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CHAOS ON FUNCTION SPACES

RICHARD M. ARON, JUAN B. SEOANE-SEPÚLVEDA AND ANDREAS WEBER

We give a sufficient condition for an operator to be chaotic and we use this condition to show that, in the Banach space $C_0[0,\infty)$ the operator $(T_{\lambda,c}f)(t) = \lambda f(t+c)$ (with $\lambda > 1$ and $c > 0$) is chaotic, with every $n \in \mathbb{N}$ being a period for this operator. We also describe a technique to construct, explicitly, hypercyclic functions for this operator.

1. INTRODUCTION AND PRELIMINARIES

Let X denote a separable infinite dimensional Banach space and $T : X \to X$ a bounded linear operator on X. We call $x \in X$ a *hypercyclic vector* for T if the orbit,

$$\{T^n x : n \in \mathbb{N}\},$$

is dense in X. If there exists such an $x \in X$ we call T a *hypercyclic operator*. T is called *chaotic* if T is hypercyclic and the set of periodic points,

$$X_p := \{x \in X \backslash \{0\} \mid \exists n \in \mathbb{N} : T^n x = x\},$$

is dense in X. This definition of the term *chaos* is due to Devaney ([**2**], see also [**1**]).

There are a number of significant criteria that imply hypercyclicity (see, for example, [**4**]). However, for the purposes of this paper, the original version of Kitai will be adequate.

KITAI'S CRITERION. Let X be a separable Banach space and T a bounded operator on X. Suppose that Y_1 and Y_2 are dense subsets of X and $Z : Y_1 \to Y_1$ is a (not necessarily linear nor continuous) map with:

1. $TZx = x$ for all $x \in Y_1$,
2. $\lim_{n\to\infty} Z^n x = 0$ for all $x \in Y_1$,
3. $\lim_{n\to\infty} T^n y = 0$ for all $y \in Y_2$.

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Then T is hypercyclic.

In this paper, we give a sufficient condition for an operator T to be chaotic using some properties of the point spectrum $\sigma_p(T)$ of T. The C_0-*semigroup version* of this result was first proved in [**3**]; here we give the result in the setting of bounded operators. In this theorem we assume that $\sigma_p(T)$ intersects the unit circle ∂D (note that this is always true for a chaotic operator). We then apply this result to show that in the Banach space $C_0[0,\infty)$ of all $\mathbb{C}$−valued continuous functions on $[0,\infty)$ that vanish at ∞, the operators

$$\begin{array}{rcccl} T_{\lambda,c} & : & C_0[0,\infty) & \longrightarrow & C_0[0,\infty) \\ & & f(t) & \mapsto & \lambda f(t+c) \end{array}$$

with $\lambda > 1$ and $c > 0$ are chaotic. In this case it is interesting that all $n \in \mathbb{N}$ are periods for $T_{\lambda,c}$, that is, for all $n \in \mathbb{N}$ there is an $0 \neq x \in X$ such that $T_{\lambda,c}^n x = x$ and n cannot be chosen smaller. We also give a technique to explicitly construct hypercyclic functions for this chaotic operator.

2. Chaotic Operators and Chaos on $C_0[0,\infty)$

Theorem 2.1. *Let X be a separable Banach space, T a bounded linear operator on X and $U \subseteq \sigma_p(T)$ an open and connected subset of the point spectrum of T.*

For all $\lambda \in U$ choose $x_\lambda \in X \setminus \{0\}$ with $Tx_\lambda = \lambda x_\lambda$. For $x^* \in X^*$ (the dual space of X), we define the function

$$\begin{array}{rcccl} F_{x^*} & : & U & \longrightarrow & \mathbb{C} \\ & & \lambda & \mapsto & \langle x^*, x_\lambda \rangle. \end{array}$$

If

1. *F_{x^*} is analytic in U for all $x^* \in X^*$,*
2. *$F_{x^*} = 0$ if and only if $x^* = 0$, and*
3. *$U \cap \partial D \neq \emptyset$,*

then T is a chaotic operator.

In the proof we shall need the following lemma.

Lemma 2.2. *Assume that the subset $\Omega \subseteq U$ contains an accumulation point. Then $Y_\Omega := \operatorname{span}\{x_\lambda \mid \lambda \in \Omega\}$ is dense in X.*

Proof of the Lemma: Assume $\overline{Y}_\Omega \neq X$. It follows from the Hahn-Banach theorem that there is an $x^* \in X^* \setminus \{0\}$ such that $F_{x^*}(\lambda) := \langle x^*, x_\lambda \rangle = 0$ for all $\lambda \in \Omega$. By the identity theorem for analytic functions, it follows that $F_{x^*} = 0$ and so $x^* = 0$, which is a contradiction. ∎

Proof of the Theorem: We define the sets

$$\Omega_1 := \big\{\lambda \in U : |\lambda| > 1\big\} = U \cap \overline{D}^c,$$
$$\Omega_2 := \big\{\lambda \in U : |\lambda| < 1\big\} = U \cap D, \quad \text{and}$$
$$\Omega_3 := \big\{\lambda \in U : \lambda \in \exp(2\pi i \mathbb{Q})\big\}.$$

Choose $\mu \in U \cap \partial D$. Since U is open, there exists a disk K with $\mu \in K$ and $K \subseteq U$. It follows that Ω_1 and Ω_2 are nonempty and contain an accumulation point. Since K contains an arc in ∂D, Ω_3 also contains an accumulation point.

With our previous lemma the sets Y_{Ω_j} $(j = 1,2,3)$ are dense in X. Now we are ready to apply Kitai's criterion. We define Z on Y_{Ω_1} as follows:

$$Z\left(\sum_{k=1}^{n} a_k x_{\lambda_k}\right) := \sum_{k=1}^{n} \frac{a_k}{\lambda_k} x_{\lambda_k}$$

Now it follows easily that T is hypercyclic. It remains to show that X_p (the set of periodic points) is dense in X.

For $x \in Y_{\Omega_3}$,

$$x = \sum_{k=1}^{n} a_k x_{\lambda_k} \quad \text{with} \quad \lambda_k := \exp\left(2\pi i \frac{m_k}{n_k}\right)$$

and $N := \prod_{k=1}^{n} n_k$, it follows that $T^N x = x$. Since Y_{Ω_3} is dense in X the same is true for X_p, and the proof is complete. □

We next apply this result to the weighted translation operator $T_{\lambda,c}$ acting on the Banach space $C_0[0,\infty)$ of continuous functions on $[0,\infty)$ that vanish at ∞ with the standard norm $\|f\| = \max_{t\in[0,\infty)} |f(t)|$.

We first show that Theorem 2.1 implies that if $\lambda > 1$ and $c > 0$, then the bounded operator

$$\begin{array}{rcccl} T_{\lambda,c} & : & C_0[0,\infty) & \longrightarrow & C_0[0,\infty) \\ & & f(t) & \mapsto & \lambda f(t+c) \end{array}$$

is chaotic.

Theorem 2.3. *For every $\lambda > 1$ and $c > 0$, $T_{\lambda,c}$ is chaotic, and every natural number is a period for $T_{\lambda,c}$.*

Proof: To simplify notation, call $T = T_{\lambda,c}$. The function $g(t) = e^{\alpha t}$ for $\operatorname{Re}(\alpha) < 0$ is in $X = C_0[0,\infty)$ and is an eigenfunction of T, since $Tg = \lambda e^{\alpha c} g$. Thus the point spectrum $\sigma_p(T)$ contains $\lambda D = \{\mu \in \mathbb{C} : |\mu| < \lambda\}$. (One can even show that equality holds but we shall not need it here.) Choose

$$U = \{\mu \in \mathbb{C} : |\mu - 1| < \min\{1/2, \lambda - 1\}\}.$$

U is an open and connected subset of $\sigma_p(T)$ which intersects ∂D.

For all $\mu \in U$ we choose $g_\mu(t) = e^{\alpha t}$ where α is defined by the relation $\mu = \lambda e^{\alpha c}$. Then $\alpha = 1/c \cdot \log \mu/\lambda$ with an analytic branch of the logarithm on U. For $\varphi \in L^1[0,\infty)$, we define

$$F_\varphi(\mu) = \langle g_\mu, \varphi\rangle = \int_0^\infty g_\mu(t)\varphi(t)\,dt = \int_0^\infty e^{\alpha t}\varphi(t)\,dt.$$

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Since the logarithm is analytic on U, F_φ is analytic on U. Furthermore, F_φ is the Laplace transform of φ, so $F_\varphi = 0$ implies $\varphi = 0$. It follows from the previous theorem that T is chaotic.

It is known that the set of periods for a chaotic operator on a Banach space,

$$\{n \in \mathbb{N} : \exists x \neq 0 \in X \text{ such that } n \text{ is the smallest number with } T^n x = x\},$$

is infinite (see [**6**]). However, in our case, every $n \in \mathbb{N}$ is a period for T:

For $n \in \mathbb{N}$ we define

$$h_n(t) = \sin\Big(\frac{2\pi}{c} \cdot t\Big) \cdot \sum_{k=0}^{\infty} \frac{1}{\lambda^{nk}} \cdot \chi_{[nkc, n(k+1)c]}(t).$$

Then $T^n h_n = h_n$ and this is not true for a smaller n. ∎

We conclude with the following explicit construction.

EXAMPLE 2.4. The construction of a hypercyclic function for $T_{\lambda,c}$ acting on $C_0[0,\infty)$.

Let $\{f_j \mid j \in \mathbb{N}\}$ be dense in $C_0[0,\infty)$. Suppose that $\|f_j\| = M_j$ and that $m_j > 0$ is such that $\big\|f_j|_{[cm_j,\infty)}\big\| < 1/j$. For each j, we choose $k_j \in \mathbb{N}$ inductively so that the following conditions are satisfied:

(i) $k_0 = k_1 = 0$.

(ii) $M_j/\lambda^{k_j - k_{j-1}} < 1/j$,

(iii) $m_j + k_j + 2 \leqslant k_{j+1}$.

It will be convenient to use the auxiliary function F_j, defined for each $j \in \mathbb{N}$ by $F_j(x) = 1/\lambda^{k_j} f_j(x - ck_j)$ if $ck_j \leqslant x \leqslant ck_j + cm_j$.

Let $f : [0,\infty) \to \mathbb{R}$ be defined as

$$(1) \qquad f(y) = \begin{cases} F_j(y) & \text{if } y \in [ck_j, ck_j + cm_j], \\ 0 & \text{if } y \in [ck_j + cm_j + c, ck_{j+1} - c], \\ \text{linear} & \text{if } y \in [cm_j + ck_j, cm_j + ck_j + c], \\ & \text{or if } y \in [ck_{j+1} - c, ck_{j+1}] \end{cases}$$

and arranged so that f is continuous. We claim that f is hypercyclic for the operator T given by $T(g)(x) = \lambda g(x + c)$. To see this, note first that $f \in C_0[0,\infty)$, since for each j and each $x \in [ck_j, ck_{j+1}]$, $|f(x)| < 1/j$.

Next, to prove that the set of iterates $\{T^n f \mid n \in \mathbb{N}\}$ is dense in $C_0[0,\infty)$, we recall that for any $k \in \mathbb{N}$, $\{f_\ell \mid \ell \geqslant k\}$ is dense. Thus it suffices to show that $\|T^{k_j} f - f_j\| \to 0$ as $j \to \infty$. Let's fix $j \in \mathbb{N}$ and examine $T^{k_j} f(x)$, as x varies in $[0,\infty)$. If $0 \leqslant x \leqslant cm_j$, then $ck_j \leqslant x + ck_j \leqslant ck_j + cm_j$, and so

$$T^{k_j} f(x) = \lambda^{k_j} f(x + ck_j) = \lambda^{k_j} F_j(x + ck_j) = f_j(x).$$

Therefore,

$$\left\|T^{k_j} f - f_j\right\| = \sup_{x \geqslant cm_j} \left|T^{k_j} f(x) - f_j(x)\right| \leqslant \sup_{x \geqslant cm_j} \left|T^{k_j} f(x)\right| + 1/j.$$

To see that $\sup\limits_{x \geqslant cm_j} \left|T^{k_j} f(x)\right| \to 0$ as $j \to \infty$, it is enough to restrict x to intervals of the form $[ck_n - ck_j, ck_n - ck_j + cm_n]$ where $n > j$. For x in this interval, we have $ck_n \leqslant x + ck_j \leqslant ck_n + cm_n$ and so

$$\begin{aligned} \left|T^{k_j} f(x)\right| &= \left|\lambda^{k_j} f(x + ck_j)\right| = \left|\lambda^{k_j} F_n(x + ck_j)\right| \\ &= \left|\lambda^{k_j}(1/\lambda^{k_n}) f_n(x + ck_j - ck_n)\right| \leqslant \lambda^{k_j}(M_n/\lambda^{k_n}) < 1/n < 1/j. \end{aligned}$$

This completes the construction.

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Math Department
Kent State University
Kent, Ohio, 44242
United States of America
e-mail: aron@math.kent.edu
jseoane@math.kent.edu

Mathematisches Institut II
Universität Karlsruhe
D-76128 Karlsruhe
Germany
e-mail: andreas.weber@math.uni-karlsruhe.de