.org/10.1017/S0017089500001269 Published online by Cambridge University Press

any P_α. It can be assumed without loss of generality that there are no containment relations among the P_α's. It follows from (F), since $S = \bigcap_{\alpha \in I} R_{P_\alpha}$ has the property (#), that $S \neq S \cap R_P = S'$. Since R has the QR property, then $S = R_M$ and $S' = R_N$. In this case both sets M and N must equal $R - \bigcup_{\alpha \in I} P_\alpha$, so that $S = S'$. Thus P is contained in some P_α and the proof of Proposition 8 is complete.

As a consequence, since (B) holds, we obtain the following generalization of (C).

COROLLARY 9. *Let R be a one-dimensional Prüfer domain. Then $Q \not\subset \bigcup_{P \in \Delta_Q} P$ for each maximal ideal Q of R if and only if R has the QR-property and property* (#).

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