12
Hot hadronic matter

We know that QCD is the formal theory of the strong interaction. In principle, its solution should yield the complete particle spectrum as well as produce the interaction terms that regulate how different particle species interact. However, this complete solution is at present impossible, partly owing to the fact that at the scale of the lighter degrees of freedom QCD is strongly coupled. To describe the interaction and the properties of hot and dense hadronic ensembles, one must turn to effective approaches. They vary in character and in philosophy. In this chapter, we shall discuss some of these techniques. They comprise effective Lagrangian theories, which aim to represent in a simple way the dynamical content of a theory in the low-energy limit. The heavier fields are integrated out, leaving a set of constants to be determined by experiment. In the specific case of QCD, the choice of low-energy effective Lagrangian is dictated by general symmetry principles, and chiral symmetry will be seen to play a special role.

A remarkably successful effective Lagrangian approach to low-energy QCD is that of chiral perturbation theory. We consider this first and study its finite-temperature behavior. Next, we will use the fact that the spectrum of strongly interacting particles is quite well known experimentally to outline a technique that enables an evaluation of in-medium self-energies directly from experimental data input. The rest of the chapter will be devoted to a discussion of the Weinberg sum rules at nonzero temperatures [1] and to investigations of the characteristics of the linear and nonlinear σ models [2].

12.1 Chiral perturbation theory

Chiral perturbation theory draws its power from the observation that the light pseudoscalar degrees of freedom in the spectrum of the confined sector of QCD can be explained in terms of a spontaneously broken

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symmetry. Let us first elaborate how and why this statement is true. Consider for example the QCD Lagrangian for massless quarks, assuming a generic quark field, ψ, for simplicity:

$$\mathcal{L} = i\bar{\psi}\not{D}\psi \tag{12.1}$$

Based on a comparison with the QCD scale, the massless approximation is a good one for u and d quarks, is less so for s quarks, and is simply bad for c, t, and b (see Table 8.2). The free-particle Dirac equation for massless fermions is

$$\not{k}\psi = 0 \tag{12.2}$$

Using the fact that $\{\gamma_\mu, \gamma_5\} = 0$, a solution is also $\gamma_5\psi$. Consequently, two solutions are $\psi_{L/R} = \frac{1}{2}(1 \mp \gamma_5)\psi$, and this establishes γ_5 as a chirality operator: $\gamma_5\psi_{L/R} = \mp\ \psi_{L/R}$. The subscripts L and R refer to the left- and right-handed solutions, respectively. Finally, this labeling is made more explicit by manipulating the free massless-particle Dirac equation (12.2) into the form

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{k}}\psi = \pm\gamma_5\psi \tag{12.3}$$

where $\gamma_5\gamma_0\boldsymbol{\gamma} = \boldsymbol{\sigma}$. Therefore, for right-handed solutions, the helicity and the sign of the energy (identified by the $\pm$ symbols) are correlated, whereas they are anticorrelated for left-handed solutions.

One can then rewrite (12.1) as

$$\mathcal{L} = i\bar{\psi}_L\not{D}\psi_L + i\bar{\psi}_R\not{D}\psi_R \tag{12.4}$$

and it is seen that the L and R sectors decouple. Consequently, symmetry transformations of the type

$$\psi_{L/R} \to \exp\left(-i\sum_j \alpha^j_{L/R}\lambda^j\right)\psi_{L/R} \tag{12.5}$$

will leave the Lagrangian invariant. Note that it is also invariant with respect to $U(1)_A$, but there is an anomaly which we will not discuss here. In the case of SU(2), the matrices λ^j are Pauli matrices and $\psi_{L/R}$ are the chiral projections of the light $\begin{pmatrix} u \\ d \end{pmatrix}$ doublet. Similarly, for SU(3) the matrices λ^j are then the Gell-Mann matrices and the chiral projections involved are those obtained from the u, d, and s fields. The elements $\alpha^j_{L/R}$ are the components of arbitrary constant vectors. Using the case of SU(2) as an example, the invariance of the Lagrangian under the symmetry transformations (12.5) is usually labeled chiral SU(2), $SU(2)_L \times SU(2)_R$, or $SU(2)_V \times SU(2)_A$; in the latter case, we have defined $\alpha^j_{V/A} =$

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$(\alpha_R^j \pm \alpha_L^j)/2$. Thus, in the case of SU(2), if chiral symmetry were realized in the conventional fashion in Nature then one would expect to have three time-independent vector charges and three time-independent axial charges. Since those charges are proportional to number operators, this leads to the prediction of parity doublets (owing to the transformation properties of γ_5), which are not observed. What has gone wrong? It turns out that this is another case where a symmetry of the Lagrangian is broken by the ground state of the theory. As we have seen in Chapter 7, this leads to the appearance of Goldstone bosons.

In the case at hand the breaking is dynamical, meaning that the Noether current associated with the axial sector is not divergenceless but receives contributions from quantum corrections. This was discussed early on by Adler and by Bell and Jackiw [3]. Another puzzling fact is that there is indeed a triplet of light particles, the pions, but these are not massless. This is to be understood in terms of the fact that our original assumption that the quarks have no bare mass is in fact incorrect. If u and d quarks were strictly massless, the pion would be a genuine Goldstone boson, with $m_\pi = 0$. To first order in the explicit symmetry breaking, the finite pion mass can be traced back to the u and d quark condensates [4].

The aim of chiral perturbation theory is to provide an effective theory that possesses the symmetries of the complete theory, QCD, and is applicable at low energies where the exact theory is strongly coupled. Then the effective theory of QCD is formulated in terms of the lightest hadron fields, the pions. Bearing in mind that the chiral symmetry is not manifest in the ground state of QCD, there is a procedure to implement a spontaneously broken symmetry in a quantum field theory [5]. In the special case of chiral symmetry, a convenient way to collect the Goldstone fields is the exponential parametrization. For SU(3) it is $\mathrm{U}(\phi) = \exp\left(i\sum_1^8 \lambda_a \phi^a/F\right)$, λ_a being a Gell-Mann matrix and F a constant. Specifically,

$$\frac{1}{\sqrt{2}}\sum_{a=1}^{8}\lambda_a\phi^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \quad (12.6)$$

Strictly speaking, the Lagrangian of the standard model is not chirally invariant. The chiral symmetry of the strong interactions is broken by the electroweak interaction owing to the quark Yukawa coupling, which generates nonzero quark masses. The basic assumption of chiral perturbation theory is that the chiral limit is a viable starting point for a perturbative expansion. This expansion is in fact a double expansion, in powers of both the momentum and the quark masses. The Goldstone bosons will decouple from each other in the low-energy limit.

An elegant technique that enables one to calculate Green's functions of quark currents is that associated with the introduction of external fields. Following Gasser and Leutwyler [6], the chirally invariant QCD Lagrangian is extended by coupling the quarks to external Hermitian matrix fields $s(x)$, $p(x)$, $v_\mu(x)$, and $a_\mu(x)$:

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \bar{\psi}\gamma^\mu(v_\mu + a_\mu\gamma_5)\psi - \bar{\psi}(s - ip\gamma_5)\psi \tag{12.7}$$

The external fields transform under parity as a scalar, a pseudoscalar, a vector, and an axial vector, respectively. They are color-neutral 3×3 matrices, where the matrix character with respect to the flavor indices u, d, and s can be illustrated, for example, by the vector field

$$v^\mu = v_0^\mu + \sum_{j=1}^{8} \tfrac{1}{2}\lambda_j v_j^\mu \tag{12.8}$$

As before, chiral fields can be defined: $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$.

The usual QCD Lagrangian is recovered in the limit $p = v_\mu = a_\mu = 0$ and $s = \text{diag}(m_u, m_d, m_s)$. The physically relevant Green's functions are functional derivatives of the usual zero-temperature generating functional $Z(s, p, v, a)$. For example,

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle = i\, \frac{\delta \ln Z}{\delta s_0(x)}\bigg|_{p=v=a=0,\, s=m} \tag{12.9}$$

where, as in the vector example above, the subscript 0 identifies the singlet component. Similarly, various currents can be obtained directly from the Lagrangian, such as the left-handed current $j_\mu^{l,a}(x)$ derived from $\partial\mathcal{L}/\partial l_a^\mu$.

Inclusion of the external fields transforms the global chiral symmetry to a local one. The invariance requirements are now contained in the following set of transformation rules. For any $g_{R/L}$ in SU(3) such that

$$\begin{aligned} \psi_R &\to g_R\psi_R \\ \psi_L &\to g_L\psi_L \end{aligned} \tag{12.10}$$

the invariance is preserved if the external fields transform as gauge fields,

$$\begin{aligned} r_\mu &\to g_R r_\mu g_R^\dagger + i g_R \partial_\mu g_R^\dagger \\ l_\mu &\to g_L l_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger \\ s + ip &\to g_R(s + ip)g_L^\dagger \end{aligned} \tag{12.11}$$

and if $U \to g_R U g_L^\dagger$. The covariant derivative which, by definition, has the same transformation properties as the object on which it is acting, is $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$.

We are now in a position to formulate the basic premises of chiral perturbation theory. At zero temperature, one can write a generating functional of QCD as

$$Z(s,p,v,a) = \int [dA_a^\mu][d\bar{\psi}][d\psi] \exp\left(i \int d^4x\, \mathcal{L}\right) \tag{12.12}$$

For simplicity, the possible ghost fields have been omitted. At low energy, the propagating modes are the Goldstone modes. In the language of effective field theory, all the heavy degrees of freedom are integrated out and are absorbed into the parameters of the effective action. Specifically,

$$Z(s,p,v,a) = \int [dU] \exp\left(i \int d^4x\, \mathcal{L}_{\text{eff}}\right) \tag{12.13}$$

One starts by constructing an effective Lagrangian in terms of derivatives and of the external fields. The limit where the external fields vanish is that of low-energy QCD. Whereas it is plausible that this procedure does reproduce the Green's functions of QCD at low energy, its formal validity has not been proven here. This was done by Leutwyler [7].

Searching for an interaction that would constitute the leading-order term in a momentum expansion, one realizes that that there are no candidates with the required invariance properties that have no derivatives and no external fields. In fact, the only candidate is $\text{Tr}(UU^\dagger) = 3$, which is simply a constant. The most general chirally invariant effective Lagrangian with the minimum number of derivatives is

$$\mathcal{L}_2 = \tfrac{1}{4} F^2\, \text{Tr}(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U) \tag{12.14}$$

where $\chi = 2B(s + ip)$. Thus at this order there are two parameters, F and B. Observe that in order to reproduce the kinetic term in the free-pion Lagrangian, the constant F above needs to be the same as that in the definition of the field matrix, $\text{U}(\phi)$ (just above (12.6)). The two constants F and B are related to the pion decay constant and to the quark condensate, up to chiral corrections [6]:

$$\begin{aligned} f_\pi &= F + \mathcal{O}(m_q) \\ \langle 0|\bar{u}u|0\rangle &= -F^2 B + \mathcal{O}(m_q) \end{aligned} \tag{12.15}$$

Using our definition for $\text{U}(\phi)$ and setting the external scalar field s equal to the quark mass matrix, one can read off from $\mathcal{L}_2$ the pseudoscalar meson masses, again up to leading order in chiral corrections. For example,

$$\begin{aligned} m_\pi^2 &= 2\bar{m}B \\ m_{K^+}^2 &= (m_u + m_s)B \\ m_{K^0}^2 &= (m_d + m_s)B \end{aligned} \tag{12.16}$$

with $\bar{m} = \frac{1}{2}(m_u + m_d)$. Those relations are consistent with the chiral counting rules, which stipulate the dimensions of the operators and of

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the fields in the chiral expansion in terms of the momentum k:

$$\begin{aligned} U &: \mathcal{O}(1) \\ D_\mu U,\, v_\mu,\, a_\mu &: \mathcal{O}(k) \\ s,\, p &: \mathcal{O}(k^2) \end{aligned}$$

With these rules, the meaning of the subscript in $\mathcal{L}_2$ is clear. The effective Lagrangian is then expanded up to order k^4 [6].

We now formulate the problem at finite temperature and calculate the pressure of a pion gas using chiral perturbation theory. The symmetry requirements will translate into exact statements for the coefficients of the expansion in powers of the temperature. Furthermore, we shall make use of the fact that pions are considerably less massive than any of their other SU(3) partners. Therefore these lighter degrees of freedom will be excited first and should play a leading role: we then restrict our discussion to SU(2). We can rewrite $\mathcal{L}_2$ in terms of the nonlinear σ model:

$$\mathcal{L}_2 = \tfrac{1}{4}F^2 \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger - M^2(U + U^\dagger)] \tag{12.17}$$

with $M^2 = (m_u + m_d)B$. The effective Lagrangian to order k^4 is written down by identifying all the independent terms to this order that have the required symmetry properties (Lorentz invariance, P, C, and chiral symmetry):

$$\begin{aligned} \mathcal{L}_4 = &-\tfrac{1}{4}l_1 \,[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2 - \tfrac{1}{4}\, l_2 \,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \\ &+ \tfrac{1}{8} l_4 M^2 \,\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\,\mathrm{Tr}(U + U^\dagger) \\ &- \tfrac{1}{16}(l_3 + l_4) M^4 \,\mathrm{Tr}(U + U^\dagger) - h_1 M^4 \end{aligned} \tag{12.18}$$

The contact term h_1 is a vacuum contribution, and isospin-breaking effects are ignored. The evaluation of the finite-temperature contribution to the thermodynamic potential proceeds as in preceding chapters, only now the chiral effective Lagrangian is used:

$$Z \approx \int_{\mathrm{periodic}} [\mathrm{d}U] \exp\left(\int_0^\beta d\tau \int d^3x\, (\mathcal{L}_2 + \mathcal{L}_4) \right) \tag{12.19}$$

The complete expansion in loop topologies that yield terms up to T^8 was worked out by Gerber and Leutwyler [8].A few of those diagrams are shown in Figure 12.1. The diagrams so obtained fall into three categories.

1 Those that generate temperature-independent contributions. These only renormalize the vacuum contribution.
2 The genuine temperature-dependent terms that will generate the thermal pressure. These are shown in Figure 12.1.

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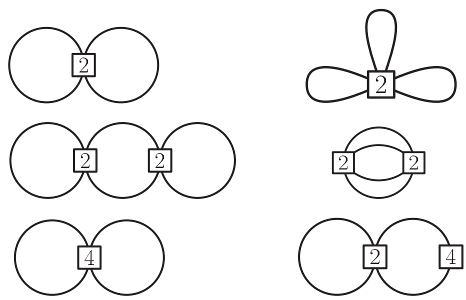

Fig. 12.1. Some of the diagrams that occur in the finite-temperature expansion of the thermodynamic potential in chiral perturbation theory, up to order T^8. The labeling of the vertices refers to the order of the chiral Lagrangian that provided the vertex.

3 Up to $\mathcal{O}(T^8)$, the thermodynamic potential will contain some diagrams with vertices coming from chiral Lagrangians of order higher than 4. These temperature-dependent contributions renormalize the bare mass M in the free-gas term, in such a way that

$$M \to M_1$$

with

$$M_1 = M^2 + 2l_3\frac{M^4}{F^2} + c_0\frac{M^6}{F^4} \tag{12.20}$$

where c_0 is a constant.

The divergences present in the zero-temperature theory are isolated using dimensional regularization then subtracted away by appropriate counter-terms. At low temperatures, the pressure will be of order $\exp(-m_\pi/T)$. Using this prescription in the expansion of the pressure enables identification of the physical pion mass with parameters of the theory:

$$m_\pi^2 = M^2 + (2l_3 + \lambda)\frac{M^4}{F^2} + c\frac{M^6}{F^4} + \mathcal{O}(M^8) \tag{12.21}$$

The constant c is a linear combination of some regularization counter-terms and l_3. When the physical pion mass is used in the theory, the parametrical dependence on counterterms disappears. Also, λ isolates the pole appearing when $d \to 4$ in the zero-temperature part of the bare-mass coordinate-space propagator, $D(x)$, when its argument vanishes:

$$\begin{aligned}\lim_{x\to 0} D(x) &= 2M^2\lambda \\ \lambda &= \frac{1}{2}(4\pi)^{-d/2}\,\Gamma(1-d/2)M^{d-4}\end{aligned} \tag{12.22}$$

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Putting all these ingredients together, the pressure can be written as [8]

$$P = 3P_0 + \frac{4}{\pi^3} a T^4 h_3^2 + \frac{24}{\pi^3} T^6 h_5 \left(8T^2 h_5 + m_\pi^2 h_3\right)\left(b - \frac{I}{\pi^3 F^4}\right) + \mathcal{O}(T^{10}) \tag{12.23}$$

The first term is the pressure of a noninteracting Bose gas of pions (the factor 3 arising from the three charged states of the pion) with

$$P_0 = \frac{4}{\pi^2} T^4 h_5 \left(\frac{m_\pi}{T}\right) \tag{12.24}$$

The functions $h_n(m/T)$ are discussed in the Appendix. It is amusing to note that h_3 is proportional to the field fluctuations of noninteracting bosons:

$$\langle \phi^2 \rangle = \frac{\partial P_0(T,m)}{\partial m^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \frac{1}{\mathrm{e}^{\beta\omega} - 1} = \frac{T^2}{\pi^2} h_3 \left(\frac{m}{T}\right) \tag{12.25}$$

The dimensionless function $I(m_\pi/T)$ in (12.23) represents a three-dimensional integral that must be calculated numerically. Its low-temperature limit is

$$I(x) = 0.6x^{-1} + \mathcal{O}(x^{-2})$$

while the high-temperature limit is

$$I(x) = -\frac{5}{8} \ln x + 0.6360 + 0.1289x^2 + \mathcal{O}(x^3)$$

Some constants, such as c in (12.21), are absent in the final result as they are absorbed into the physical pion mass. The two constants that do appear explicitly in the pressure, a and b, are functions of the renormalized Lagrangian parameters:

$$\begin{aligned} a &= -\frac{3M^2}{32\pi F^2} + \frac{5M^4}{128\pi^3 F^4}\left(\bar{l}_1 + 2\bar{l}_2 - \frac{3}{10}\bar{l}_3 + \frac{9}{8}\right) \\ b &= \frac{1}{16\pi^3 F^4}\left(\bar{l}_1 + 4\bar{l}_2 - \frac{29}{24}\right) \end{aligned} \tag{12.26}$$

where

$$l_i = \gamma_i \left(\lambda + \frac{1}{32\pi^2}\bar{l}_i\right) \quad \gamma_1 = \frac{1}{3} \quad \gamma_2 = \frac{2}{3} \quad \gamma_3 = -\frac{1}{2} \quad \gamma_4 = 2 \tag{12.27}$$

The quantity λ, defined in (12.21), contains the singularity.

It is extremely satisfying to verify that the expression for the pressure, derived in finite-temperature chiral perturbation theory, agrees with a treatment based on the virial expansion [8]. This represents an important consistency check.

12.2 Self-energy from experimental data

At this point, it is clear that any quantum field in interaction with other fields will see its vacuum properties modified. A rigorous formalism for calculating these changes was set up in the previous chapters and, given an interaction Lagrangian, it mainly consists of calculating an in-medium self-energy. This can in turn be related to in-medium masses and decay widths, through the real and imaginary parts, respectively. Up to now we have seen that a satisfactory way of organizing the perturbation expansion was to follow the topology of the multiloop diagrams. However, this procedure becomes questionable when large coupling constants are involved, as the very validity of the perturbation expansion is called into question. Owing to asymptotic freedom, QCD in its nonperturbative sector will involve just such large constants, and a calculation of hadronic properties from first principles becomes prohibitively difficult. Nevertheless, data does exist on the scattering of the different QCD bound states among themselves. As those measurements carry some information on the underlying interaction, it should be possible to infer from them how the fundamental characteristics of a specific field get changed in a strongly interacting medium. Relying on experimental measurements to the extent that they are available will help to develop a procedure that is as model independent as possible. A method that is applicable to dilute media is described in what follows.

For a particle of type a traversing a medium with a de Broglie wavelength less than the interparticle spacing of target particles of type b, there is a direct proportionality between the scattering amplitude and the energy. The dispersion relation of a boson is determined by

$$E^2 = m^2 + p^2 + \Pi \tag{12.28}$$

In the nonrelativistic limit we may wish to express the energy in terms of an optical potential U as

$$E = m + \frac{p^2}{2m} + U \tag{12.29}$$

The optical potential will in general have both real and imaginary parts. This leads to real and imaginary parts of the energy: $E = E_{\rm R} - i\Gamma/2$. The imaginary part is related to the mean free path $1/\rho\sigma$, where σ is the scattering cross section and ρ is the density of scatterers, and to the velocity, as $\Gamma = v\rho\sigma$. Using the forward scattering amplitude f and the optical theorem $p\sigma = 4\pi f$ gives

$$\operatorname{Im}\Pi = 2m \operatorname{Im} U = -4\pi\rho \operatorname{Im} f \tag{12.30}$$

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In the low-energy limit the mean potential energy of the particle is

$$\mathrm{Re}\, U = \rho \int d^3x\, V(\mathbf{x}) \tag{12.31}$$

where V is the two-body potential. In this limit the Born approximation gives

$$\mathrm{Re}\, f = -\frac{m}{2\pi} \int d^3x\, V(\mathbf{x}) \tag{12.32}$$

Hence both the real and imaginary parts fit a simple formula,

$$\Pi = -4\pi\rho f \tag{12.33}$$

This formula has a wider range of applicability than this derivation might suggest; it is the leading term in a multiple-scattering expansion [9].

The generalization to target particles that are moving, and to relativistic kinematics, is straightforward. For meson a scattering from hadron b in the medium, the contribution to the self-energy is:

$$\begin{aligned}\Pi_{ab}(E,p) &= -4\pi \int \frac{d^3k}{(2\pi)^3}\, n_b(\omega)\, \frac{\sqrt{s}}{\omega}\, f_{ab}^{(\mathrm{cm})}(s) \\ &= -\frac{1}{2\pi p} \int_{m_b}^{\infty} d\omega\, n_b(\omega) \int_{s_-}^{s_+} ds\, \sqrt{s} f_{ab}^{(\mathrm{cm})}(s)\end{aligned} \tag{12.34}$$

where E and p are the energy and momentum of the particle, $\omega^2 = m_b^2 + k^2$,

$$s_{\pm} = E^2 - p^2 + m_b^2 + 2(E\omega \pm pk) \tag{12.35}$$

n_b is either a Bose–Einstein or Fermi–Dirac occupation number, and f_{ab} is the forward scattering amplitude. The normalization of the amplitude corresponds to the standard form of the optical theorem,

$$\sigma = \frac{4\pi}{q_{\mathrm{cm}}}\, \mathrm{Im}\, f^{(\mathrm{cm})}(s) \tag{12.36}$$

where q_{cm} is the momentum in the cm frame. The dispersion relation is determined by the poles of the propagator after summing over all target species and including the vacuum contribution to the self-energy:

$$E^2 - m_a^2 - p^2 - \Pi_a^{\mathrm{vac}}(E,p) - \sum_b \Pi_{ab}(E,p) = 0 \tag{12.37}$$

The applicability of (12.34) is limited to those cases where interference between sequential scatterings is negligible.

Taking various limits of (12.34) is instructive. First of all, we note that the cross section is invariant under longitudinal boosts. It is convenient to know how the scattering amplitude transforms. They are related to each other as follows.

$$m_a f_{ab}^{(a\text{'s rest frame})} = m_b f_{ab}^{(b\text{'s rest frame})} = \sqrt{s} f_{ab}^{(\text{cm})} \tag{12.38}$$

In the limit that the target particles b move nonrelativistically we can approximate ω in the first line of (12.34) by m_b, in which case

$$\Pi_{ab} = -4\pi f_{ab}^{(b\text{'s rest frame})} \rho_b \tag{12.39}$$

where ρ_b is the spatial density. Next consider the chiral limit when pions serve as the target particles, relevant for low-temperature baryon-free matter. From (12.38) $\sqrt{s} f_{a\pi}^{(\text{cm})} = m_a f_{a\pi}^{(a\text{'s rest frame})}$. Since $f_{a\pi}^{(a\text{'s rest frame})}$ involves two derivative couplings of the pion to the massive state a (Adler's theorem) one sees from (12.34) that $\Pi_{a\pi} \sim T^4$. Finally, if the self-energy is evaluated in the rest frame of a it is possible to do all the integrations but one:

$$\Pi_{ab}(E,p) = -\frac{m_a^2 T}{\pi p} \int_{m_b}^{\infty} d\omega \ln\left(\frac{1-\exp(-\omega_+/T)}{1-\exp(-\omega_-/T)}\right) f_{ab}^{(a\text{'s rest frame})}(\omega) \tag{12.40}$$

Here $\omega_\pm = (E\omega \pm pk)/m_a$. This assumes that b is a boson; a similar formula ensues if it is a fermion.

As a specific application, we will estimate the ρ meson dispersion relation for finite temperature and baryon density and for momenta up to 1 GeV/c. This is of special interest, as vector mesons can couple directly to the photon [10] and therefore the in-medium modification of vector meson properties can in principle be inferred from the measurement of electromagnetic observables. This direct conversion of a vector meson to a photon (real or virtual) is often referred to as vector meson dominance (VMD). The low-energy part of the ρ meson scattering amplitude will be dominated by coupling to resonances. The physical context assumed here is that ρ mesons are formed during the last stage of the evolution of hadronic matter created in a heavy ion collision. The matter there is approximated as a weakly interacting gas of pions and nucleons. This stage is formed when the local temperature is of the order of 100 to 150 MeV and when the local baryon density is of the order of the normal nucleon density in a nucleus. The main ingredients of the calculation are $\rho\pi$ and ρN forward scattering amplitudes and total cross sections.

We will consider the momentum p to be real and evaluate the scattering amplitudes on-shell, that is, evaluate the self-energy at $E = \sqrt{p^2 + m_\rho^2}$. In this case (12.37) takes the form

$$E^2 = m_\rho^2 + p^2 + \Pi_\rho^{\text{vac}} + \Pi_{\rho\pi}(p) + \Pi_{\rho N}(p) \tag{12.41}$$

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Since the self-energy has real and imaginary parts, so does $E(p) = E_{\rm R}(p) - i\Gamma(p)/2$. In the narrow-width approximation the dispersion relation is determined from

$$\begin{aligned} E_{\rm R}^2(p) &= p^2 + m_\rho^2 + {\rm Re}\,\Pi_{\rho\pi}(p) + {\rm Re}\,\Pi_{\rho N}(p) \\ \Gamma(p) &= -\left[{\rm Im}\,\Pi_\rho^{\rm vac} + {\rm Im}\,\Pi_{\rho\pi}(p) + {\rm Im}\,\Pi_{\rho N}(p)\right] / E_{\rm R}(p) \end{aligned} \tag{12.42}$$

The width of the ρ meson in vacuum, $\Gamma_\rho^{\rm vac} = -{\rm Im}\,\Pi_\rho^{\rm vac}/m_\rho$, is 150 MeV.

We can also define a mass shift and an optical potential:

$$\begin{aligned} \Delta m_\rho(p) &= \sqrt{m_\rho^2 + {\rm Re}\,\Pi_{\rho\pi}(p) + {\rm Re}\,\Pi_{\rho N}(p)} - m_\rho \\ {\rm U}(p) &= E_{\rm R}(p) - \sqrt{m_\rho^2 + p^2} \end{aligned} \tag{12.43}$$

These will be evaluated for temperatures of 100 and 150 MeV and nucleon densities of 0, 1, and 2 times the normal nuclear matter density (0.155 nucleons per fm^3). Recall that one needs a Bose–Einstein distribution for pions and a Fermi–Dirac distribution for nucleons. The pion chemical potentials are zero and the nucleon chemical potentials are 745 and 820 MeV for densities of 1 and 2 times normal at $T = 100$ MeV, and 540 and 645 MeV at $T = 150$ MeV. Antinucleons are not considered here. For a ρ meson scattering from a particle a and going to a resonance R, the forward scattering amplitude can be written in its usual nonrelativistic form, in the center of mass:

$$f_{\rho a}^{\rm cm}(s) = \frac{1}{2q_{\rm cm}} \sum_R W_{\rho a}^R \frac{\Gamma_{R\to\rho a}}{M_R - \sqrt{s} - \frac{1}{2}i\Gamma_R} - \frac{q_{\rm cm} r_{\rm P}^{\rho a}}{4\pi s} \frac{\left(1 + \exp^{-i\pi\alpha_{\rm P}}\right)}{\sin \pi\alpha_{\rm P}} s^{\alpha_{\rm P}} \tag{12.44}$$

In familiar notation, the subscript P refers to the Pomeron, $\sqrt{s}$ is the total cm energy and the magnitude of the cm momentum is

$$q_{\rm cm} = \frac{1}{2\sqrt{s}} \sqrt{[s - (m_\rho + m_a)^2][s - (m_\rho - m_a)^2]} \tag{12.45}$$

The statistical averaging factor for spin and isospin is

$$W_{\rho a}^R = \frac{(2s_R + 1)}{(2s_\rho + 1)(2s_a + 1)} \frac{(2t_R + 1)}{(2t_\rho + 1)(2t_a + 1)} \tag{12.46}$$

The second part of the forward scattering amplitude is a nonresonant background contribution, a description of which goes beyond this text. See, for example, Collins [11] for a detailed discussion. It suffices here to state that the parameters are determined by high-energy scattering phenomenology. Also, the real and imaginary parts of the scattering amplitude are related by a dispersion relation. This constraint turns out to be better satisfied in the presence of the background term [12].

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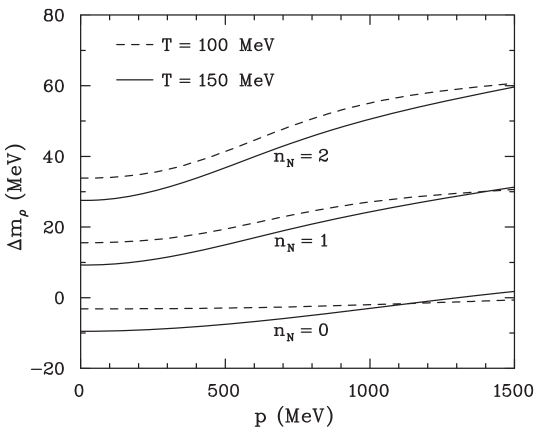

Fig. 12.2. The vector meson mass shift as a function of momentum for various temperatures and nucleon densities n_N (expressed in units of equilibrium nuclear matter density).

For the case of ρN scattering, the intermediate resonance can be one of several species of N^* or Δ resonances. One then needs to know the width of that resonance in the channel where there is a ρ meson and a nucleon. Because of kinematical constraints, this width is often not measured, but the radiative decays often are. These can be related to the width one is after, using the VMD relationship of the scattering amplitudes:

$$f_{\gamma N} = 4\pi\alpha \left(\frac{1}{g_\rho^2} f_{\rho N} + \frac{1}{g_\omega^2} f_{\omega N} + \frac{1}{g_\phi^2} f_{\phi N} \right) \tag{12.47}$$

where α is the fine structure constant. From measurements of ϕ photoproduction, the last term is small and can be neglected. In the spirit of the quark model, one further assumes that $f_{\omega N} \approx f_{\rho N}$. This assumption may in fact be examined more closely [13] The direct vector-meson–photon coupling can be deduced from $V \to l^+ l^-$ measurements. With these ingredients, the widths in the ρN channel can be directly extracted from the radiative decay widths. The details of the procedure outlined here, along with specific parameter values and relevant references, can be found in Eletsky *et al.* [12] Note that the calculation of the real and imaginary parts of the in-medium self-energy of any species can proceed in the same way, provided that enough experimental data can map its interaction with other fields. The mass shift and width of the ρ meson, as defined in (12.43) and (12.42), are shown in Figures 12.2 and 12.3 for different temperatures and densities (n_N is in units of n_0, the equilibrium nuclear matter density). The width is systematically larger at larger temperatures and densities. The change in mass is numerically less important. Any interaction

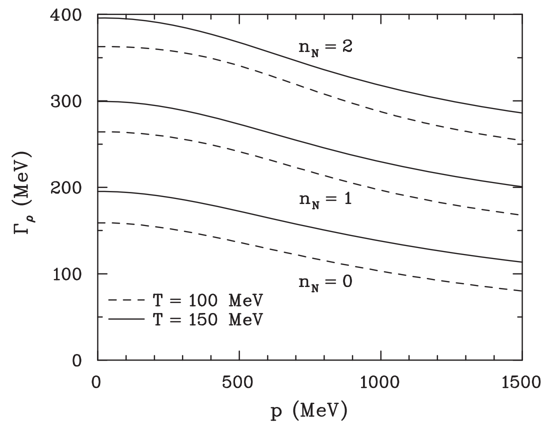

Fig. 12.3. The vector meson width as a function of momentum for various temperatures and densities.

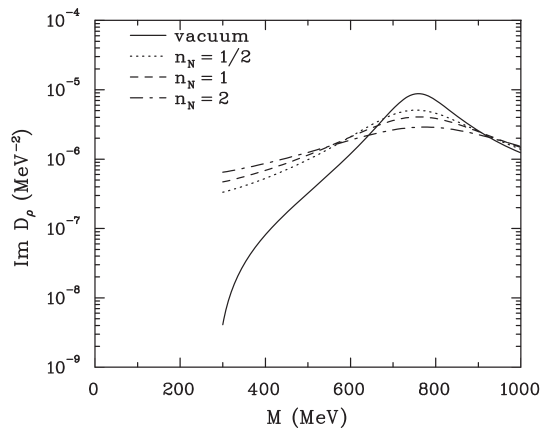

Fig. 12.4. The imaginary part of the vector meson propagator as a function of invariant mass at a momentum 300 MeV and temperature 150 MeV.

will contribute to a larger width, but the real part of the self-energy can be less affected owing to cancellations between different channels. The information in the mass shift and in the width is also contained in a plot of the imaginary part of the ρ propagator, shown in Figure 12.4, and is directly related to the in-medium spectral density. Note that since the thermal medium constitutes a preferred rest frame (that in which temperature is defined), the self-energy in general depends on the energy and the momentum separately. Alternatively, one may fix the momentum at

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a specific value (300 MeV here) and study the self-energy as a function of invariant mass, since $E = \sqrt{p^2 + M^2}$.

Alternatively, a method complementary to the one presented here consists of using effective hadronic Lagrangians (i.e., those whose basic symmetries are consistent with that of QCD), with parameters fitted to measured properties [14–16]. Because they are both constrained by experimental data, the two techniques should of course yield comparable results unless ones deviates significantly from the on-shell condition for the vector field.

12.3 Weinberg sum rules

Spectral sum rules were in use before the advent of QCD as the theory of the strong interaction. Weinberg had in fact proposed two sum rules based on current algebra, relating moments of the spectral density of vector and axial-vector currents [17]. These relied on the validity of chiral symmetry. It is instructive to revisit these sum rules in the language of QCD and then to pursue a finite-temperature extension, in order to explore the implications of the approach to chiral symmetry restoration at finite temperature that follow from sum rules of the Weinberg type [1]. Note that the up and down quark masses are then implicitly assumed to be zero, so that chiral symmetry is indeed exact.

12.3.1 Sum rules at zero temperature

One first defines vector and axial-vector currents (using an explicit notation for the current operators):

$$V_\mu^a = \bar{\psi}\gamma_\mu(\tau^a/2)\psi \tag{12.48}$$

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5(\tau^a/2)\psi \tag{12.49}$$

where $\tau^a/2$ is the isospin generator. With this normalization the current algebra of charges obeys the equal-time commutation relations

$$\left[Q_V^a, Q_V^b\right] = i\varepsilon^{abc}Q_V^c \tag{12.50}$$

$$\left[Q_V^a, Q_A^b\right] = i\varepsilon^{abc}Q_A^c \tag{12.51}$$

$$\left[Q_A^a, Q_A^b\right] = i\varepsilon^{abc}Q_V^c \tag{12.52}$$

Each charge is the volume integral of the zeroth component of the corresponding current operator. We now write the vector and axial-vector spectral densities. They are positive definite quantities defined for

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positive s:

$$\langle 0|V_a^\mu(x)V_b^\nu(0)|0\rangle = -\frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\mathrm{e}^{ip\cdot x}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\rho_V(s) \tag{12.53}$$

$$\langle 0|A_a^\mu(x)A_b^\nu(0)|0\rangle = -\frac{\delta^{ab}}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\mathrm{e}^{ip\cdot x}\left[\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\rho_A(s) + f_\pi^2\delta(s)p^\mu p^\nu\right] \tag{12.54}$$

The dimension of the spectral densities is energy-squared. Note that the pion contribution to the axial-vector correlator has been written out explicitly in the second term in (12.54).

Imaginary time is used, so that all distances are space-like, or Euclidean: $x^2 = t^2 - r^2 = -\tau^2$. In this domain the spectral representation of the correlation functions is as follows:

$$\begin{aligned}\Delta D_\mu^{ab\mu}(\tau) &\equiv \langle 0|T_\tau\left[V^{a\mu}(x)V_\mu^b(0) - A^{a\mu}(x)A_\mu^b(0)\right]|0\rangle \\ &= -\frac{\delta^{ab}}{4\pi^2\tau}\int_0^\infty ds\,\sqrt{s}\left[3\rho_V(s)-3\rho_A(s)-s\,f_\pi^2\delta(s)\right]K_1(\sqrt{s}\tau)\end{aligned} \tag{12.55}$$

and

$$\begin{aligned}\Delta D_{ab}^{00}(\tau) &\equiv \langle 0|T_\tau\left[V_a^0(x)V_b^0(0) - A_a^0(x)A_b^0(0)\right]|0\rangle \\ &= -\frac{\delta_{ab}}{4\pi^2\tau}\int_0^\infty ds\,\sqrt{s}\left[\rho_V(s)-\rho_A(s)-s\,f_\pi^2\delta(s)\right] \\ &\quad\times\left[\frac{K_0(\sqrt{s}\tau)}{\sqrt{s}\tau}+\left(\frac{2}{s\tau^2}+1\right)K_1(\sqrt{s}\tau)\right]\end{aligned} \tag{12.56}$$

Notice that the integrands essentially involve the standard Feynman propagator for a particle of mass m, which, in the Euclidean domain, is

$$D(m,\tau)_{\text{free scalar}} = \frac{m}{4\pi^2\tau}K_1(m\tau) \tag{12.57}$$

Exponential decay of the Bessel function K_1 at large values of the argument ensures the convergence of such integrals for any QCD correlation functions, except probably at $\tau = 0$.

Each sum rule will correspond to a particular term in the small-distance asymptotic expansion of the correlation function. In the limit $\tau \to 0$ the product of currents can be expanded according to the operator product expansion (OPE), a very successful means of connecting vacuum expectation values (VEVs) of quark and gluon operators with experimentally observable hadronic properties. We will refer the reader to the original

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literature for a discussion of this powerful theoretical method, but a general description can be given as follows. Consider for example the current–current correlator in real time and its expansion:

$$i\int d^4x\; \mathrm{e}^{iq\cdot x}T_t\{\bar{\psi}(x)\gamma_\mu\psi(x),\,\bar{\psi}(0)\gamma_\nu\psi(0)\} = (q_\mu q_\nu - q^2 g_{\mu\nu})\sum_d C_d(q^2)O_d \tag{12.58}$$

The O_d are local operators and the $C_d(q^2)$ are c-numbers called Wilson coefficients. The operator expansion is organized according to dimension. When considering a vacuum matrix element of the current–current correlator, one might simply expect all operators except the unit operator to have a vanishing expectation value. However, long-distance nonperturbative effects will make this expectation unrealized. In principle, all vacuum expectation values, often called vacuum condensates, should be calculable in lattice gauge theory. The initial terms in this expansion were first computed perturbatively by Shifman, Vainshtein, and Zakharov [18]. For the contracted polarization tensor the result is

$$\begin{aligned} D_\mu^{ab\mu}(\tau) &\equiv \langle 0|T_\tau\left[V^{a\mu}(x)V_\mu^b(0)\right]|0\rangle \\ &= -\frac{3\delta^{ab}}{\pi^4\tau^6}\left(1+\frac{\alpha_\mathrm{s}(\tau)}{\pi} - \frac{\langle 0|\left(gF_{\mu\nu}^c\right)^2|0\rangle\tau^4}{3\times 2^7}\right. \\ &\qquad \left. -\frac{\pi^2\tau^6}{8}\ln(\mu\tau)\,\langle 0|\mathcal{O}_\rho|0\rangle + \cdots\right) \end{aligned} \tag{12.59}$$

where, in the argument of the logarithm, $\mu \ll 1/\tau$ is the renormalization scale, and $\mathcal{O}_\rho$ is a complicated four-quark operator. There is a similar expression for the correlator of two axial-vector currents but it has a different four-quark operator $\mathcal{O}_{a_1}$. For our purposes we only need their difference, which is given below.

Since chiral symmetry breaking is a long-wavelength phenomenon, at very short distances or at very high energies the difference between vector and axial-vector correlators should go to zero. Indeed, taking this difference one finds that all terms except for the four-quark operators in (12.59) drop out. One can now look for consequences of this statement for the spectral density. Expanding the Bessel function in (12.55) for small values of τ we get

$$\begin{aligned} \Delta D_\mu^{ab\mu}(\tau) = -\frac{3\delta^{ab}}{4\pi^2}\int_0^\infty ds\,[\rho_V(s)-\rho_A(s)] \\ \times\left[\frac{1}{\tau^2}+\frac{s}{2}\ln\left(\frac{\sqrt{s}\tau}{2}\mathrm{e}^{\gamma_\mathrm{E}-1/2}\right)+\mathcal{O}\left(\tau^2,\tau^2\ln\tau\right)\right] \end{aligned} \tag{12.60}$$

where $\gamma_{\rm E}$ is Euler's constant. The OPE has no power divergence in τ in the difference $\Delta D_\mu^{ab\mu}$. Therefore the coefficient of $1/\tau^2$ in (12.60) must vanish. This gives the second Weinberg sum rule (see below). In the OPE framework it simply follows from the observation that the first covariant operators which are not chirality blind are four-quark ones that have dimension 6 or more. Similarly expanding (12.56) for small τ and applying the observation of chirality blindness we get

$$\int_0^\infty \frac{ds}{s}\left[\rho_V(s) - \rho_A(s) - s\, f_\pi^2 \delta(s)\right]\left(\frac{1}{\tau^4} + \frac{s}{4\tau^2}\right) = 0 \tag{12.61}$$

The first and second terms in the last parentheses give

$$\text{I} \qquad \int_0^\infty \frac{ds}{s}\left[\rho_V(s) - \rho_A(s)\right] = f_\pi^2 \tag{12.62}$$

and

$$\text{II} \qquad \int_0^\infty ds\left[\rho_V(s) - \rho_A(s)\right] = 0 \tag{12.63}$$

respectively. These are Weinberg's first and second sum rules.

The phenomenological implications of the zero-temperature sum rules have been discussed numerous times in the literature and we will therefore not do so here.

12.3.2 Sum rules at finite temperature

Weinberg's two sum rules can be extended to finite temperature using essentially the same methods as he used without any specific reference to QCD. As seen in other applications, earlier in this text, the introduction of a thermal medium will complicate some expressions as Lorentz invariance is no longer manifest. This preferred rest frame will cause functions that previously depended only on $\sqrt{s}$ to depend separately on energy and momentum, and the number of Lorentz tensors will increase because there is a new vector available, namely, the vector $u_\mu = (1, 0, 0, 0)$ that specifies the rest frame of the matter.

We now define the longitudinal and transverse spectral densities for the vector current as

$$\langle V_a^\mu(x) V_b^\nu(0)\rangle = \frac{\delta^{ab}}{(2\pi)^3}\int d^4p\, {\rm e}^{ip\cdot x}[1 + N_{\rm B}(p_0)]\left(\rho_V^{\rm L} P_{\rm L}^{\mu\nu} + \rho_V^{\rm T} P_{\rm T}^{\mu\nu}\right) \tag{12.64}$$

and for the axial-vector current as

$$\langle A_a^\mu(x) A_b^\nu(0)\rangle = \frac{\delta^{ab}}{(2\pi)^3}\int d^4p\, {\rm e}^{ip\cdot x}[1 + N_{\rm B}(p_0)]\left[\rho_A^{\rm L} P_{\rm L}^{\mu\nu} + \rho_A^{\rm T} P_{\rm T}^{\mu\nu}\right] \tag{12.65}$$

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In these expressions the angle brackets refer to the thermal average. The longitudinal and transverse projection tensors were defined in Chapter 5. These spectral densities are the ρ^{n} discussed in Section 6.2. In general the spectral densities depend on p^0 and $\mathbf{p}$ separately as well as on the temperature (and chemical potential). In the vacuum we can always go to the rest frame of a massive particle and in this frame there can be no difference between longitudinal and transverse polarizations, so that $\rho_{\mathrm{L}} = \rho_{\mathrm{T}} = \rho$. Since $P_{\mathrm{L}}^{\mu\nu} + P_{\mathrm{T}}^{\mu\nu} = -(g^{\mu\nu} - p^\mu p^\nu/p^2)$, (12.64) and (12.65) collapse to (12.53) and (12.54). The pion, being a massless Goldstone boson, is special. It contributes to the longitudinal axial spectral density and not to the transverse one. In fact, we could write

$$f_\pi^2 \delta(p^2) p^\mu p^\nu \;=\; f_\pi^2 p^2 \delta(p^2) P_{\mathrm{L}}^{\mu\nu} \tag{12.66}$$

This should not be done at finite temperature because the contribution of the pion to the longitudinal spectral density cannot be assumed to be a delta function in p^2. In general the pion's dispersion relation will be more complicated and will develop a width at nonzero momentum. Therefore, we do not try to separate out the pionic contribution but subsume it into the spectral density ρ_A^{L}, without any loss of generality.

Following Weinberg, we define a three-point function by

$$-i\epsilon_{abc} M^{\mu\nu\lambda}(q,p) \;=\; \int d^4x\, d^4y\, \mathrm{e}^{-i(q\cdot x + p\cdot y)} \left\langle T_t \left[A_a^\mu(x) A_b^\nu(y) V_c^\lambda(0) \right] \right\rangle \tag{12.67}$$

We multiply both sides by q_μ. On the right-hand side we can use

$$q_\mu \mathrm{e}^{-i(q\cdot x + p\cdot y)} \;=\; i\frac{\partial}{\partial x^\mu} \mathrm{e}^{-i(q\cdot x + p\cdot y)} \tag{12.68}$$

Both the vector and axial-vector currents are conserved. We assume that we can integrate by parts and that the surface term is zero. The nonzero contribution comes from

$$\begin{aligned} &\frac{\partial}{\partial x^\mu} \left\{ T_t \left[A_a^\mu(x) A_b^\nu(y) V_c^\lambda(0) \right] \right\} \\ &= \delta(x^0 - y^0) \left\{ \theta(x^0) \left[A_a^0(x), A_b^\nu(y) \right] V_c^\lambda(0) + \theta(-x^0) V_c^\lambda(0) \left[A_a^0(x), A_b^\nu(y) \right] \right\} \\ &\quad + \delta(x^0) \left\{ \theta(y^0) A_b^\nu(y) \left[A_a^0(x), V_c^\lambda(0) \right] + \theta(-y^0) \left[A_a^0(x), V_c^\lambda(0) \right] A_b^\nu(y) \right\} \end{aligned} \tag{12.69}$$

From this expression we see the need for knowledge of the equal-time commutators. Consistently with the normalization of (12.50)–(12.52) we have

$$\delta(z^0)\left[A_a^0(x), A_b^\nu(y)\right] = i\epsilon_{abd}V_d^\nu(x)\delta(\mathbf{z}) + S_{Vab}^{\nu j}(\mathbf{x})\frac{\partial}{\partial z^j}\delta(\mathbf{z})$$
$$\delta(z^0)\left[A_a^0(x), V_b^\nu(y)\right] = i\epsilon_{abd}A_d^\nu(x)\delta(\mathbf{z}) + S_{Aab}^{\nu j}(\mathbf{x})\frac{\partial}{\partial z^j}\delta(\mathbf{z}) \tag{12.70}$$

Here $z = x - y$ and the S's denote the Schwinger terms. These terms do not vanish, in general, and they need to appear to guarantee the self-consistency of the current algebra.

Consider now the contribution of the Schwinger terms to the thermal average. Generically they will be of the form

$$\langle SJ\rangle \;=\; Z^{-1}\sum_{m,n}\mathrm{e}^{-K_n/T}\langle n|S|m\rangle\langle m|J|n\rangle \tag{12.71}$$

where $K = H - \mu N$ is the Hamiltonian minus the chemical potential times the conserved particle number, the states are chosen to be eigenstates of H, N, and isospin, and J is either the vector or the axial-vector current. J has isospin 1, so we get zero if either (i) S is a c-number, or (ii) S is an operator with no isospin-1 component. We shall assume that one of these holds. Then

$$\frac{\partial}{\partial x^\mu}\left\langle T_t\left[A_a^\mu(x)A_b^\nu(y)V_c^\lambda(0)\right]\right\rangle = i\epsilon_{abd}\delta(x-y)\left\langle T_t\left[V_d^\nu(x)V_c^\lambda(0)\right]\right\rangle + i\epsilon_{acd}\delta(x)\left\langle T_t\left[A_b^\nu(y)A_d^\lambda(0)\right]\right\rangle \tag{12.72}$$

It is now a simple matter to show that

$$\tfrac{1}{2}q_\mu M^{\mu\nu\lambda}(q,p) \;=\; D_V^{\nu\lambda}(q+p) - D_A^{\nu\lambda}(p) \tag{12.73}$$

where the D's are the propagators for the currents; for example,

$$\delta_{ab}D_A^{\nu\lambda}(p) \;=\; \int d^4y\,\mathrm{e}^{-ip\cdot y}\left\langle T_t\left[A_a^\nu(y)A_b^\lambda(0)\right]\right\rangle \tag{12.74}$$

Similarly, one can show that

$$\tfrac{1}{2}(q+p)_\lambda M^{\mu\nu\lambda}(q,p) \;=\; D_A^{\mu\nu}(q) - D_A^{\mu\nu}(p) \tag{12.75}$$

These Ward identities have exactly the same form as at zero temperature [17].

With a similar consideration of the three-point function

$$-i\epsilon_{abc}N^{\mu\nu\lambda}(q,p) \;=\; \int d^4x\,d^4y\,\mathrm{e}^{-i(q\cdot x+p\cdot y)}\left\langle T_t\left[V_a^\mu(x)V_b^\nu(y)V_c^\lambda(0)\right]\right\rangle \tag{12.76}$$

one can prove two more Ward identities,

$$\tfrac{1}{2}q_\mu N^{\mu\nu\lambda}(q,p) = D_V^{\nu\lambda}(q+p) - D_V^{\nu\lambda}(p) \tag{12.77}$$

and

$$\tfrac{1}{2}(q+p)_\lambda N^{\mu\nu\lambda}(q,p) = D_V^{\mu\nu}(q) - D_V^{\mu\nu}(p) \tag{12.78}$$

Multiply (12.75) by $(q+p)_\lambda$ and (12.77) by q_μ. Doing the same for the other two Ward identities, one obtains the constraints

$$(q+p)_\lambda D_V^{\nu\lambda}(q+p) \;=\; q_\lambda D_V^{\nu\lambda}(q) \;+\; p_\lambda D_V^{\nu\lambda}(p) \;=\; q_\lambda D_A^{\nu\lambda}(q) \;+\; p_\lambda D_A^{\nu\lambda}(p) \tag{12.79}$$

The equation above holds for all values of q and p. This implies

$$k_\lambda D_V^{\nu\lambda}(k) \;=\; k_\lambda D_A^{\nu\lambda}(k) \;=\; C^{\nu\lambda} k_\lambda \tag{12.80}$$

where $C^{\nu\lambda}$ is momentum independent (but can depend on temperature) and is the same for the vector and axial-vector channels. By taking the Fourier transform of these relations we can find the thermal averages of the equal-time commutators,

$$\delta(x^0)\big\langle\, \big[V_a^\nu(x), V_b^0(0)\big]\,\big\rangle \;=\; \delta(x^0)\big\langle\, \big[A_a^\nu(x), A_b^0(0)\big]\,\big\rangle \;=\; \delta_{ab} C^{\nu\lambda} \frac{\partial}{\partial x^\lambda}\delta(x) \tag{12.81}$$

The commutators above can be expressed in terms of the spectral densities from (12.64) and (12.65). Taking their difference one obtains the finite-temperature generalization of the first Weinberg sum rule,

$$\text{I} \qquad \int_0^\infty \frac{d\omega\,\omega}{\omega^2 - \mathbf{p}^2}\left[\rho_V^{\mathrm{L}}(\omega,\mathbf{p}) - \rho_A^{\mathrm{L}}(\omega,\mathbf{p})\right] \;=\; 0 \tag{12.82}$$

Here (6.44) has been used to write the integral over positive ω only. Notice that this sum rule involves only the longitudinal spectral densities and not the transverse ones. At zero temperature the spectral densities depend only on $p^2 = s = \omega^2 - \mathbf{p}^2$. Then this equation reduces to (12.62) once we remember to separate out the pion part of ρ_A^{L}, namely, $s f_\pi^2 \delta(s)$. At finite temperature, the spectral densities in general will depend on ω and $\mathbf{p}$ separately and not just on the combination s. Then this sum rule must be satisfied at each value of the momentum.

For the second sum rule, we follow a method due to Das, Mathur, and Okubo [19]. Omitting the index V or A the explicit expressions for the time-ordered propagator are

$$D^{00}(p^0,\mathbf{p}) = \mathbf{p}^2 D_{\mathrm{L}}(p^0,\mathbf{p}) \tag{12.83}$$

$$D^{0j}(p^0,\mathbf{p}) = p^0 p^j D_{\mathrm{L}}(p^0,\mathbf{p}) \tag{12.84}$$

$$D^{ij}(p^0,\mathbf{p}) = \left(\delta^{ij} - \frac{p^i p^j}{\mathbf{p}^2}\right) D_{\mathrm{T}}(p^0,\mathbf{p}) \;+\; \frac{p^i p^j}{\mathbf{p}^2} D'_{\mathrm{L}}(p^0,\mathbf{p}) \tag{12.85}$$

where

$$D_{\rm L}(p^0,\mathbf{p}) = 2i\int_{-\infty}^{\infty}\frac{d\omega\,\omega}{\omega^2-\mathbf{p}^2}\left[\frac{\rho^{\rm L}(\omega,\mathbf{p})}{(\omega+i\epsilon)^2-p_0^2}\right][1+N_{\rm B}(\omega)] \quad (12.86)$$

$$D'_{\rm L}(p^0,\mathbf{p}) = 2i\int_{-\infty}^{\infty}\frac{d\omega\,\omega^3}{\omega^2-\mathbf{p}^2}\left|\frac{\rho^{\rm L}(\omega,\mathbf{p})}{(\omega+i\epsilon)^2-p_0^2}\right|[1+N_{\rm B}(\omega)] \quad (12.87)$$

$$D_{\rm T}(p^0,\mathbf{p}) = 2i\int_{-\infty}^{\infty}d\omega\,\omega\left|\frac{\rho^{\rm T}(\omega,\mathbf{p})}{(\omega+i\epsilon)^2-p_0^2}\right|[1+N_{\rm B}(\omega)] \quad (12.88)$$

and for the Schwinger term they are

$$C^{00} = C^{0j} = C^{j0} = 0 \qquad C^{ij}(\mathbf{p}) = \delta^{ij}D_{\rm S}(\mathbf{p}) \quad (12.89)$$

where

$$D_{\rm S}(\mathbf{p}) = 2i\int_{-\infty}^{\infty}\frac{d\omega\,\omega}{\omega^2-\mathbf{p}^2}\,\rho^{\rm L}(\omega,\mathbf{p})\,[1+N_{\rm B}(\omega)] \quad (12.90)$$

The first observation we can make concerns the thermally averaged generic Schwinger term C. Since it is the same for the vector and the axial-vector correlators, by (12.80), the factor $D_{\rm S}(\mathbf{p})$ must be the same as well. Equating them reproduces the first finite-temperature sum rule (12.82).

The essence of the argument of Das, Mathur, and Okubo is that spontaneous chiral symmetry breaking is a low-energy phenomenon. At very high energy it must disappear, at least in the limit that quark masses are zero and chiral symmetry is exact. Thus the difference between the vector and axial-vector propagators should go to zero at very high energy,

$$\lim_{p^0\to\infty,\ \mathbf{p}\,\text{fixed}}\left[D_V^{\mu\nu}(p^0,\mathbf{p}) - D_A^{\mu\nu}(p^0,\mathbf{p})\right] = 0 \quad (12.91)$$

If we do this for the time–time or time–space components of the propagators, that is, for the $D_{\rm L}$, we again reproduce the first finite-temperature sum rule. Expanding to the next order in $1/p_0^2$ we obtain a finite-temperature generalization of the second zero-temperature sum rule, which is

$$\text{II-L} \qquad \int_0^{\infty}d\omega\,\omega\left[\rho_V^{\rm L}(\omega,\mathbf{p}) - \rho_A^{\rm L}(\omega,\mathbf{p})\right] = 0 \quad (12.92)$$

Like the first, this sum rule involves only the longitudinal spectral densities, and so we call it II-L. Also like the first, it reduces to the original Weinberg sum rule as the temperature and/or chemical potential go to zero.

Next we consider the space–space components of the propagators. Examination of the $D'_{\rm L}$ in the infinite-energy limit gives us the sum rule II-L and nothing new. Examination of the $D_{\rm T}$ in the infinite-energy limit

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gives us another sum rule, which we call II-T because it involves the transverse spectral densities,

$$\text{II-T} \qquad \int_0^\infty d\omega\,\omega\left[\rho_V^{\rm T}(\omega,\mathbf{p})-\rho_A^{\rm T}(\omega,\mathbf{p})\right]\;=\;0 \tag{12.93}$$

The finite-temperature sum rules II-L and II-T should become degenerate at $\mathbf{p}=\mathbf{0}$ because there ought not to be any difference between longitudinal and transverse excitations at rest. The sum rule II-T then reduces to the original second sum rule in the vacuum.

We want to emphasize that the sum rules derived in this section, I, II-L, and II-T, must be satisfied for every value of the momentum. Furthermore, our derivation is more general than QCD; any theory that satisfies the assumptions we have made must obey these sum rules.

Low-temperature behavior

As we are taking the zero-quark-mass limit here the pion is massless below any critical temperature for chiral symmetry restoration and/or deconfinement, and thus at parametrically low temperatures the heat bath is dominated by pions. In [20] the so-called Dey–Eletsky–Ioffe mixing theorem was proven, which says that, to order T^2, there is no change in the masses of vector and axial-vector mesons. What does change are the couplings to the currents. The finite-temperature correlators can be described by a mixing between the vector and axial-vector $T=0$ correlators with a temperature-dependent coefficient:

$$D_V^{\mu\nu}(p,T)=(1-\epsilon)D_V^{\mu\nu}(p,0)\;+\;\epsilon D_A^{\mu\nu}(p,0) \tag{12.94}$$
$$D_A^{\mu\nu}(p,T)=(1-\epsilon)D_A^{\mu\nu}(p,0)\;+\;\epsilon D_V^{\mu\nu}(p,0) \tag{12.95}$$

These are valid to first order in $\epsilon\equiv T^2/6f_\pi^2$. This implies the same mixing of the spectral densities, namely,

$$\rho_V(p^0,\mathbf{p},T)=(1-\epsilon)\rho_V(s,0)\;+\;\epsilon\rho_A(s,0) \tag{12.96}$$
$$\rho_A(p^0,\mathbf{p},T)=(1-\epsilon)\rho_A(s,0)\;+\;\epsilon\rho_V(s,0) \tag{12.97}$$

with the appropriate longitudinal and transverse subscripts. The temperature dependence of the pion decay coupling was thus proven to be $f_\pi^2(T)=(1-\epsilon)f_\pi^2$ for small T, consistent with the prediction of chiral perturbation theory [21]. Therefore, the finite-temperature sum rules I (12.82), II-L (12.92), and II-T (12.93), reduce to the original zero-temperature sum rules but with both sides of (12.62) and (12.63) multiplied by the factor $1-2\epsilon$. This satisfies the Dey–Eletsky–Ioffe mixing theorem.

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The approach to chiral-symmetry restoration

Chiral transformations are rotations of the quark field with γ_5, and they may or may not have the $SU(N_f)$ (isospin) generators. The corresponding $U(1)_A$ and $SU(N_f)_A$ generators have different fates in QCD; the former is explicitly violated by the anomaly, the latter is broken spontaneously at low temperature and is restored at some critical temperature T_c, provided that the quark mass is strictly zero, as is assumed for the purposes of the current discussion. The ρ and a_1 currents are both unchanged by the $U(1)_A$ transformation but are mixed under $SU(N_f)_A$. Therefore, if this symmetry is restored at high temperatures then there should be no difference between the vector and the axial-vector correlators. In this section we speculate on exactly how this difference goes to zero with increasing temperature. Generally, one may suggest many different scenarios. Let us discuss the following three.

The simplest scenario is that the T-dependence factorizes. It means that the vector and axial-vector spectral densities mix, without changing their shape, as in the low-temperature limit considered in the previous section, only with a more general function $\epsilon(T)$. When the mixing becomes maximal, $\epsilon = 1/2$, chiral symmetry is restored. It is interesting to see the temperature at which this occurs using the lowest-order formula, $\epsilon = T^2/6f_\pi^2$. This estimate gives $T_{\text{complete mixing}} = \sqrt{3}f_\pi \approx 164$ MeV, which is indeed roughly equal to the expected critical temperature T_c.

The second scenario assumes that the ρ and a_1 mesons retain their identities and dominate the correlation function. However, their parameters change with temperature. In particular, the masses may move towards each other [22] or go to zero [23]. At T_c they become degenerate, and chiral symmetry is restored.

It is instructive then to look at the sum rules. Let us assume that vector meson dominance is a good approximation for the spectral densities and not worry about the continuum contribution for the time being. We focus on zero momentum for the sake of simplicity. When a pole mass is defined at finite temperature, it is usually defined as the energy of the excitation at zero momentum.

The vector spectral density is (note that there is no difference between the longitudinal and transverse cases at zero momentum)

$$\text{sign}(\omega)\ \rho_V(\omega) = -\frac{1}{\pi}\frac{m_\rho^4}{g_\rho^2}\,\text{Im}\,\frac{1}{\omega^2 - m_\rho^2 - \Pi_R^\rho(\omega) - i\Pi_I^\rho(\omega)} \qquad (12.98)$$

where Π_R^ρ and Π_I^ρ are the real and imaginary parts of the ρ self-energy at temperature T. In the narrow-width approximation this

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becomes

$$\mathrm{sign}(\omega)\ \rho_V(\omega) \ = \ \frac{m_\rho^4}{g_\rho^2}\delta\big(\omega^2 - m_\rho^2 - \Pi_{\mathrm{R}}^\rho(\omega)\big) \tag{12.99}$$

The pole mass is determined self-consistently from $m_\rho^2(T) = m_\rho^2 + \Pi_{\mathrm{R}}^\rho(m_\rho(T))$. Then the spectral density can be rewritten as

$$\mathrm{sign}(\omega)\ \rho_V(\omega) \ = \ Z_\rho(T)\frac{m_\rho^4}{g_\rho^2}\delta\big(\omega^2 - m_\rho^2(T)\big) \tag{12.100}$$

where the temperature-dependent residue is

$$Z_\rho^{-1}(T) \ = \ \left|1 - \frac{d}{d\omega^2}\Pi_{\mathrm{R}}^\rho(\omega)\right| \tag{12.101}$$

The normalization is $Z_\rho(0) = 1$. Similarly

$$\mathrm{sign}(\omega)\ \rho_A(\omega) \ = \ Z_a(T)\frac{m_{a_1}^4}{g_a^2}\delta\big(\omega^2 - m_{a_1}^2(T)\big) \ + \ Z_\pi(T) f_\pi^2 \omega^2 \delta\big(\omega^2\big) \tag{12.102}$$

Substituting these spectral densities into the finite-temperature sum rules I, II-L, and II-T tells us that the ρ and a_1 residues are equal:

$$Z_\rho(T) \ = \ Z_a(T) \tag{12.103}$$

and that the pion residue is

$$Z_\pi(T) \ = \ 2Z_\rho(T)\left(\frac{m_\rho^2}{m_\rho^2(T)} - \frac{m_\rho^2}{m_{a_1}^2(T)}\right) \tag{12.104}$$

We expect that $m_{a_1}^2(T) - m_\rho^2(T) \to 0$ as the temperature increases. Three types of behavior can be distinguished: both the ρ and the a_1 masses decrease with T, both masses increase with T, or the ρ mass increases while the a_1 mass decreases with T. The sum rules do not appear to rule out any of these possibilities. In any case, the result is that $Z_\pi(T) \to 0$ unless $Z_\rho(T) \to \infty$, which seems rather unphysical.

As distinct from the previous two scenarios, it may be that particles are not well defined as we approach a chiral-symmetry-restoring phase transition. That is, the imaginary part of the self-energy may become larger with increasing temperature. This broadening would also decrease the maximum peak value of the spectral density. Picturesquely, the vector and axial-vector mesons melt away in a very broad distribution of strength in the spectral densities.

Concluding this section, we say once more that the sum rules by themselves cannot of course tell which scenario is preferable. However, they can be used to restrict significantly the parametrization of the spectral densities at nonzero temperature.

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12.4 Linear and nonlinear σ models

The O(N) model as a quantum field theory in $d+1$ dimensions [24] is a basis or prototype for many interesting physical systems. The bosonic field $\mathbf{\Phi}$ has N components. When the Lagrangian is such that the vacuum state exhibits spontaneous symmetry breaking, it is known as a sigma model. This is the case of interest to us here. In $d = 3$ space dimensions the linear sigma model has the potential

$$\tfrac{1}{4}\lambda\left(\mathbf{\Phi}^2 - f_\pi^2\right)^2$$

where λ is a positive coupling constant and f_π is the pion decay constant. The model is renormalizable. In the limit $\lambda \to \infty$ the potential goes over to a delta-function constraint on the length of the field vector and is then known as a nonlinear sigma model.

When $N = 4$ one has a model for the low-energy dynamics of quantum chromodynamics (QCD). More explicitly, it is essentially the unique description of the dynamics of very soft pions. This is basically due to the isomorphism between the groups O(4) and SU(2) $\times$ SU(2), the latter being the appropriate group for two flavors of massless quarks in QCD. The linear sigma model, including the nucleon, goes back to the work of Gell-Mann and Levy [25]. This subject has a vast literature.

As we have seen earlier in this chapter, much work has been done on chiral perturbation theory that starts with the nonlinear sigma model and adds higher-order, nonrenormalizable, terms to the Lagrangian; these are determined by the dimensionality of the coefficients or field derivatives [26]. The goal is to construct an effective Lagrangian that describes the low-energy properties of QCD to the desired accuracy. This whole program owes a considerable amount to the classic works of Weinberg [27, 28]

Finally, the standard model of the electroweak interactions, due to Weinberg, Salam, and Glashow, has an SU(2) doublet scalar Higgs field responsible for spontaneous symmetry breaking. If one neglects spin-1 gauge fields then the Higgs sector is also an O(4) field theory.

Since both linear and nonlinear σ models are prototypical field theories in many respects, one expects that much insight on the nature of the chiral-restoring phase transition, for example, can be had by studying those at finite temperature.

12.4.1 Linear σ model at finite temperature

The linear σ model Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \mathbf{\Phi})^2 - \tfrac{1}{4}\lambda\big(\mathbf{\Phi}^2 - f_\pi^2\big)^2 \tag{12.105}$$

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where λ is a positive coupling constant. The bosonic field $\mathbf{\Phi}$ has N components. Rather arbitrarily, we define the first $N-1$ components to represent a pion field $\boldsymbol{\pi}$ and the last, Nth, component to represent the sigma field. Since the O(N) symmetry is broken to an O($N-1$) symmetry at low temperatures, we immediately allow for a sigma condensate v whose value is temperature-dependent and yet to be determined. We write

$$\begin{aligned} \Phi_i(\mathbf{x},t) &= \pi_i(\mathbf{x},t) \qquad i = 1,\ldots,N-1 \\ \Phi_N(\mathbf{x},t) &= v + \sigma(\mathbf{x},t) \end{aligned} \tag{12.106}$$

In terms of these fields the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu \boldsymbol{\pi}\right)^2 + \tfrac{1}{2}\left(\partial_\mu \sigma\right)^2 - \tfrac{1}{4}\lambda\left(v^2 - f_\pi^2 + 2v\sigma + \sigma^2 + \boldsymbol{\pi}^2\right)^2 \tag{12.107}$$

The action at finite temperature is obtained by rotating to imaginary time, $\tau = it$, and integrating τ from 0 to $\beta = 1/T$. The action is defined as

$$\begin{aligned} S = {} & -\tfrac{1}{4}\lambda\left(f_\pi^2 - v^2\right)^2 \beta V \\ & + \int_0^\beta d\tau \int_V d^3x \big\{ \tfrac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 - \bar{m}_\pi^2 \boldsymbol{\pi}^2 + (\partial_\mu \sigma)^2 - \bar{m}_\sigma^2 \sigma^2\right] \\ & \qquad + \tfrac{1}{2}\lambda v\left(v^2 - f_\pi^2\right)\sigma - \lambda v \sigma(\boldsymbol{\pi}^2 + \sigma^2) - \tfrac{1}{4}\lambda(\sigma^2 + \boldsymbol{\pi}^2)^2 \big\} \end{aligned} \tag{12.108}$$

where the effective masses are

$$\begin{aligned} \bar{m}_\pi^2 &= \lambda\left(v^2 - f_\pi^2\right) \\ \bar{m}_\sigma^2 &= \lambda\left(3v^2 - f_\pi^2\right) \end{aligned} \tag{12.109}$$

At any temperature v is chosen such that $\langle\sigma\rangle = 0$. This eliminates any one-particle reducible (1PR) diagrams in perturbation theory, leaving only one-particle irreducible (1PI) diagrams.

At zero temperature the potential is minimized when $v = f_\pi$. The pion is massless and the σ particle has a mass of $\sqrt{2\lambda} f_\pi$. The Goldstone theorem is satisfied.

Lin and Serot [29] argued that the σ meson should not be identified with the attractive s-wave interaction in the $\pi - \pi$ interaction, which is responsible for nuclear attraction. Rather, they argue that the σ meson should have a mass which is at least 1 GeV if not more. This means that λ is of order 50 or greater.

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The simplest approximation at finite temperature is the mean field approximation. One allows for v to be temperature dependent; hence the effective masses are temperature dependent as well. However, interactions among the particles or collective excitations are neglected. The pressure includes only the contribution of the condensate and of the thermal motion of the independently moving particles. Thus

$$P = \frac{T}{V} \ln Z = -\frac{\lambda}{4} \left(f_\pi^2 - v^2 \right)^2 + P_0(T, m_\sigma) + (N-1) P_0(T, m_\pi) \tag{12.110}$$

The pressure of a free relativistic boson gas can be written in two ways:

$$P_0 = -T \int \frac{d^3p}{(2\pi)^3} \ln(1 - \mathrm{e}^{-\beta\omega}) = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\omega} \frac{1}{\mathrm{e}^{\beta\omega} - 1} \tag{12.111}$$

As pointed out earlier, this is a relatively simple but surprisingly powerful first approximation, which allows one to gain much insight into the behavior of relativistic quantum field theories at high temperature.

One expects that, as the temperature is raised, thermal fluctuations will tend to disorder the condensate field v, and at sufficiently high temperature it may even disappear. If there is a second-order phase transition then the correlation length should go to infinity, which is equivalent to the effective σ mass going to zero. With such an expectation one may expand the free-boson gas pressure about zero mass to obtain

$$P_0(T, m) = \frac{\pi^2}{90} T^4 - \frac{m^2 T^2}{24} + \frac{m^3 T}{12\pi} + \cdots \tag{12.112}$$

Since the masses are proportional to the square root of λ it is generally inconsistent to retain the cubic term in m because there exist loop diagrams which are not included in the mean field approximation but which contribute to the same order in λ. Therefore we take

$$P(T, v) = N \frac{\pi^2}{90} T^4 + \frac{\lambda}{2} v^2 \left(f_\pi^2 - \frac{N+2}{12} T^2 \right) - \frac{\lambda}{4} v^4 \tag{12.113}$$

where the pion and σ masses have been expressed in terms of λ, v, and f_π. Maximizing the pressure with respect to v gives

$$v^2 = f_\pi^2 - \frac{N+2}{12} T^2 \tag{12.114}$$

This result is easily understood. Going back to (12.108), we can differentiate $\ln Z$ with respect to v with the result that

$$v^2 = f_\pi^2 - 3\langle \sigma^2 \rangle - \langle \boldsymbol{\pi}^2 \rangle \tag{12.115}$$

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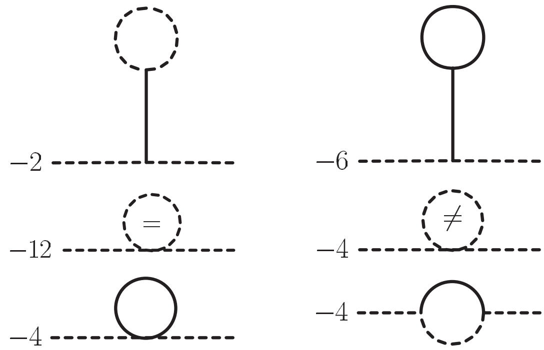

Fig. 12.5. The diagrams contributing to the one-loop pion self-energy, in the linear σ model. The broken lines represent the pion whereas the solid lines represent the σ. The overall sign and combinatoric factors are shown. In the contributions involving the pion four-point vertex, the signs $=$ and $\neq$ stand for cases where the pion loop and the external field have the same, or different, quantum numbers.

as long as we choose $\langle\sigma\rangle = 0$. For any free bosonic field ϕ with mass m,

$$\langle\phi^2\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1} \tag{12.116}$$

where $\omega = \sqrt{p^2+m^2}$. In the limit where the temperature is greater than the mass, $\langle\phi^2\rangle \to T^2/12$. This yields (12.114) directly.

The condensate goes to zero at a critical temperature given by

$$T_c^2 = \frac{12}{N+2}f_\pi^2 \tag{12.117}$$

Above this temperature thermal fluctuations are too large to allow a nonzero condensate. It is a straightforward exercise to show that the pressure and its first derivative are continuous at T_c but that the second derivative is discontinuous. This is therefore a second-order phase transition.

There are two major problems with the mean field approximation as described. The first is that the pion has a negative mass-squared at every temperature greater than zero. Not only is the Goldstone theorem not satisfied, but there are tachyons as well! The sigma particle also gets a negative mass-squared at temperatures above $\sqrt{8/(N+2)}\,f_\pi < T_c$. Recalling the analysis in Section 7.3, this violation of basic physical principles is resolved by recognizing that the finite-temperature corrections to the squared masses are proportional to λT^2 and that one-loop self-energy corrections, not included in the mean field analysis, are of the same order. This can be understood from the following analysis.

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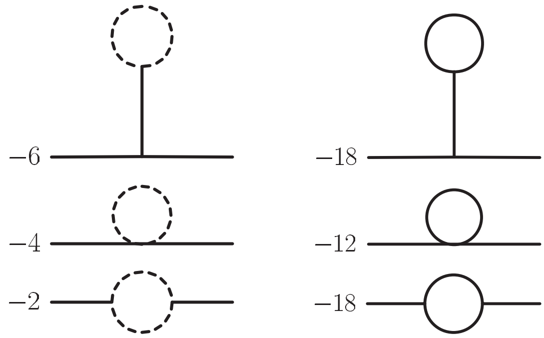

Fig. 12.6. The diagrams contributing to the one-loop σ self-energy.

At high temperatures, when the masses can be neglected in the loops, the mean field result is obtained by combining (12.110) and (12.114):

$$\begin{aligned} \bar{m}_\pi^2 &= -\frac{N+2}{12}\lambda T^2 \\ \bar{m}_\sigma^2 &= 2\lambda f_\pi^2 - \frac{N+2}{4}\lambda T^2 \end{aligned} \tag{12.118}$$

The full one-loop self-energies for pions and the σ meson are shown in Figures 12.5 and 12.6. If one chooses $\langle\sigma\rangle = 0$ then there are no 1PR diagrams and the tadpoles should not be included; they are already included in the temperature dependence of v. One may check this by fixing $v = f_\pi$ and then computing the tadpole contributions to the effective masses. One gets precisely (12.118). The diagrams involving the four-point vertices contribute an amount $(N+2)\lambda T^2/12$ to both the pion and σ meson self-energies. When evaluated in the high-temperature approximation and at low frequency and momentum the 1PI diagrams involving three-point vertices may be neglected. (This follows from power counting. These diagrams involve two propagators instead of one, and so are only logarithmically divergent in the ultraviolet in the vacuum. The other diagrams are quadratically divergent, which leads to a T^2 behavior at finite temperature.) When all contributions of order λT^2 are included, the pole positions of the pion and σ meson propagators move, with the result that below $T_{\rm c}$

$$\begin{aligned} m_\pi^2 &= \bar{m}_\pi^2 + \Pi_\pi = 0 \\ m_\sigma^2 &= \bar{m}_\sigma^2 + \Pi_\sigma = 2\lambda f_\pi^2 \left(1 - \frac{T^2}{T_{\rm c}^2}\right) \end{aligned} \tag{12.119}$$

and above $T_{\rm c}$

$$m_\pi^2 = m_\sigma^2 = m_\Phi^2 = -\lambda f_\pi^2 + \Pi_\Phi = \frac{N+2}{12}\lambda\left(T^2 - T_{\rm c}^2\right) \tag{12.120}$$

The Goldstone theorem is satisfied, there are no tachyons, and restoration of the full symmetry of the Lagrangian above T_c is evident.

It must be recognized that the results (12.118)–(12.120) are valid to order λ and cannot be extrapolated to $\lambda \to \infty$. At low temperature, where pions scatter from each other sequentially and there is essentially no propagation off mass shell between scatterings because of the low particle density, one may take the point of view that λ is a parameter to be adjusted to fit π–π scattering data and it does not matter how large λ is. This point of view cannot be taken at high temperature, where the pion number density is large, for then multiple scatterings will occur and they cannot be factorized into independent scatterings. This means that multiloop self-energy diagrams will be important at high temperature if λ is not perturbatively small.

The second major problem is that long-wavelength fluctuations very near the phase transition cannot be treated with perturbation theory because the self-interacting boson fields become massless just at the transition. Although this is a well-known problem in the statistical mechanics of second-order phase transitions, exactly how it affects the critical temperature is not known for the linear σ model in $3+1$ dimensions. The result presented here must be accepted for what it is: a one-loop estimate of the critical temperature.

12.4.2 Nonlinear σ model at finite temperature

The nonlinear σ model may be defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} \left(\partial_\mu \mathbf{\Phi}\right)^2 \tag{12.121}$$

together with the constraint

$$f_\pi^2 = \mathbf{\Phi}^2(\mathbf{x}, t) \tag{12.122}$$

The partition function is

$$Z = \int [d\mathbf{\Phi}]\, \delta\big(f_\pi^2 - \mathbf{\Phi}^2\big) \exp\left(\int_0^\beta d\tau \int d^3x\, \mathcal{L}\right) \tag{12.123}$$

Because the length of the chiral field is fixed and cannot be changed by thermal fluctuations it is often said that on the one hand chiral symmetry-breaking is built into this model and therefore there can be no chiral-symmetry-restoring phase transition. On the other hand, the linear sigma model does undergo a symmetry-restoring phase transition. Taking the quartic coupling constant λ to infinity essentially constrains the length of the chiral field to be f_π, just as in the nonlinear model. The critical temperature, however, is independent of λ at least in the mean field approximation. So it would seem that the phase transition survives. If this is true then one ought to be able to derive it entirely within the context of the nonlinear model. That is what we shall do, although it involves a lot more effort than the treatment of the linear model in the mean field approximation. Since the only parameter in the model is f_π and we are interested in temperatures comparable with it, we cannot make an expansion in powers of T/f_π. The only other parameter is N, the number of field components. This suggests an expansion in $1/N$.

We begin by representing the field-constraining delta function by an integral,

$$Z = \int [d\mathbf{\Phi}]\,[db']\exp\left\{\int_0^\beta d\tau \int d^3x\,\left[\mathcal{L} + ib'\left(\mathbf{\Phi}^2 - f_\pi^2\right)\right]\right\} \tag{12.124}$$

As with the linear model, we define the first $N-1$ components of $\mathbf{\Phi}$ to be the pion field and the last component to be the sigma field. We allow for a zero-frequency and zero-momentum condensate of the sigma field, referred to as v. Following Polyakov [30] we also separate out explicitly the zero-frequency and zero-momentum mode of the auxiliary field b'. Integrating over all the other modes will give us an effective action involving the constant part of the fields. We will then minimize the free energy with respect to these constant parts, which gives us a saddle point approximation. Integrating over fluctuations about the saddle point is a finite-volume correction and of no consequence in the thermodynamic limit. The Fourier expansions are

$$\begin{aligned}
\Phi_i(\mathbf{x},\tau) &= \pi_i(\mathbf{x},\tau) = \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} \mathrm{e}^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)}\,\tilde{\pi}_i(\mathbf{p},n)\\
\Phi_N(\mathbf{x},\tau) &= v + \sigma(\mathbf{x},\tau) = v + \sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} \mathrm{e}^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)}\,\tilde{\sigma}(\mathbf{p},n)\\
b'(\mathbf{x},\tau) &= \tfrac{1}{2}im^2 + b(\mathbf{x},\tau) = \tfrac{1}{2}im^2 + T\sqrt{\frac{\beta}{V}}\sum_n\sum_{\mathbf{p}} \mathrm{e}^{i(\mathbf{x}\cdot\mathbf{p}+\nu_n\tau)}\,\tilde{b}(\mathbf{p},n)
\end{aligned} \tag{12.125}$$

One must remember to exclude the zero-frequency and zero-momentum mode from the summations. The field $\mathbf{\Phi}$ must be periodic in imaginary time for the usual reasons, but there is no such requirement on the auxiliary field b, hence we must have $\omega_n = 2\pi nT$ and $\nu_n = \pi nT$. Since the field b has dimensions of inverse length squared we have inserted another factor of T so as to make its Fourier amplitude dimensionless, as is the

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case for the other fields. The action then becomes

$$S = \int_0^\beta d\tau \int_V d^3x \left\{ \tfrac{1}{2}\left[(\partial_\mu \boldsymbol{\pi})^2 - m^2\boldsymbol{\pi}^2 + (\partial_\mu\sigma)^2 - m^2\sigma^2 \right] - ib\left(2v\sigma + \boldsymbol{\pi}^2 + \sigma^2\right) \right\} + \tfrac{1}{2}m^2\left(f_\pi^2 - v^2\right)\beta V \qquad (12.126)$$

Note that terms linear in the fields integrate to zero because $\langle \pi_i \rangle = \langle \sigma \rangle = \langle b \rangle = 0$.

An effective action is derived by expanding $\exp(S)$ in powers of b and integrating over the pion and σ fields. The term linear in b vanishes on account of $\tilde{b}(\mathbf{0}, 0) \propto \langle b \rangle = 0$. The term proportional to b^2 is nonzero and is exponentiated, thus summing a whole series of contributions. The term proportional to b^3 is also nonzero and it, too, may be exponentiated, summing an infinite series of higher-order terms left out of the order-b^2 exponentiation. After making the scaling $b \to b/\sqrt{2N}$ the effective action becomes

$$\begin{aligned} S_{\rm eff} = &-\tfrac{1}{2}\sum_n \sum_{\mathbf{p}} \left(\omega_n^2 + p^2 + m^2\right)\left[\tilde{\boldsymbol{\pi}}(\mathbf{p}, n)\cdot\tilde{\boldsymbol{\pi}}(-\mathbf{p}, -n) + \tilde{\sigma}(\mathbf{p}, n)\tilde{\sigma}(-\mathbf{p}, -n)\right] \\ &-\tfrac{1}{2}\sum_n \sum_{\mathbf{p}} \left(\Pi(p, \omega_n, T, m) + \frac{2}{N}\frac{v^2}{\omega_n^2 + p^2 + m^2} \right)\tilde{b}(\mathbf{p}, 2n)\tilde{b}(-\mathbf{p}, -2n) \\ &+\tfrac{1}{2}m^2\left(f_\pi^2 - v^2\right)\beta V + \mathrm{O}\left(\frac{\tilde{b}^3}{\sqrt{N}}\right) \end{aligned} \qquad (12.127)$$

Note that only even Matsubara frequencies contribute in the b-field: $\nu_n = 2\pi n T$. This may have been anticipated. There appears the one-loop function

$$\Pi(p, \omega_n, T, m) = T\sum_l \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n - \omega_l)^2 + (\mathbf{p} - \mathbf{k})^2 + m^2} \frac{1}{\omega_l^2 + k^2 + m^2} \qquad (12.128)$$

The effective action is an infinite series in b. The coefficients are frequency and momentum dependent, arising from one-loop diagrams. In addition, each successive term is suppressed by $1/\sqrt{N}$ compared with the previous one. This is the large-N expansion.

The propagators for the π and σ fields are of the usual form,

$$\mathcal{D}_0^{-1}(p, \omega_n, m) = \omega_n^2 + p^2 + m^2 \qquad (12.129)$$

with an effective mass m yet to be determined. The propagator for the b-field is more complicated:

$$\mathcal{D}_b^{-1}(p, \omega_n, m) = \Pi(p, \omega_n, T, m) + \frac{2}{N}\frac{v^2}{\omega_n^2 + p^2 + m^2} \qquad (12.130)$$

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The value of the condensate v is not yet determined, either.

Keeping only the terms up to order b^2 in $S_{\rm eff}$ (the rest vanish in the limit $N \to \infty$) allows us to obtain an explicit expression for the partition function and the pressure; this includes the next-to-leading order terms in N:

$$P = \frac{T}{V}\ln Z = \tfrac{1}{2}m^2\big(f_\pi^2 - v^2\big) - \tfrac{1}{2}N\,T\sum_n \int \frac{d^3p}{(2\pi)^3}\,\ln\big[\beta^2\big(\omega_n^2 + p^2 + m^2\big)\big] - \tfrac{1}{2}T\sum_n \int \frac{d^3p}{(2\pi)^3}\,\ln\left(\Pi(p,\omega_n,T,m) + \frac{2}{N}\,\frac{v^2}{\omega_n^2 + p^2 + m^2}\right) \tag{12.131}$$

The second term in the argument of the last logarithm should and will be set to zero at this order. It may be needed at higher order in the large-N expansion to regulate infrared divergences.

The pressure is extremized with respect to the mass parameter m. Therefore $\partial P/\partial m^2 = 0$. From the initial expression for Z this is seen to be equivalent to the thermal average of the constraint:

$$f_\pi^2 = \langle \mathbf{\Phi}^2\rangle = v^2 + \langle \boldsymbol{\pi}^2\rangle + \langle \sigma^2\rangle \tag{12.132}$$

If an approximation to the exact partition function is made, such as the large-N expansion, this constraint should still be satisfied. It may, in fact, single out a preferred value of m.

To leading order in N we may neglect the term involving Π entirely. The pressure is then

$$P = \tfrac{1}{2}m^2\left(f_\pi^2 - v^2\right) + N\,P_0(T,m) \tag{12.133}$$

The pressure must be a maximum with respect to variations in the condensate v. This means that

$$\frac{\partial P}{\partial v} = -m^2 v = 0 \tag{12.134}$$

which is equivalent to the condition $\langle\sigma\rangle = 0$. There are two possibilities.

1 $m = 0$ There exist massless particles, or Goldstone bosons, and the value of the condensate is determined by the thermally averaged constraint. This is the symmetry-broken phase.

2 $v = 0$ The thermally averaged constraint is satisfied by a nonzero temperature-dependent mass. There are no Goldstone bosons. This is the symmetry-restored phase.

Evidently there is a chiral-symmetry-restoring phase transition!

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In the leading order of the large-N approximation the particles are represented by free fields with a potentially temperature-dependent mass m. Again, we may use

$$\frac{\partial P_0(T,m)}{\partial m^2} = \langle \phi^2 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega}-1} \tag{12.135}$$

with $\omega = \sqrt{p^2+m^2}$. Thus extremizing the pressure with respect to m^2 is equivalent to satisfying the thermally averaged constraint

$$f_\pi^2 = v^2 + \langle \boldsymbol{\pi}^2 \rangle + \langle \sigma^2 \rangle \tag{12.136}$$

Note however that the pion and σ fields have the same mass and therefore $\langle \boldsymbol{\pi}^2 \rangle = (N-1)\langle \sigma^2 \rangle$. Consider now the two different phases.

In the asymmetric, symmetry-broken, phase the mass is zero. The above constraint is satisfied by a temperature-dependent condensate:

$$v^2(T) = f_\pi^2 - \frac{NT^2}{12} \tag{12.137}$$

This condensate goes to zero at a critical temperature

$$T_c^2 = \frac{12}{N} f_\pi^2 \qquad \text{(leading-}N\text{ approximation)} \tag{12.138}$$

Exactly at T_c the thermally averaged constraint is satisfied by the fluctuations of N massless degrees of freedom without the help of a condensate.

In the symmetric phase the condensate is zero. The constraint is satisfied by thermal fluctuations alone:

$$f_\pi^2 = N \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega}-1} \tag{12.139}$$

Thermal fluctuations decrease with increasing mass at fixed temperature. The constraint is only satisfied by massless excitations at one temperature, namely, T_c. At temperatures $T > T_c$ the mass must be greater than zero. Near the critical temperature the mass should be small, and the fluctuations may be expanded about $m = 0$ as

$$f_\pi^2 = NT^2 \left[\frac{1}{12} - \frac{m}{4\pi T} - \frac{m^2}{8\pi^2 T^2} \ln\left(\frac{m}{4\pi T}\right) - \frac{m^2}{16\pi^2 T^2} + \cdots \right] \tag{12.140}$$

As T approaches T_c from above, the mass approaches zero as follows:

$$m(T) = \frac{\pi}{3T}\left(T^2 - T_c^2\right) + \cdots \tag{12.141}$$

This is a second-order phase transition since there is no possibility of metastable supercooled or superheated states.

The mass must grow faster than the temperature at very high temperatures in order to keep the field fluctuations fixed and equal to f_π^2.

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Asymptotically the particles move nonrelativistically. This allows us to compute the fluctuations analytically. We get

$$f_\pi^2 = N \left(\frac{T}{2\pi} \right)^{3/2} \sqrt{m}\, \mathrm{e}^{-m/T} \tag{12.142}$$

This is a transcendental equation for $m(T)$. It can also be written as

$$m = T \, \ln \left(\frac{NT}{2\pi f_\pi} \sqrt{\frac{mT}{2\pi f_\pi^2}} \right) \tag{12.143}$$

Roughly, the solution behaves as follows:

$$m \sim T \, \ln \left(\frac{T^2}{T_\mathrm{c}^2} \right) \tag{12.144}$$

It is rather amusing that, at the leading order of the large-N approximation, the elementary excitations are massless below T_c, become massive above T_c, and at asymptotically high temperatures move nonrelativistically.

The result to first order of the large-N expansion provides good insight into the nature of the two-phase structure of the nonlinear σ model, but it is not quite satisfactory for two reasons. First, it predicts N massless Goldstone bosons in the broken-symmetry phase when in fact we know there ought to be only $N-1$. Second, the square of the critical temperature is $12 f_\pi^2/N$ whereas it is $12 f_\pi^2/(N+2)$ in the linear σ model in the mean field approximation; we expect them to be the same in the limit $\lambda \to \infty$. Both these problems can be rectified by inclusion of the next-to-leading-order term in N, which gives the contribution of the b-field.

It is natural to expect that the b-field will contribute essentially one negative degree of freedom to the T^4 term in the pressure so as to give $N-1$ Goldstone bosons in the low-temperature phase. Therefore we move one of the N degrees of freedom and put it together with the b-field contribution as

$$\begin{aligned} P = \tfrac{1}{2} m^2 \left(f_\pi^2 - v^2 \right) - \tfrac{1}{2}(N-1)\, T \sum_n \int \frac{d^3p}{(2\pi)^3} \, \ln\!\left[\beta^2 \left(\omega_n^2 + p^2 + m^2 \right)\right] \\ - \tfrac{1}{2} T \sum_n \int \frac{d^3p}{(2\pi)^3} \, \ln\!\left[\beta^2 \left(\omega_n^2 + p^2 + m^2 \right) \Pi \right] \end{aligned} \tag{12.145}$$

The function $\Pi(p, \omega_n, T, m)$ can be reduced to a one-dimensional integral:

$$\Pi = \frac{1}{8\pi^2 p} \int_0^\infty \frac{dk\, k}{\omega} \ln \left(\frac{k^2 + pk + \Lambda^2}{k^2 - pk + \Lambda^2} \right) \frac{1}{\mathrm{e}^{\beta\omega} - 1} \tag{12.146}$$

where

$$\Lambda^2 = \Lambda^2(p, \omega_n, m) = \frac{(\omega_n^2 + p^2)^2 + 4m^2\omega_n^2}{4(\omega_n^2 + p^2)} \tag{12.147}$$

but unfortunately Π cannot be simplified any further. In any case, to the order in N to which we are working, the pressure is

$$P = \tfrac{1}{2}m^2\left(f_\pi^2 - v^2\right) + (N-1)\,P_0(T,m) + P_{\rm I}(T,m) \tag{12.148}$$

The pressure can be thought of, in the low-temperature phase, as due to $N-1$ Goldstone bosons with an interaction term $P_{\rm I}$.

Because of the logarithm, the main contribution to the interaction pressure will come when Π is very small compared to unity. This corresponds to very large values of the parameter Λ; in other words, to very high momentum, Matsubara frequency, or mass. In this limit,

$$\Pi \to \frac{1}{4\pi^2\Lambda^2}\int_0^\infty \frac{dk\,k^2}{\omega}\,\frac{1}{{\rm e}^{\beta\omega}-1} = \frac{T^2}{2\pi^2\Lambda^2}h_3\left(\frac{m}{T}\right) \tag{12.149}$$

This may be considered as a high-energy approximation, and we shall henceforth refer to it as such. Then

$$\begin{aligned} P_{\rm I} &= \tfrac{1}{2}\,T\sum_n\int\frac{d^3p}{(2\pi)^3}\,\ln\left[\beta^2\left(\omega_n^2+p^2+m^2\right)\Pi\right] \\ &\approx -\tfrac{1}{2}\,T\sum_n\int\frac{d^3p}{(2\pi)^3}\,\ln\left(\frac{h_3}{\pi^2}\,\frac{(\omega_n^2+p^2)(\omega_n^2+p^2+m^2)}{(\omega_n^2+\omega_+^2)(\omega_n^2+\omega_-^2)}\right) \end{aligned} \tag{12.150}$$

with dispersion relations

$$\omega_\pm^2 = p^2 + 2m^2 \pm 2m\sqrt{p^2+m^2} \tag{12.151}$$

The interaction pressure can now be determined in the usual way to be

$$\begin{aligned} P_{\rm I} = -T\int\frac{d^3p}{(2\pi)^3}\,\Big[&\ln(1-{\rm e}^{-\beta p}) + \ln(1-{\rm e}^{-\beta\omega(p)}) \\ &- \ln(1-{\rm e}^{-\beta\omega_+(p)}) - \ln(1-{\rm e}^{-\beta\omega_-(p)})\Big] \end{aligned} \tag{12.152}$$

Note that $h_3(m/T)$ has no effect within this approximation. Note also that in the broken-symmetry phase where $m=0$ the contribution of the b-field cancels one of the massless degrees of freedom to give $N-1$ Goldstone bosons.

Now we are prepared to examine the behavior of the system near the critical temperature with the inclusion of next-to-leading terms in N. We make an expansion in m/T as before. The pressure is, up to and including

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order m^3,

$$P = (N-1)\frac{\pi^2}{90}T^4 - \frac{N+2}{24}m^2T^2 + \frac{1}{2}m^2\left(f_\pi^2 - v^2\right) + \frac{N}{12\pi}m^3T \tag{12.153}$$

In the high-temperature phase, where $v = 0$, maximization with respect to m yields

$$f_\pi^2 = T^2\left(\frac{N+2}{12} - \frac{N}{4\pi}\frac{m}{T}\right) \tag{12.154}$$

This gives the same critical temperature as in the mean field treatment of the linear σ model.

$$T_{\mathrm{c}}^2 = \frac{12}{N+2}f_\pi^2 \qquad \text{(sub-leading-}N\text{ approximation)} \tag{12.155}$$

The mass approaches zero from above as follows:

$$m(T) = \frac{\pi(N+2)}{3NT}\left(T^2 - T_{\mathrm{c}}^2\right) \tag{12.156}$$

In the results obtained immediately above, an approximation for Π to which we have referred as a high-energy approximation has been used. Relaxing this approximation can be done, albeit at the cost of a numerical calculation. Of course, one should also go beyond the mean field approximation in the linear model.

12.4.3 Finite-temperature behavior of f_π

Consideration of correlation functions at finite temperature is more involved than at zero temperature. Lorentz invariance is not manifest because there is a preferred frame of reference, the frame in which the matter is at rest. Thus spectral densities and other functions may depend on energy and momentum separately and not just on their invariant s. Also, the number of Lorentz tensors is greater because there is a new vector available, namely, the vector $u_\mu = (1, 0, 0, 0)$ that specifies the rest frame of the matter.

In the usual fashion one may construct a Green's function for the axial-vector current $\mathcal{A}_a^\mu$:

$$G_{ab}^{\mu\nu}(z, \mathbf{q}) = \int_{-\infty}^{\infty}\frac{d\omega}{\omega - z}\,\rho_{ab}^{\mu\nu}(\omega, \mathbf{q}) \tag{12.157}$$

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where the spectral density tensor is

$$\rho^{\mu\nu}_{ab}(\omega,\mathbf{q}) = \frac{1}{Z}\sum_{m,n}(2\pi)^3\delta(\omega - E_m + E_n)\delta(\mathbf{q} - \mathbf{p_m} + \mathbf{p_n}) \times \left(\mathrm{e}^{-E_n/T} - \mathrm{e}^{-E_m/T}\right)\langle n|\mathcal{A}^\mu_a(0)|m\rangle\langle m|\mathcal{A}^\nu_b(0)|n\rangle \quad (12.158)$$

The summation is over a complete set of energy eigenstates.

Owing to current conservation the spectral density tensor can be decomposed into longitudinal and transverse terms:

$$\rho^{\mu\nu}_{ab}(q) \;=\; \delta_{ab}\left[\rho^{\mathrm{L}}_A(q)P^{\mu\nu}_{\mathrm{L}} + \rho^{\mathrm{T}}_A(q)P^{\mu\nu}_{\mathrm{T}}\right] \quad (12.159)$$

In general the spectral densities depend on q^0 and $\mathbf{q}$ separately as well as on the temperature. In the vacuum we can always go to the rest frame of a massive particle and in that frame there can be no difference between longitudinal and transverse polarizations, so that $\rho_{\mathrm{L}} = \rho_{\mathrm{T}} = \rho$. We also observe that $P^{\mu\nu}_{\mathrm{L}} + P^{\mu\nu}_{\mathrm{T}} = -(g^{\mu\nu} - q^\mu q^\nu/q^2)$. The pion, being a massless Goldstone boson, is special. It contributes to the longitudinal axial spectral density and not to the transverse one. In vacuum,

$$\rho^{\mu\nu}(q) \;=\; \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\rho_A(q^2) \;+\; f_\pi^2\delta(q^2)q^\mu q^\nu \quad (12.160)$$

This may be taken to be the definition of the pion decay constant at zero temperature. In fact, one can write the pion's contribution as

$$f_\pi^2\delta(q^2)q^\mu q^\nu \;=\; f_\pi^2 q^2\delta(q^2)P^{\mu\nu}_{\mathrm{L}} \quad (12.161)$$

This cannot be taken as the definition of the pion decay constant at finite temperature because the contribution of the pion to the longitudinal spectral density cannot be assumed to be a delta function in q^2. In general, as mentioned previously, the pion's dispersion relation will be more complicated and will develop a width at nonzero momentum. This smears out the delta function into something like a relativistic Breit–Wigner distribution. Fortunately, the Goldstone theorem [31] requires that there be a zero-frequency excitation when the momentum is zero (see Chapter 7). This implies that the width must go to zero at $\mathbf{q} = \mathbf{0}$, which results in a delta function at zero frequency. Explicit calculations support this assertion [32, 33]. Therefore it would seem to make sense to define

$$f_\pi^2(T) \equiv 2\lim_{\epsilon\to 0}\int_0^\epsilon \frac{dq_0^2}{q_0^2}\,\rho^{\mathrm{L}}_A(q_0,\,\mathbf{q} = \mathbf{0}) \quad (12.162)$$

Physically this means that the pion decay constant at finite temperature measures the strength of the coupling of the Goldstone boson to the longitudinal part of the retarded axial-vector response function in the limit of zero momentum.

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We shall first study the pion's contribution to the spectral density at temperatures small compared with f_π. We shall study both the nonlinear and linear σ models. At low temperatures the σ meson's contribution as a material degree of freedom is frozen out and one might expect the same dynamics to be operative in both models; in other words, one might expect the result to be the same and so independent of λ.

The nonlinear σ model

The nonlinear σ model was defined at the beginning of subsection 12.4.2. One can make a nonlinear redefinition of the field without changing the physical content of the theory. Various redefinitions may be found in the literature. First we will list the most common ones and then we will compute $f_\pi(T)$ for each of them, thereby illustrating that one always gets the same result. It is interesting to see how this comes about; it is also reassuring that it does.

A convenient way to express the sigma and pion fields that explicitly contains the constraint is

$$\begin{aligned} \sigma &= f_\pi \; \cos(\phi/f_\pi) \\ \boldsymbol{\pi} &= f_\pi \hat{\boldsymbol{\phi}} \; \sin(\phi/f_\pi) \end{aligned} \tag{12.163}$$

where $\phi = |\boldsymbol{\phi}|$ and $\hat{\boldsymbol{\phi}} = \boldsymbol{\phi}/\phi$. The Lagrangian may then be expressed in terms of the fields of choice:

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} + \tfrac{1}{2}\partial_\mu \sigma \, \partial^\mu \sigma \\ &= \tfrac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} + \tfrac{1}{2}\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\,(\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi})}{f_\pi^2 - \pi^2} \\ &= \tfrac{1}{2}\,\frac{f_\pi^2}{\phi^2}\,\sin^2\left(\frac{\phi}{f_\pi}\right)\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} + \tfrac{1}{2}\left[1 - \frac{f_\pi^2}{\phi^2}\sin^2\left(\frac{\phi}{f_\pi}\right)\right]\partial_\mu\,\partial^\mu\phi \end{aligned} \tag{12.164}$$

Another representation to consider is due to Weinberg [27], who makes the definition

$$\mathbf{p} = 2\,\frac{f_\pi^2}{\pi^2}\left(1 - \sqrt{1 - \frac{\pi^2}{f_\pi^2}}\right)\boldsymbol{\pi} \tag{12.165}$$

or inversely

$$\boldsymbol{\pi} = \frac{\mathbf{p}}{1 + p^2/4f_\pi^2} \tag{12.166}$$

In terms of Weinberg's field definition the Lagrangian is very compact:

$$\mathcal{L} = \frac{1}{2}\,\frac{\partial_\mu\mathbf{p}\cdot\partial^\mu\mathbf{p}}{(1 + p^2/4f_\pi^2)^2} \tag{12.167}$$

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The $(\sigma, \boldsymbol{\pi})$ representation is cumbersome because of the constraint, although it can be handled by the Lagrange multiplier method of subsection 12.4.2. However, it is inconvenient for exposing the physical particle content and for doing perturbation theory in terms of physical particles. Among the three physical representations we choose to work with here, it is interesting to note the range of allowed values of the fields. The magnitude of the $\mathbf{p}$-field can range from zero to infinity, the magnitude of the $\boldsymbol{\pi}$-field can range from 0 to f_π, and the magnitude of the $\boldsymbol{\phi}$-field can range from 0 to πf_π. This distinction is important when dealing with nonperturbative large-amplitude motion; whether it makes any difference in low orders of perturbation theory is not known.

The first step in our quest to extract the temperature dependence of f_π from the theory is to obtain the form of the axial-vector current in terms of the chosen fields. Starting from

$$\mathcal{A}_\mu = -\sigma\, \partial_\mu \boldsymbol{\pi} + \boldsymbol{\pi}\, \partial_\mu \sigma \tag{12.168}$$

one directly computes

$$\begin{aligned}
\mathcal{A}_\mu &= -\sigma \left(\partial_\mu \boldsymbol{\pi} + \frac{\boldsymbol{\pi}\ (\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi})}{f_\pi^2 - \pi^2} \right) \\
&= -\frac{f_\pi^2}{2\phi} \sin\left(\frac{2\phi}{f_\pi}\right) \partial_\mu \boldsymbol{\phi} - f_\pi \hat{\boldsymbol{\phi}} \left[1 - \frac{f_\pi}{2\phi} \sin\left(\frac{2\phi}{f_\pi}\right) \right] \hat{\boldsymbol{\phi}} \cdot \partial_\mu \boldsymbol{\phi} \\
&= -\frac{1}{f_\pi} \frac{1}{(1 + p^2/4f_\pi^2)^2} \left[\left(f_\pi^2 - \frac{1}{4} p^2 \right) \partial_\mu \mathbf{p} + \frac{1}{2} \mathbf{p}\, (\mathbf{p} \cdot \partial_\mu \mathbf{p}) \right]
\end{aligned} \tag{12.169}$$

Every form of the axial-vector current is an odd function of the pion field.

Obviously it is not possible to compute the axial-vector correlation function exactly. We will restrict our attention to low temperatures. Roughly speaking, a loop expansion of the correlation function is an expansion in powers of T^2/f_π^2, each additional loop contributing one more such factor. To one-loop order, we need the axial-vector current to third order in the pion field:

$$\begin{aligned}
\mathcal{A}_\mu &= -f_\pi\, \partial_\mu \boldsymbol{\pi} + \frac{\pi^2}{2f_\pi}\, \partial_\mu \boldsymbol{\pi} - \frac{1}{f_\pi}\, \boldsymbol{\pi}\, (\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi}) \\
&= -f_\pi\, \partial_\mu \boldsymbol{\phi} + \frac{2\phi^2}{3f_\pi}\, \partial_\mu \boldsymbol{\phi} - \frac{2}{3f_\pi}\, \boldsymbol{\phi}\, (\boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi}) \\
&= -f_\pi\, \partial_\mu \mathbf{p} + \frac{3p^2}{4f_\pi}\, \partial_\mu \mathbf{p} - \frac{1}{2f_\pi}\, \mathbf{p}\, (\mathbf{p} \cdot \partial_\mu \mathbf{p})
\end{aligned} \tag{12.170}$$

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We will also need the Lagrangian to fourth order in the pion field:

$$\begin{aligned}\mathcal{L}_4 &= \frac{1}{2f_\pi^2}\,(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})\,(\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}) \\ &= \frac{1}{6f_\pi^2}\left[(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})\,(\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}) - \phi^2\,\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}\right] \\ &= -\frac{1}{4f_\pi^2}\,p^2\,\partial_\mu\mathbf{p}\cdot\partial^\mu\mathbf{p}\end{aligned} \tag{12.171}$$

The correlation function $\langle\mathcal{A}_\mu^i(x)\,\mathcal{A}_\nu^j(y)\rangle$ will have a zero-loop contribution from the π–π correlation function $\langle\partial_\mu\pi^i(x)\,\partial_\nu\pi^j(y)\rangle$, a one-loop self-energy correction to the same π–π correlation function, and a one-loop contribution from the correlation function involving four pions, $\langle\partial_\mu\pi^i(x)\,\pi^j(y)\pi^k(y)\partial_\nu\pi^l(y)\rangle$.

The contribution of the bare-pion propagator $\mathcal{D}_0$ to the longitudinal spectral density is easily found to be

$$\rho_A^{\mathrm{L}}(q_0,\mathbf{q}) = f_\pi^2\,q^2\,\delta\left(q^2\right) \tag{12.172}$$

At zero temperature this is just the definition of the pion decay constant.

The one-loop pion self-energy may be computed by standard diagrammatic or functional integral techniques. The results are:

$$\begin{aligned}\Pi_{\boldsymbol{\pi}}(q) &= -\frac{T^2}{12f_\pi^2}\,q^2 \\ \Pi_{\mathbf{p}}(q) &= (N-1)\,\frac{T^2}{24f_\pi^2}\,q^2 \\ \Pi_{\boldsymbol{\phi}}(q) &= \frac{1}{3}\,\Pi_{\boldsymbol{\pi}}(q) + \frac{2}{3}\,\Pi_{\mathbf{p}}(q)\end{aligned} \tag{12.173}$$

These are rather dependent on the definition of the pion field! Nevertheless, it is worth noting that the Goldstone theorem is satisfied on account of the fact that the self-energy is always proportional to q^2.

The final contribution comes from the correlation function for a pion at point x with three pions at point y. Again, standard diagrammatic or functional integral techniques may be used. To express the answers, we gather together the contributions from the bare propagator, from the one-loop self-energy, and from this correlation function and quote the coefficient of the term $f_\pi^2\,q^2\,\delta\left(q^2\right)$ in the longitudinal part of the axial-vector spectral density:

$$
\begin{aligned}
\boldsymbol{\pi} :& \quad \left(1 - \frac{T^2}{12 f_\pi^2}\right) - (N-3)\frac{T^2}{12 f_\pi^2} \\
\mathbf{p} :& \quad \left(1 + (N-1)\frac{T^2}{24 f_\pi^2}\right) - \left(N - \frac{5}{3}\right)\frac{T^2}{8 f_\pi^2} \\
\boldsymbol{\phi} :& \quad \left(1 + (N-2)\frac{T^2}{36 f_\pi^2}\right) - (N-2)\frac{T^2}{9 f_\pi^2}
\end{aligned}
\tag{12.174}
$$

In all three cases the results are the same and amount to a temperature dependence of

$$
f_\pi^2(T) = f_\pi^2 \left(1 - \frac{N-2}{12}\frac{T^2}{f_\pi^2}\right) \tag{12.175}
$$

This agrees with the analysis of Gasser and Leutwyler [21] for the only case where they can be compared, $N_\mathrm{f}^2 = N = 4$.

The linear σ model

It is now not surprising to discover that the linear σ model gives the same result for $f_\pi(T)$ at low temperatures as the nonlinear sigma model. The reason is that the σ meson is very heavy at low temperatures and cannot contribute materially in the way that the pions do. However, the way in which the linear σ model works out is very different.

Let us go back to the axial-vector current before shifting the sigma field:

$$
\mathcal{A}_\mu = -\sigma\, \partial_\mu \boldsymbol{\pi} + \boldsymbol{\pi}\, \partial_\mu \sigma \tag{12.176}
$$

After making the shift $\sigma \to v + \sigma$ the current takes the form

$$
\mathcal{A}_\mu = -v\, \partial_\mu \boldsymbol{\pi} - \sigma\, \partial_\mu \boldsymbol{\pi} + \boldsymbol{\pi}\, \partial_\mu \sigma \tag{12.177}
$$

By maximizing the pressure (which is equivalent to minimizing the effective potential) with respect to v at each temperature we effectively sum all tadpole diagrams, leaving only 1PI diagrams in any subsequent perturbative treatment. If this is done, one's inclination is to identify $v(T)$ with $f_\pi(T)$. This is wrong; $f_\pi(T)$ has additional contributions, as we shall now see.

The first contribution to $f_\pi^2(T)$ does come from $v^2(T)$ since it involves the cross term of $\partial_\mu \pi^a(x)$ with $\partial_\nu \pi^a(y)$. Following the analysis of subsection 12.4.1, but at low temperature rather than high, we simply leave out

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the contribution of the heavy σ meson. This gives

$$P(T, v) = (N-1)\frac{\pi^2}{90}T^4 + \frac{\lambda}{2}v^2\left(f_\pi^2 - \frac{N-1}{12}T^2\right) - \frac{\lambda}{4}v^4 \qquad (12.178)$$

Maximizing with respect to v gives

$$v^2(T) = f_\pi^2 - \frac{N-1}{12}T^2 \qquad (12.179)$$

There is another, nonlocal, contribution to the vertex, corresponding to the emission and absorption of a virtual σ meson. One might think that it would be suppressed by the large σ mass, $m_\sigma^2 = 2\lambda f_\pi^2$, but in fact this is compensated by the coupling constant λ in the extra vertex. Evaluation of this diagram gives a contribution to $f_\pi^2(T)$ of $T^2/6f_\pi^2$.

Finally there is a contribution coming from the dressed pion propagator analogous to that in the nonlinear σ model. The full one-loop 1PI pion self-energy diagrams have been shown already in Figure 12.5. We know that the sum of the momentum-independent terms is zero on account of Goldstone's theorem. We just need the contribution that is quadratic in the energy and momentum of the pion. This can arise only from the so-called exchange diagram involving two $\sigma\pi\pi$ vertices. In imaginary time (Euclidean space) it is

$$\Pi_{\rm ex}(\omega_n, \mathbf{q}) = -4\lambda^2 f_\pi^2\, T \sum_l \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_l^2 + \mathbf{k}^2}\frac{1}{(\omega_l+\omega_n)^2 + (\mathbf{k}+\mathbf{q})^2 + m_\sigma^2} \qquad (12.180)$$

Since $T \ll m_\sigma$ it is easy to extract the part that is quadratic in the momentum. Analytically continuing to Minkowski space ($\omega_n \to iq_0$), it is $q^2T^2/12f_\pi^2$.

The residue of the pion pole in the axial-vector correlation function can now be obtained by adding the vacuum contribution, the pion self-energy correction, and the tadpole and nonlocal vertex corrections as follows:

$$\left(1 - \frac{1}{12}\frac{T^2}{f_\pi^2}\right) - \frac{N-1}{12}\frac{T^2}{f_\pi^2} + \frac{1}{6}\frac{T^2}{f_\pi^2}$$

The final result,

$$f_\pi^2(T) = f_\pi^2\left(1 - \frac{N-2}{12}\frac{T^2}{f_\pi^2}\right) \qquad (12.181)$$

is identical to that of the nonlinear σ model. We remark that this cannot be used to compute the critical temperature since it was obtained under the condition that $T \ll f_\pi$.

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The approach to chiral-symmetry restoration

Calculation of $f_\pi(T)$ as $T \to T_c$ is more involved than in the low-temperature limit. It was done for the nonlinear model by Jeon and Kapusta [34]. Here we just quote the result:

$$f_\pi^2(T) = f_\pi^2 - \frac{N+2}{12} T^2 \tag{12.182}$$

It goes to zero at the correct critical temperature. Notice that the coefficient of the T^2 term is different from that in the low-temperature limit. A relatively simple Padé approximation may be used to extrapolate smoothly from low temperatures to the critical temperature:

$$\frac{f_\pi^2(T)}{f_\pi^2} \approx \frac{1 - \dfrac{T^2}{T_c^2}}{1 - \dfrac{4}{(N+2)} \dfrac{T^2}{T_c^2} \left(1 - \dfrac{T^2}{T_c^2}\right)} \tag{12.183}$$

12.4.4 Finite-temperature scalar condensate

The scalar condensate is defined as $|\langle \mathbf{\Phi} \rangle|$. Our convention has been to allow the last, Nth, component of the field to condense, and to refer to this as either v, if the field is shifted, or $\langle \sigma \rangle$ if the field is not shifted. In this section we use the latter convention.

It is interesting to ask what happens to this condensate as a function of temperature in the nonlinear model. The constraint as an operator equation is $f_\pi^2 = \mathbf{\Phi}^2$ and as a thermal average is $f_\pi^2 = \langle \mathbf{\Phi}^2 \rangle$; it is not $f_\pi = |\langle \mathbf{\Phi} \rangle|$. The condensate can indeed change with temperature. In fact we can quite easily compute it to two-loop order. Before doing so, we first discuss the connection of this condensate with the quark condensate $\langle \bar{\psi}\psi \rangle$.

In two-flavor QCD one often associates the sigma and pion fields with certain bilinear forms of the quark fields:

$$\begin{aligned} \bar{\psi}\psi &\sim \sigma \\ i\bar{\psi}\gamma_5 \boldsymbol{\tau}\psi &\sim \boldsymbol{\pi} \end{aligned}$$

This association is made because the quark bilinear forms transform in the same way under $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as the corresponding meson fields. The dimensions do not match, so there must be some dimensional coefficient relating them; this coefficient could even be a function of the group invariant $\sigma^2 + \boldsymbol{\pi}^2 \sim \left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\right)^2$. Does this particular combination of four-quark condensates change with temperature? The temperature dependence of the four-quark condensates at low temperatures was first

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calculated in [35] with the help of the fluctuation–dissipation theorem. The contribution of pions alone was later discussed in [36] using soft-pion techniques. From [35, 36] one can state the two condensates separately:

$$\left\langle\, (\bar\psi\psi)^2 \,\right\rangle = \left(1 - \frac{T^2}{4f_\pi^2}\right) \langle 0|(\bar\psi\psi)^2|0\rangle - \frac{T^2}{12f_\pi^2}\langle 0|(\bar\psi\gamma_5\boldsymbol{\tau}\psi)^2|0\rangle \quad (12.184)$$

and

$$\left\langle(\bar\psi\gamma_5\boldsymbol{\tau}\psi)^2\right\rangle = \left(1 - \frac{T^2}{12f_\pi^2}\right) \langle 0|(\bar\psi\gamma_5\boldsymbol{\tau}\psi)^2|0\rangle - \frac{T^2}{4f_\pi^2}\langle 0|(\bar\psi\psi)^2|0\rangle \quad (12.185)$$

Therefore there is no correction to this group invariant to order T^2/f_π^2 inclusive:

$$\left\langle(\bar\psi\psi)^2 - (\bar\psi\gamma_5\boldsymbol{\tau}\psi)^2\right\rangle \;=\; \langle 0|(\bar\psi\psi)^2 - (\bar\psi\gamma_5\boldsymbol{\tau}\psi)^2|0\rangle \quad (12.186)$$

This result is consistent with our analysis of the nonlinear σ model in previous sections.

Now let us return to the business of computing the temperature dependence of the scalar condensate to one- and two-loop order. In terms of the three representations used in the discussion of the nonlinear σ model in Section 12.4.3 the σ field is given by

$$\begin{aligned}\frac{\sigma}{f_\pi} &= \sqrt{1 - \frac{\boldsymbol{\pi}^2}{f_\pi^2}} = 1 - \frac{\boldsymbol{\pi}^2}{2f_\pi^2} - \frac{\left(\boldsymbol{\pi}^2\right)^2}{8f_\pi^4} + \cdots \\ &= \left(1 - \frac{\mathbf{p}^2}{2f_\pi^2} + \frac{(\mathbf{p}^2)^2}{16f_\pi^4}\right)^{1/2}\left(1 + \frac{\mathbf{p}^2}{4f_\pi^2}\right)^{-1} = 1 - \frac{\mathbf{p}^2}{2f_\pi^2} + \frac{(\mathbf{p}^2)^2}{8f_\pi^4} + \cdots \\ &= \cos\left(\frac{\phi}{f_\pi}\right) = 1 - \frac{\boldsymbol{\phi}^2}{2f_\pi^2} + \frac{(\boldsymbol{\phi}^2)^2}{24f_\pi^4} + \cdots \end{aligned} \quad (12.187)$$

To second order in the pion field all three representations are the same. Using the free-field expression for the thermal average of the field squared we get

$$\frac{\langle\sigma\rangle}{f_\pi} = 1 - \frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right) + \cdots \quad (12.188)$$

For $N = 4$, the only value for that we can quantitatively compare with QCD, this agrees with the result of Gasser and Leutwyler [21].

The coefficient of the term that is fourth order in the pion field differs in sign and magnitude among the three representations. It would be a miracle if the thermal average of $\sqrt{1 - \boldsymbol{\pi}^2/f_\pi^2}$, $\cos(\phi/f_\pi)$, and the Weinberg expression were all the same! But regarding the order $(T^2/12f_\pi^2)^2$ we must recognize that the term that is second order in the pion field gets modified owing to a one-loop self-energy. This was computed for each representation in Section 12.4.3 and the results were listed in (12.173). The term that is fourth order in the pion field can be evaluated using free fields. The result is

$$\left\langle(\phi^2)^2\right\rangle = (N^2-1)\left(\frac{T^2}{12}\right)^2 \tag{12.189}$$

and is obviously representation independent. The contributions for each representation are

$$\begin{aligned}
\boldsymbol{\pi}:&\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1-\left(\frac{T^2}{12f_\pi^2}\right)\right]-\frac{N^2-1}{8}\left(\frac{T^2}{12f_\pi^2}\right)^2\\
\mathbf{p}:&\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1+\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\right]+\frac{N^2-1}{8}\left(\frac{T^2}{12f_\pi^2}\right)^2\\
\boldsymbol{\phi}:&\ 1-\frac{N-1}{2}\left(\frac{T^2}{12f_\pi^2}\right)\left[1+\frac{N-2}{3}\left(\frac{T^2}{12f_\pi^2}\right)\right]+\frac{N^2-1}{24}\left(\frac{T^2}{12f_\pi^2}\right)^2
\end{aligned} \tag{12.190}$$

where the second term in each line comes from the square of the pion field and the last term comes from the pion field in fourth order. The sum of all terms is identical in all three representations; it is

$$\frac{\langle\sigma\rangle}{f_\pi} = 1-(N-1)\left(\frac{T^2}{24f_\pi^2}\right)-\frac{(N-1)(N-3)}{2}\left(\frac{T^2}{24f_\pi^2}\right)^2+\cdots \tag{12.191}$$

The miracle happens. It is a consequence of the fact that physical quantities must be independent of field redefinition. What is more, for $N = 4$ it agrees with the previously obtained result of Gasser and Leutwyler. However, we emphasize once more that this expression should not be used to infer a critical temperature because it has been derived under the assumption that the temperature is small compared with f_π.

A calculation of the scalar condensate in the nonlinear model near the critical temperature was made by Jeon and Kapusta [34]. The result is exactly the same as in the linear model, (12.114), namely

$$\langle\sigma\rangle^2 = v^2(T) = f_\pi^2 - \frac{N+2}{12}T^2 \tag{12.192}$$

This expression has corrections of order $v^2(T)/N$ and T^2/N in the large-N expansion.

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12.5 Exercises

12.1 Use the exponential representation of the pseudoscalar fields (just above (12.6)) in the leading-order chiral Lagrangian $\mathcal{L}_2$ to calculate the four-pion interaction.

12.2 Use the four-pion interaction calculated in the first exercise to calculate the two-loop contribution to the pressure of a pion gas. Compare with (12.23).

12.3 Use the chiral Lagrangian $\mathcal{L}_2$ to compute the π–π scattering amplitude. Use it to calculate the pion self-energy as in Section 12.2. Compare your result with (12.173).

12.4 Read the paper by Dey, Eletsky, and Ioffe and rederive the mixing rule for vector and axial-vector correlators at finite temperature.

12.5 Derive (12.116).

12.6 Construct a Padé approximation for $\langle\sigma\rangle = v(T)$ to extrapolate from $T \ll f_\pi$ to $T \to T_c$.

12.7 Do the linear and nonlinear σ models satisfy the Weinberg sum rules at finite temperature? Explain your answer.

12.8 How are conditions (12.103) and (12.104) modified if the ρ and a_1 spectral densities are taken to be relativistic Breit–Wigner distributions with momentum-independent but temperature-dependent widths instead of delta functions?

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