SOME NON-CRITICAL IDEMPOTENTS IN THE CLOSURE OF THE CHARACTERS IN THE MAXIMAL IDEAL SPACE OF $M(D_2)$

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Abstract

This paper shows that the idempotent generalized characters associated with a Raikov System generated by a $K_2$ set in $D_2 = \prod_{i=1}^{\infty}(Z_2)$, is contained in the closure of the characters $D_2^*$ in $\Delta M(D_2)$. 1980 Mathematics subject classification (Amer. Math. Soc.): 43 A 46.

1. Introduction

We will be working with the compact totally disconnected abelian group $D_2 = \prod_{i=1}^{\infty}(Z_2)$, which has dual group $D_2^* = \Theta_{i=1}^{\infty}(Z_2)$, where $Z_2$ is the multiplicative group of order 2, $Z_2 = \{1, -1; \cdot\}$. The dual group $D_2^*$ is canonically embedded in $\Delta M(D_2)$, the maximal ideal space of $M(D_2)$.

A compact perfect subset $K$ of $D_2$ is called a $K_2$ subset of $D_2$ if for any continuous function $f: K \to Z_2$ there is a character $\phi \in D_2^*$ such that $\phi$ restricted to $K$ is equal to $f$.

It will be shown that the idempotent associated with any Raikov System generated by a $K_2$ set is contained in the closure of the characters $D_2^*$ in $\Delta M(D_2)$. Dunkl and Ramirez (1972) have shown that the idempotent associated with the Raikov System generated by a closed subgroup is in the closure of the characters.

As the maximal ideal space $\Delta M(D_2)$ has the weak topology from the Fourier-Stieltjes transforms of the measures in $M(D_2)$, an idempotent associated with a Raikov System is in the closure of the characters if and only if the Fourier-Stieltjes
transforms of the measures satisfy the following condition:

For all measures \( \mu \) concentrated on the Raikov System and for all measures \( \nu \) which annihilate all the sets in the Raikov System

\[ ||\mu||_\infty \leq ||(\mu + \nu)\||_\infty \]

where the sup norm is taken over \( D_2 \).

We will prove that Raikov Systems generated by \( K_2 \) subsets of \( D_2 \) satisfy this bound by constructing a series of positive definite functions such that, for each measure \( \mu \) and \( \nu \) as above, there is a positive definite function \( P_\gamma \) such that

\[ \left| \int_{D_2} P_\gamma d\mu - \mu(1) \right| < \epsilon \quad \text{and} \quad \left| \int_{D_2} P_\gamma d\nu \right| < \epsilon. \]

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2. Raikov systems and idempotents

Let \( G \) be a locally abelian group and let \( A \) be a subset of \( G \). We define the Raikov System of sets of \( G \) generated by \( A \), \( \mathcal{R}_A \), to be the collection of all measurable subsets of some countable union of translates of sums of \( A \).

That is to say

\[
\mathcal{R}_A = \left\{ B \subseteq G : \begin{array}{l} 1. \ B \text{ is measurable;} \\
2. \ \exists x_i \in G, m_i \in \mathbb{Z}^+, \text{ for } i \in \mathbb{Z}^+, \text{ such that } B \subseteq \bigcup_{i=1}^{\infty} x_i + (m_i)A \end{array} \right\}
\]

where for \( m \in \mathbb{Z}^+ \),

\[
(m)A = A + A + \cdots + A = \left\{ \sum_{i=1}^{m} x_i : x_i \in A, i = 1, \ldots, m \right\}.
\]

We notice that \( \mathcal{R}_A \) is closed under translation, intersection, countable unions and addition of sets.

The Raikov System \( \mathcal{R}_A \) is now used to define a direct sum splitting of \( M(G) \) into the \( L \)-algebra \( \mathcal{D}_A \) of measures concentrated on the sets in the Raikov System, and the \( L \)-ideal \( \mathcal{I}_A \) of measures which annihilate all the sets in the Raikov System.
The maximal ideal space of \( M(D_2) \)

That is to say,

\[ \mathcal{A} = \{ \mu \in M(G) : \exists B \in \mathcal{B} \text{ such that } \mu \text{ is concentrated on } B \}, \quad \text{and} \]

\[ \mathcal{A} = \{ \nu \in M(G) : \nu(B) = 0 \text{ for all } B \in \mathcal{A} \}. \]

The idempotent \( I_A \) associated with this Raikov system is the projection from \( M(G) \) onto \( \mathcal{A} \) and hence is a homomorphism and an element of the maximal ideal space of \( M(G) \).

The group of characters on \( G \), \( G^\ast \), is canonically embedded in \( \Delta M(G) \) and the idempotent \( I_A \) is contained in the closure of the characters \( G^\ast \) in \( \Delta M(G) \) if and only if the projection

\[ I_A : M(G) \to \mathcal{A} \]

is bounded in the Fourier-Stieltjes transform norm, that is to say, for any measure \( \mu \in M(G) \)

\[ \| (I_A \mu) \|_{\infty} \leq \| \mu \|_{\infty} \]

where the sup norm is taken over \( G^\ast \). If we have two direct sum splittings of \( M(G) \) into an \( L \)-subalgebra and \( L \)-ideal associated with idempotents in the closure of the characters, then the direct sum splitting

\[ \mathcal{A}_1 \cap \mathcal{A}_2 \oplus \mathcal{A}_1 + \mathcal{A}_2 \]

is also a splitting of \( M(G) \) associated with an idempotent in the closure of the characters of \( G \).

3. Properties of \( K_2 \) sets

Let \( K \subseteq D_2 \) be a compact perfect \( K_2 \) subset of \( D_2 \). \( \overline{Gp K} \) is a closed subgroup of \( D_2 \) and so \( D_2 = \overline{Gp K} \oplus H \) where \( H \) is also a closed subgroup of \( D_2 \). \( \overline{Gp K} \) is isomorphic to \( D_2 \) and the idempotent associated with the Raikov System generated by \( Gp K \) is in the closure of the characters (Dunkl and Ramirez (1972)).

If we have a Raikov splitting of \( M(D_2) \) generated by a set \( A \subseteq \overline{Gp K} \) then the idempotent associated with this splitting is in the closure of the characters if and only if the condition

\[ \|(I_A \mu)\|_{\infty} \leq \| \mu \|_{\infty} \]

holds for all measures \( \nu \) in \( M(\overline{Gp K}) \). For convenience therefore we assume that \( \overline{Gp K} = D_2 \).

For any continuous function \( \phi : K \to \mathbb{Z}_2 \) there is a character \( \chi \in D_2^\ast \) such that \( \chi|_K = \phi \). For each character \( \chi \) in \( D_2^\ast \) we let \( P_\chi = \{ x \in K : \chi(x) = -1 \} \). We say
that a set of characters \( \{ \chi_1, \cdots, \chi_n \} \) determines a partition of \( K \) if

1. \( P_{\chi_i} \cap P_{\chi_j} = \emptyset \quad \forall i \neq j, 1 \leq i, j \leq n; \)

2. \[ K = \bigcup_{i=1}^{n} P_{\chi_i}. \]

Given two partitions \( \mathcal{P} = \{ P_{\chi_1}, P_{\chi_2}, \ldots, P_{\chi_n} \} \) and \( \mathcal{P}' = \{ P_{\phi_1}, P_{\phi_2}, \ldots, P_{\phi_m} \} \) of \( K \) determined by the characters \( \{ \chi_1, \chi_2, \ldots, \chi_n \} \) and \( \{ \phi_1, \ldots, \phi_m \} \) respectively we say \( \mathcal{P} \) is an everywhere finer partition of \( K \) than \( \mathcal{P}' \) if for each \( 1 \leq i \leq m \) there exists an \( I_i \subseteq \{1, \ldots, n\} \) with \( \#I_i \geq 2 \) such that

\[ P_{\phi_i} = \bigcup_{j \in I_i} P_{\chi_j}. \]

Since \( \varphi K = D_2 \) this implies that

\[ \phi_i = \prod_{j \in I_i} \chi_j. \]

As \( K \) is a \( K_2 \) subset of \( D_2 \), for any continuous function \( f \) from \( K \) into \( Z_2 \) there is a partition (in fact, trivial) \( \mathcal{P} = \{ P_{\chi_1}, P_{\chi_2}, \ldots, P_{\chi_n} \} \) such that

\[ f = \prod_{j \in I} \chi_j \bigg|_K \quad \text{for some } I \subseteq \{1, 2, \ldots, m\}. \]

We say that the function \( f \) can be generated by the partition \( \mathcal{P} \).

Given two partitions of \( K \), \( \mathcal{P} = \{ P_{\phi_1}, P_{\phi_2}, \ldots, P_{\phi_m} \} \) and \( \mathcal{P}' = \{ P_{\phi_1}, P_{\phi_2}, \ldots, P_{\phi_m} \} \), there exists a partition \( \mathcal{P}''' \) which is everywhere finer than \( \mathcal{P} \) and \( \mathcal{P}' \) since, for each \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), if we have that \( P_{\chi_i} \cap P_{\phi_j} \neq \emptyset \) then there exists a character \( \omega_{ij} \) such that

\[ \omega_{ij} = \begin{cases} -1 & \text{on } P_{\chi_i} \cap P_{\phi_j} \\ 1 & \text{elsewhere on } K \end{cases} \]

and so the partition determined by the characters

\( \{ \omega_{ij} : P_{\chi_i} \cap P_{\phi_j} \neq \emptyset, 1 \leq i \leq m, 1 \leq j \leq n \} \)

is finer than \( \mathcal{P} \) and \( \mathcal{P}' \). Since \( K \) is totally disconnected there exists a partition \( \mathcal{P}''' \) which is everywhere finer than \( \{ \omega_{ij} : P_{\chi_i} \cap P_{\phi_j} \neq \emptyset, 1 \leq i \leq m, 1 \leq j \leq n \} \) and so is everywhere finer than both \( \mathcal{P} \) and \( \mathcal{P}' \).

We define a sequence of partitions of \( K \), \( \{ \mathcal{P}_i \}_{i \in \mathbb{Z}^+} \) where

\[ \mathcal{P}_i = \{ P_{\phi_{i1}}, P_{\phi_{i2}}, \ldots, P_{\phi_{in}} \} \]

determined by a set of characters \( \{ \phi_{i1}, \phi_{i2}, \ldots, \phi_{in} \} \), to be a “separating sequence of partitions of \( K \)” if

1. \( \forall n \in \mathbb{Z}^+ \) \( \mathcal{P}_{n+1} \) is an everywhere finer partition of \( K \) than \( \mathcal{P}_n \):

2. for each continuous function \( f : K \to \mathbb{Z}_2 \) there is an \( N \in \mathbb{Z}^+ \) such that \( f \) can
be generated by the partition $\mathcal{P}_N$ and hence $f$ can be generated by each partition $\mathcal{P}_n$ for $n \geq N$.

**Lemma 1.** Let $K$ be a $K_2$ subset of $D_2$. Then there is a separating sequence of partitions of $K$.

**Proof.** The set of continuous functions from $K$ into $Z_2$ is countable: denote it by $\{f_i : i \in Z^+\}$. We will define the sequence of partitions inductively. Let $\mathcal{P}_1$ be a partition of $K$ which generates $f_1$. Let $\mathcal{P}_2$ be a partition of $K$ everywhere finer than $\mathcal{P}_1$ which also generates $f_2$. Inductively, let $\mathcal{P}_{n+1}$ be an everywhere finer partition than $\mathcal{P}_n$ that generates $f_{n+1}$ and hence also generates $f_1, f_2, \ldots, f_n$.

We can now characterize elements of $K$, $(m)K$ and $\overline{Gp_K}$ using a separating sequence of partitions of $K$. Let $\{\mathcal{P}_n\}_{n \in Z^+}$, where $\mathcal{P}_n = \{P_{\phi_1^n}, P_{\phi_2^n}, \ldots, P_{\phi_{|n|}^n}\}$, be a separating sequence of partitions of $K$. We define a sequence of characters $\{\phi_k^m\}_{m \in Z^+}$ where $1 \leq k(m) \leq j(m)$ to be a "chain" of characters from the separating sequence $\{\mathcal{P}_n\}_{n \in Z^+}$ if

$$P_{\phi_{k(1)}^1} \supset P_{\phi_{k(2)}^1} \supset P_{\phi_{k(3)}^1} \supset \cdots \supset P_{\phi_{k(n)}^1} \supset P_{\phi_{k(n+1)}^1} \supset \cdots.$$  

Obviously if we have two chains $\{\phi_{k(i)}^i\}_{i \in Z^+}$ and $\{\phi_{m(i)}^i\}_{i \in Z^+}$ such that for some $N \in Z^+$, $\phi_{k(N)}^N \neq \phi_{m(N)}^N$, then $\phi_{k(n)}^n \neq \phi_{m(n)}^n$ for all $n \geq N$. We also have the following lemma.

**Lemma 2.** Given $K$ a $K_2$ subset of $D_2$ with $\overline{Gp_K} = D_2$, and $\{\mathcal{P}_i\}_{i \in Z^+}$ a separating sequence of partitions of $K$, then $K$ is equal to the set $H$ where

$$H = \left\{ x \in C_2 : \begin{array}{l}
\exists \text{ chain } \{\phi_{k(i)}^i\}_{i \in Z^+} \text{ of characters from the separating sequence such that}
1. \phi_{k(i)}^i(x) = -1 \quad \forall i \in Z^+;
2. \phi_{m}^m(x) = 1 \quad \forall m \neq k(i), 1 \leq m \leq j(i).
\end{array} \right\}$$

**Proof.** Obviously $K \subseteq H$. Let $\{\phi_{k(i)}^i\}_{i \in Z^+}$ be a chain of characters from the separating sequence of $K$. By definition $P_{\phi_{k(i)}^i} \subseteq K$, and so we have $\bigcap_{i \in Z^+} P_{\phi_{k(i)}^i}$ is non-empty, as $\{\phi_{k(i)}^i\}_{i \in Z^+}$ is a chain, and so $\bigcap_{i \in Z^+} P_{\phi_{k(i)}^i} = \{x\}$ for some $x \in K$ as $\{\mathcal{P}_i\}_{i \in Z^+}$ is a separating sequence of partitions of $K$. So we have

$$\phi_{k(i)}^i(x) = -1 \quad \forall i \in Z^+$$

and

$$\phi_{m}^m(x) = 1, \quad m \neq k(i), \forall 1 \leq m \leq j(i).$$
To show that $K = H$, we need to show that for each chain $\{\phi_{k(i)}\}_{i \in \mathbb{Z}^+}$ from the separating sequence there is a unique $x \in D_2$ with

$$\phi_{k(i)}(x) = -1 \quad \forall \, i \in \mathbb{Z}^+$$

and

$$\phi_m(x) = 1 \quad \forall \, m \neq k(i), \, 1 \leq m \leq j(i).$$

Assume $x, y \in D_2$ with

$$\phi_{k(i)}(x) = \phi_{k(i)}(y) = -1 \quad \forall \, i \in \mathbb{Z}^+$$

and

$$\phi_m(x) = \phi_m(y) = 1 \quad \forall \, m \neq k(i), \, 1 \leq m \leq j(i),$$

so if $x$ and $y$ are distinct there must exist a character $\chi \in D_2^*$ such that $\chi(x) \neq \chi(y)$, but $\overline{\text{Gp}} K = D_2$ so $\chi|_K$ is not identically equal to 1. As $\{\phi_i\}_{i \in \mathbb{Z}^+}$ is a separating sequence of partitions of $K$ there must be an $i \in \mathbb{Z}^+$ such that $\chi$ is generated by the functions $\{\phi_1, \phi_2, \ldots, \phi_{j(i)}\}$ on $K$. Thus

$$\phi_m(x) \neq \phi_m(y) \quad \text{for some} \, 1 \leq m \leq j(i)$$

and so $x = y$.

As $\overline{\text{Gp}} K = D_2$, every character on $D_2$ is uniquely determined by its restrictions to $K$, so given $\{\phi_i\}_{i \in \mathbb{Z}^+}$, a separating sequence of partitions of $K$, we have that every element of $D_2$ is uniquely determined by the values of the $\phi_k(x)$ where $\phi_k$ are the characters from the separating sequence $\{\phi_i\}_{i \in \mathbb{Z}^+}$.

For $x \in \overline{\text{Gp}} K = D_2$ we define the length of $x$ on the $n$th partition of $K$ to be

$$l(n, x) = \sum_{i=1}^{u(n)} \frac{1}{2} (1 - \phi_i^n(x)) = \# \{ \phi_i^n : \phi_i^n(x) = -1, \, 1 \leq i \leq j(n) \}.$$
PROOF. 1. We can see from Lemma 2 that each \( x \in K \) is uniquely associated with a chain of characters, say \( \{ \phi_{k(i)}^j \}_{i \in \mathbb{Z}^+} \), from the separating sequence with
\[
\phi_{k(i)}^j(x) = -1 \quad \forall \, i \in \mathbb{Z}^+
\]
and
\[
\phi_m^j(x) = 1 \quad \forall \, 1 \leq m \leq j(i), \, m \neq k(i),
\]
and so \( l(n, x) = 1 \).

2. \( x = x_1 + x_2 + \cdots + x_m, \, x_i \in K, \) all distinct. Each \( x_i \) is uniquely associated with a chain \( \{ \phi_{k(n,i)}^n \}_{n \in \mathbb{Z}^+} \) from the separating sequence with
\[
\phi_{k(n,i)}^n(x_i) = -1 \quad \forall \, n \in \mathbb{Z}^+
\]
and
\[
\phi_{j}^n(x_i) = 1, \quad 1 \leq j \leq j(n), \, j \neq k(n, i).
\]
As the \( x_i \) are distinct there exists an \( N \in \mathbb{Z}^+ \) with
\[
\phi_{k(N,i)}^N \neq \phi_{k(N,j)}^N \quad \forall \, i \neq j, \, 1 \leq i, j \leq m
\]
and so
\[
l(n, x) = m \quad \text{for all} \, n \geq N.
\]

3. a) As \( \text{Gp} K = D_2 \) and \( \{ \phi_i \}_{i \in \mathbb{Z}^+} \) is a separating sequence of partitions of \( K \), letting
\[
H_n = \{ x \in D_2 : l(i, x) = 0, \, i = 1, \ldots, n \}
\]
we have that \( \{ H_n : n \in \mathbb{Z}^+ \} \) forms a base of open neighbourhoods of zero in \( D_2 \). Let \( x \in \text{Gp} K = D_2 \) be such that \( l(n, x) = 0 \). We can write
\[
x = x_1 + x_2 + \cdots + x_r + h
\]
where \( h \in H_{n+1} \) and \( x_i \in K, \, i = 1, \ldots, r \) are distinct, so each \( x_i \) is uniquely associated with a chain \( \{ \phi_{k(m,i)}^m \}_{m \in \mathbb{Z}^+} \) where \( \phi_{k(m,i)}^m(x_i) = -1 \) and
\[
\phi_{j}^m(x_i) = 1, \quad 1 \leq j \leq j(m), \, j \neq k(m, i).
\]
Now \( l(n, x) = l(n, x_1 + x_2 + \cdots + x_r) = 0 \) so we must be able to group the \( x_i \) in pairs \( x_i, x_j \) with \( \phi_{k(n,i)}^n = \phi_{k(n,j)}^n \), and so
\[
\phi_{k(m,i)}^m = \phi_{k(m,j)}^m \quad \text{for all} \, m \leq n.
\]
Thus
\[
l(m, x_1 + x_2 + \cdots + x_r) = 0 \quad \forall \, m \leq n
\]
so \( l(m, x) = 0 \) for all \( m \leq n \).

3. b) For \( x \in \text{Gp} K = D_2 \) let \( l(n + 1, x) = m \). Then we can write \( x = x_1 + x_2 + \cdots + x_m + h \) where \( l(n + 1, h) = 0 \), so
\[
l(i, h) = 0, \quad 1 \leq i \leq n + 1,
\]
and so
\[ l(n, x) = l(n, x_1 + x_2 + \cdots + x_m) \leq m \leq l(n + 1, x). \]

4. Let \( x \in \overline{\text{Gp} K} \) and suppose that \( \lim_{n \to \infty} l(n, x) = m \), so for each \( n \in \mathbb{Z}^+ \) we can find an \( h \in H_n \) and \( x_1 \cdots x_m \in K \) so that
\[ x = x_1 + x_2 + \cdots + x_m + h. \]

As \( \{ H_n : n \in \mathbb{Z}^+ \} \) forms a base of open neighbourhoods of zero we have that
\[ x \in (m)K = (m)K. \]

4. Positive definite functions

We will now use the separating sequence of partitions of \( K \), \( \{ \phi_m \}_{m \in \mathbb{Z}^+} \), with generating characters \( \{ \phi_1^m, \phi_2^m, \ldots, \phi_{s(n)}^m \} \), for \( m \in \mathbb{Z}^+ \), to construct a sequence of positive definite functions on \( \mathbb{D}_2 \).

**LEMMA.** Let \( r \in (0, 1) \) and \( n \in \mathbb{Z}^+ \). Then the function
\[ F_r^n : \text{Gp} \bar{K} \to \mathbb{R} \]
\[ : x \mapsto r^{l(n, x)} \]
is a positive definite function on \( \mathbb{D}_2 = \overline{\text{Gp} K} \).

**PROOF.** Consider the \( n \)th partition of \( K \), \( \phi_n = \{ P_{\phi_1^n}, P_{\phi_2^n}, \ldots, P_{\phi_{s(n)}^n} \} \), generated by the characters \( \{ \phi_1^n, \phi_2^n, \ldots, \phi_{s(n)}^n \} \). The measure \( \mu_{n,r} \) on \( \mathbb{D}_2 \)
\[ \mu_{n,r} = \sum_{i=1}^{s(n)} \left( \frac{1 + r}{2} \right) \delta(1) + \left( \frac{1 - r}{2} \right) \delta(\phi_i^n) \]
is a positive measure for \( r \in (0, 1) \) and has Fourier transform
\[ \hat{\mu}_{n,r}(x) = r^{l(n, x)} \]
and so \( F_r^n : \mathbb{D}_2 \to \mathbb{R} : x \mapsto r^{l(n, x)} \) is a positive definite function on \( \mathbb{D}_2 = \overline{\text{Gp} K} \).

We now have the main theorem of this section.

**THEOREM 1.** Let \( K \subseteq \mathbb{D}_2 \) be a \( K_2 \) subset of \( \mathbb{D}_2 \) such that \( \overline{\text{Gp} K} = \mathbb{D}_2 \). Then, for each \( m \in \mathbb{Z}^+ \) and \( \epsilon, \delta > 0 \) we can choose an \( h \in \mathbb{Z}^+ \) such that, for any open neighbourhood \( H \) of zero, there exists a positive definite function \( F \) with \( F(0) = 1 \).
and

1. $F(x) > 1 - \varepsilon$ for $x \in \bigcup_{i=1}^{m} (i)K$,

2. $|F(x)| < \delta$ for $x \in D_{2} \setminus \left( \bigcup_{j=1}^{m+h} (j)K + H \right)$.

**Proof.** Choose an $r \in (0, 1)$ such that $(1 - r^{m}) < \varepsilon$ and choose an $h \in \mathbb{Z}^{+}$ so that $r^{m+h} < \delta$. Now let $H'$ be an open neighbourhood of zero of the form

$$H' = \left\{ x \in D_{2} : \phi_{j}(x) = 1 \forall 1 \leq j \leq j(i), 1 \leq i \leq I \right\} = H_{I},$$

for some $I \in \mathbb{Z}^{+}$. So we have

$$\left\{ \left( \bigcup_{i=1}^{m+h} (i)K \right) + H' \right\} = \left\{ x \in D_{2} : l(p, x) \leq m + h \text{ for } 1 \leq p \leq I \right\} = \left\{ x \in D_{2} : l(I, x) \leq m + h \right\}.$$

Now observe that $F_{r}^{I}(x) = r^{l(I, x)}$; so, for $x \in \bigcup_{i=1}^{m} (i)K$,

$$|1 - F_{r}^{I}(x)| \leq |1 - r^{m}| < \varepsilon.$$

For $x \in D_{2} \setminus \left( \bigcup_{i=1}^{m+h} (i)K + H' \right)$ we have $l(I, x) > m + h$, so $F_{r}^{I}(x) \leq r^{m+h} < \delta$.

5.

We now prove a general theorem about Raikov idempotent generalized characters in the closure of the characters of $D_{2}$, given the existence of positive definite functions with certain properties. The main result is then a corollary of Theorem 1 and the following theorem.

**Theorem 2.** Let $A \subseteq D_{2}$ be a compact perfect subset of $D_{2}$ such that, for every $m \in \mathbb{Z}^{+}$, $\varepsilon, \delta > 0$ and open neighbourhood $H$ of zero, there exists an integer $h \in \mathbb{Z}^{+}$ independent of the neighbourhood $H$ and a positive definite function $F$ on $D_{2}$ with

1. $F(0) = 0$;

2. $|F(x) - 1| < \varepsilon \forall x \in \bigcup_{i=1}^{m} (i)A$;

3. $|F(x)| < \delta \forall x \in D_{2} \setminus \left( \bigcup_{i=1}^{m+h} (i)A + H \right)$. 

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Then the idempotent generalized character $I_A$ associated with the Raikov System generated by $A$ on $D_2$ is in the closure of the characters $\hat{D}_2$ in $\Delta M(D_2)$.

**Proof.** $\Delta M(D_2)$ has the weak topology induced from the Fourier-Stieltjes transforms of the measures in $M(D_2)$, so the idempotent $I_A$ is in $\hat{D}_2$ if and only if

$$\|(I_A\mu)\|_\infty \leq \|\mu\|_\infty \quad \forall \mu \in M(D_2)$$

where the sup norms are taken over $\hat{D}_2$.

Let $\mu \in \mathbb{C}A$ and $\epsilon > 0$. We can find an $l \in \mathbb{Z}^+$ so that

$$\mu = \sum_{i=1}^{n} \delta_{x_i} \ast \mu_i + \mu'$$

where $\mu_i \in M(\bigcup_{j=1}^{l-1} (j)A)$, $\|\mu\| < \epsilon$ and $x_i \in D_2$. We will consider the measure $\Sigma_{i=1}^{n} \delta_{x_i} \ast \mu_i$ which is concentrated on

$$\bigcup_{j=1}^{n} \left( x_j + \bigcup_{i=1}^{l} (i)A \right).$$

We can assume (without loss of generality) that $S = \{x_1, x_2, \ldots, x_n\}$ is a finite subgroup of $D_2$. We can find a subgroup $S_0$ of $S$ such that

$$S + \text{Gp}A = S_0 + \text{Gp}A \quad \text{and} \quad S_0 \cap \text{Gp}A = \{0\}$$

and can find an $m \in \mathbb{Z}^+$ so that

$$\bigcup_{x \in S} \left( x + \bigcup_{i=1}^{l} (i)A \right) \subset \bigcup_{y \in S_0} \left( y + \bigcup_{i=1}^{m} (i)A \right).$$

Now we have for each $q \in \mathbb{Z}^+$ and $x \neq y \in S_0$ that

$$\left\{ x + \bigcup_{i=1}^{q} (i)A \right\} \cap \left\{ y + \bigcup_{i=1}^{q} (i)A \right\} = \emptyset$$

so we can choose an open neighbourhood $H(q)$ of zero such that for all $x, y \in S_0$, $x \neq y$,

$$\left\{ \left( x + \bigcup_{i=1}^{q} (i)A \right) + H(q) \right\} \cap \left\{ \left( y + \bigcup_{i=1}^{q} (i)A \right) + H(q) \right\} = \emptyset.$$

Now choose an $h \in \mathbb{Z}^+$ such that for every open neighbourhood $H$ of zero there exists a positive definite function $F$ on $D_2$ with

1. $F(0) = 1$;
2. $|F(x) - 1| < \epsilon \quad \forall x \in \bigcup_{i=1}^{m} (i)A$;
3. $|F(x)| < \frac{\epsilon}{|S_0|} \quad \forall x \in D_2 \setminus \left\{ \left( \bigcup_{i=1}^{m+h} (i)A \right) + H \right\}$. 

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Let $H$ be an open neighbourhood of zero contained in $H(m + h)$. Then we can find a positive definite function $F$ satisfying

1. $F(0) = 1$;
2. $|F(x) - 1| < \epsilon$ for all $x \in \bigcup_{i=1}^{m} (i)A$;
3. $|F(x)| < \frac{\epsilon}{|S_0|}$ for all $x \in D_2 \setminus \left\{ \bigcup_{i=1}^{m+h} (i)A \right\} + H$.

Now the measure $|S_0| \cdot m_{S_0}$ where $m_{S_0}$ is haar measure on $S_0$ has positive Fourier transform, so $\mathcal{F} = |S_0| \cdot m_{S_0} \ast F$ is a positive definite function on $D_2$, and

(1) \[ \mathcal{F} = \sum_{x \in S_0} \delta_x \ast F. \]

$\mathcal{F}$ has the following properties:

1. $\mathcal{F}(0) \leq 1 + \epsilon$.
2. $|\mathcal{F}(x) - 1| < 2\epsilon$, $x \in \bigcup_{y \in S_0} \left( y + \bigcup_{i=1}^{m} (i)A \right)$.
3. $|\mathcal{F}(x)| < \epsilon$, $x \in D_2 \setminus \left\{ \bigcup_{y \in S_0} y + \bigcup_{i=1}^{m+h} (i)A \right\} + H$.

Let $\nu$ be a measure in $I_A$. Then

$$ |\nu| \left( \bigcup_{y \in S_0} \left( y + \bigcup_{i=1}^{m+h} (i)A \right) \right) = 0. $$

So we can choose an open neighbourhood $H$ of zero contained in $H(m + h)$ such that

$$ |\nu| \left( \bigcup_{y \in S_0} y + \bigcup_{i=1}^{m+h} (i)A \right) + H \right\} < \epsilon $$

and let $\mathcal{F}$ be the associated positive definite function as in (1).
We then have, for $\gamma \in \mathcal{D}_2$,

$$|\hat{\mu}(\gamma)| \leq \left| \left( \sum_{i=1}^{n} \delta x_i * \mu_i \right)(\gamma) \right| + \varepsilon$$

$$\leq \int_{\mathcal{D}_2} \gamma d \left( \sum_{i=1}^{n} \delta x_i * \mu_i \right) + 2\varepsilon\|\mu\| + \varepsilon$$

$$\leq \int_{\mathcal{D}_2} \gamma d \left( \sum_{i=1}^{n} \delta x_i * \mu_i + \nu \right) + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon\|\nu\| + 2)$$

$$\leq \left\| \left( \sum_{i=1}^{n} \delta x_i * \mu_i + \nu \right) \right\|_{\infty} + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon\|\nu\| + 2)$$

(where the sup norm is taken over $\mathcal{D}_2$)

$$\leq (1 + \varepsilon)\|\mu + \nu\|_{\infty} + (1 + \varepsilon)(\varepsilon) + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon\|\nu\| + 2)$$

and so

$$\|\hat{\mu}\|_{\infty} \leq \|\mu + \nu\|_{\infty}$$

where the supremum norm is taken over $\mathcal{D}_2$.

From this we have the corollary.

**Corollary 1.** Let $K \subseteq \mathcal{D}_2$ be a compact perfect $K_2$ subset of $\mathcal{D}_2$ such that $\text{Gp} K = \mathcal{D}_2$. Then the idempotent associated with the Raikov System generated by $K$ is contained in the closure of the characters $\mathcal{D}_2^\circ$ in $\Delta M(\mathcal{D}_2)$.

**Corollary 2.** Let $K \subseteq \mathcal{D}_2$ be a compact perfect $K_2$ subset of $\mathcal{D}_2$. Then the idempotent associated with the Raikov System generated by $K$ is contained in the closure of the characters $\mathcal{D}_2^\circ$ in $\Delta M(\mathcal{D}_2)$.

**Proof.** $\mathcal{D}_2 = \overline{\text{Gp} K} \oplus H$ for some closed subgroup $H$ of $\mathcal{D}_2$. We can give $\mathcal{D}_2$ a finer l.c.a. topology $\mathcal{T}$ where

$$(\mathcal{D}_2)_\mathcal{T} = \overline{\text{Gp} K} \oplus H_d$$

where $H_d$ is the group $H$ with the discrete topology. The positive definite function $F$ with

1. $F(x) = 1 \quad \forall x \in \overline{\text{Gp} K}$
2. $F(x) = 0 \quad$ elsewhere

The positive definite function $F$ is bounded on $\mathcal{D}_2$.

The set $\mathcal{D}_2$ is bounded, and so $F$ is bounded on $\mathcal{D}_2$.
is continuous on \((D_2)_\sigma\) and so there exist continuous positive definite functions on \((D_2)_\sigma\) as required in Theorem 2. Hence the idempotent \(I_K \in \overline{(D_2)_\sigma^2}\) but \(\overline{(D_2)_\sigma^2}\) \(\subseteq D_2\), so \(I_K \in D_2\).

References


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